THE BRAUER-SEVERI VARIETY ASSOCIATED WITH A CENTRAL SIMPLE ALGEBRA: A SURVEY

by

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Abstract. — The article describes the one-to-one correspondence between central simple algebras and Brauer-Severi varieties over a field. Non-abelian group cohomology is recalled and Galois descent is worked out in detail. The classifications of central simple algebras as well as of Brauer-Severi varieties by one and the same Galois cohomology set are explained. A whole section is devoted to the discussion of functoriality. Finally, the functor of points of the Brauer-Severi variety associated with a central simple algebra is described in terms of the central simple algebra thus giving a link to another approach to the subject.

Introductory remarks

This article is devoted to the connection between central simple algebras and Brauer-Severi varieties.

Central simple algebras were studied intensively by many mathematicians at the end of the 19th and in the first half of the 20th century. We refer the reader to N. Bourbaki [**Bou**, Note historique] for a detailed account on the history of the subject and mention only a few important milestones here. The structure of central simple algebras (being finite dimensional over a field K) is fairly easy. They are full matrix rings over division algebras the center of which is equal to K. This was finally discovered by J. H. Maclagan-Wedderburn in 1907 [**MWe08**] after several special cases had been treated before. T. Molien [**Mo**] had considered the case of $\mathbb{C}$-algebras already in 1893 and the case of $\mathbb{R}$-algebras had been investigated by E. Cartan [**Ca**]. J. H. Maclagan-Wedderburn himself had proven the structure theorem for central simple algebras over finite fields in 1905 [**MWe05, Di**].

In 1929, R. Brauer ([**Br**], see also [**De**], [**A/N/T**]) found the group structure on the set of similarity classes of central simple algebras over a field K using the ideas of E. Noether about crossed products of algebras. He proved, in today's language, that it is isomorphic to the Galois cohomology group $H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), (K^{\mathrm{sep}})^*)$. Further, he discusses the structure of this group in the case of a number field.

Relative versions of central simple algebras over base rings instead of fields were introduced for the first time by G. Azumaya [**Az**] and M. Auslander and O. Goldmann [**A/G**]. The case of an arbitrary base scheme was considered by A.

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Grothendieck in his famous Bourbaki talks "Le groupe de Brauer" [**GrBrI, GrBrII, GrBrIII**]. Using the étale topology there is given a cohomological description of the Brauer group in the case that the base is not necessarily a field.

Central simple algebras over fields should be a subject in any course on algebra. For that reason, there is a lot of literature on them and the information necessary to follow this article can be found in many different sources. Among them there are the standard textbooks on algebra as S. Lang's book [**La**, section XVII, Corollary 3.5 and section 5] or N. Bourbaki [**Bou**, §5 and §10] but also more specialized literature like I. Kersten's book [**Ke**] on Brauer groups.

Brauer-Severi varieties are twisted forms of the projective space. The term "varieté de Brauer" appeared for the first time in 1944 in the article [**Ch44**] by F. Châtelet. Nevertheless, F. Severi had proven already in 1932 that a Brauer-Severi variety over a field K admitting a K-valued point is necessarily isomorphic to the projective space.

It should be mentioned that Brauer-Severi varieties are important in a number of applications. The first one that has to be mentioned and definitely the most striking one is the proof of the theorem of Merkurjev-Suslin [**M/S82b**] (see also [**So**], [**Sr**], [**Ke**]) on the cotorsion of K_2 of fields. One has to adopt the point of view introduced by S. A. Amitsur in his work on generic splitting fields of central simple algebras [**Am55**] and applies the computation of the Quillen K-theory of Brauer-Severi varieties which was done by D. Quillen himself in his ground-breaking paper [**Qu**]. With that strategy one proves the so-called theorem Hilbert 90 for K_2. The remaining part of the proof is more elementary but it still requires work. Clearly, this looks like a very indirect approach. It would definitely be desirable to have an elementary proof for the Merkurjev-Suslin theorem, i.e. one that does neither use Brauer-Severi varieties nor Quillen's K-theory.

The Merkurjev-Suslin theorem has an interesting further application to a better understanding of the (torsion part of the) Chow groups of certain algebraic varieties. The reader should consult the work of J.-L. Colliot-Thélène [**CT91**] for information about that.

One should notice that M. Rost [**Ro98, Ro99**] found a proof for the general Bloch-Kato conjecture on the cotorsion of K_n of fields. It depends on V. Voevodsky's construction of motivic cohomology [**V, V/S/E, S/V**] as well as on his unpublished work on homotopy theory of schemes. M. Rost does no more work with Brauer-Severi varieties but more general norm varieties play a prominent role instead. This work should include a new proof for the Merkurjev-Suslin theorem. Surely, it cannot be an elementary one.

As a second kind of application, Brauer-Severi varieties appear in complex algebraic geometry when one deals with varieties being somehow close to rational varieties. In particular, the famous example due to M. Artin and D. Mumford [**A/M**] of a threefold which is unirational but not rational is a variety fibered over a rational surface such that the generic fiber is a conic without rational points. For more historical details, especially on the work of F. Châtelet, we refer the reader to the article of J.-L. Colliot-Thélène [**CT88**].

The connection between central simple algebras and Brauer-Severi varieties was first observed by E. Witt [**Wi35**] and H. Hasse in the special case of quaternion algebras and plane conics. To that connection in its general form there are (at least) three approaches. Here we are going to present in detail the most elementary one which was promoted by J.-P. Serre in his books Corps locaux [**Se62**, chap. X, §§5,6] and Cohomologie Galoisienne [**Se73**, Remarque III.1.3.1] but was known to F. Châtelet, already.

This approach is based on non-abelian group cohomology. The main observation is that central simple algebras of dimension n^2 over a field K as well as $(n-1)$-dimensional Brauer-Severi varieties over K can both be described by classes in one and the same cohomology set $H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_n(K^{\mathrm{sep}}))$.

A second approach is closer to A. Grothendieck's style. One can give a direct description of a functor of points $P_{X_A}\colon \{K\text{−schemes}\} \to \{\text{sets}\}$ in terms of data of the central simple algebra A and prove its representability by a projective scheme using brute force. This approach is presented explicitly in [**Ke**]. For a very detailed account the reader can consult the Ph.D. thesis of F. Henningsen [**Hen**]. We are going to prove here that these two approaches are equivalent. In fact, we will compute the functor of points of the variety given by the first approach and show that it is naturally isomorphic to the functor usually taken as the starting point for the second approach.

There is a third approach which we only mention here. It works via algebraic groups and can be used to produce twisted forms not only of the projective space but of any homogeneous space G/P where G is a semisimple algebraic group and $P \subset G$ a parabolic subgroup. For information about that we suggest the reader to consult the paper of I. Kersten and U. Rehmann [**K/R**].

The article is organized as follows. In section 1 we recall non-abelian group cohomology. In particular, we state the exact sequence associated with a short sequence of G-groups. Section 2 is devoted to Galois descent which is our main algebro-geometric tool. We decided to develop the theory in an elementary way. We do not aim at maximal generality and ignore A. Grothendieck's faithful flat descent. In sections 3 and 4 we develop the description of central simple algebras and Brauer-Severi varieties, respectively, by Galois cohomology classes. Section 5 contains an explicit procedure how to associate a Brauer-Severi variety to a central simple algebra based on the results presented in the sections before. In section 6 we deal with the question how to modify this procedure in order to make it functorial. We did not find that point in the literature but a discussion with K. Künnemann convinced the author that the material presented should be well-known among experts. Section 7 describes the functor of points of the Brauer-Severi variety associated with a given central simple algebra. Thus, it gives the link to the second approach mentioned above.

It is clear that a certain background from Algebraic Geometry will be necessary to follow the text. In order to make the subject as accessible as possible for a reader who is not an Algebraic or Arithmetic Geometer, we present the material in such a way that the knowledge of Algebraic Geometry needed is reduced to a minimum. For that purpose, we even decided in several cases to include a certain statement in detail although there is a good reference for it in the literature. For

example, Lemma 2.12 is due to A. Grothendieck and J. Dieudonné and can be found in [**EGA**]. Essentially, all the results that are used can be found in the second chapter of R. Hartshorne's book [**Ha**] except for a number of facts from Commutative Algebra which are taken from H. Matsumura [**Ma**].
We note finally that we will work over a base field in the entire article. We do not consider the relative versions of central simple algebras and Brauer-Severi varieties over arbitrary base schemes. For information about that we refer the reader to A. Grothendieck [**GrBrI, GrBrII, GrBrIII**].

Notations and conventions. We will follow the standard notations and conventions from Algebra and Algebraic Geometry unless stated otherwise. More precisely,

- all rings are assumed to be associative.
- If R is a ring with unit then R^* denotes the multiplicative group of invertible elements in R.
- All homomorphisms between rings with unit are supposed to respect the unit elements.
- By a field we always mean a commutative field, i.e. a commutative ring with unit every non-zero element of which is invertible. If K is a field then K^{sep} will denote a fixed separable closure of it.
- Nevertheless, a ring with unit every non-zero element of which is invertible is called a skew field.
- If R is a commutative ring with unit then an R-algebra is always understood as a ring homomorphism $j\colon R \to A$ whose image is contained in the center of A.
- An R-algebra $j\colon R \to A$ is denoted simply by A when there seems to be no danger of confusion.
- If $\sigma : R \to R$ is an automorphism of R then A^σ denotes the R-algebra $R \xrightarrow{\sigma} R \xrightarrow{j} A$. If M is an R-module then we put $M^\sigma := M \otimes_R R^\sigma$. M^σ is an R^σ-module as well as an R-module.
- If R is a ring then R^{op} denotes the opposite ring, i.e. the ring that coincides with R as an abelian group but where one has $xy = z$ if one had $yx = z$ in R.
- All central simple algebras are assumed to be finite dimensional over a base field.

1. Non-abelian group cohomology (H^0 and H^1)

In this section we recall elementary facts about what is called non-abelian group cohomology, i.e. cohomology with non-abelian coefficients of discrete groups. All the results presented can be found in detail in Cohomologie Galoisienne [**Se73**]. Non-abelian Galois cohomology will turn out to be the central tool for the purposes of this article.

Definition 1.1. — Let G denote a finite group.

i) A *G-set E* is a set equipped with a G-operation from the left. Following [**Se73**] we will use the notation ${}^g x := g \cdot x$ for $x \in E$ and $g \in G$. A *morphism of G-sets*, a

G-morphism for short, is a map $i\colon E \to F$ of G-sets such that the diagram

$$\begin{tikzcd}
G \times E \arrow[r, "\cdot"] \arrow[d, "\mathrm{id} \times i"'] & E \arrow[d, "i"] \\
G \times F \arrow[r, "\cdot"] & F
\end{tikzcd}$$

commutes.

ii) A *G-group A* is a G-set carrying a group structure such that ${}^{g}(xy) = {}^{g}x{}^{g}y$ for every $g \in G$ and $x, y \in A$. If A is abelian then it is called a *G-module*.

Definition 1.2. — Let G be a finite group.

i) If E is a G-set then one puts $H^0(G, E) := E^G$, i.e. the *zeroth cohomology set of G with coefficients in E* is just the subset of G-invariants in E. If E is a G-group then $H^0(G, E)$ is a group.

ii) If A is a G-group then a *cocycle* from G to A is a map $G \to A, g \mapsto a_g$ such that $a_{gh} = a_g \cdot {}^{g}a_h$ for each $g, h \in G$. Two cocycles a, a' are said to be *cohomologous* if there exists some $b \in A$ such that $a'_g = b^{-1} \cdot a_g \cdot {}^{g}b$ for every $g \in G$. This is an equivalence relation and the quotient set, *the first cohomology set of G with coefficients in A*, is denoted by $H^1(G, A)$. This is a pointed set as the map $g \mapsto e$ defines a cocycle, the so-called *trivial* cocycle.

Remarks 1.3. — i) If a is a cocycle then a' with $a'_g := b^{-1} \cdot a_g \cdot {}^{g}b$ for each $g \in G$ is a cocycle, too.

ii) $H^0(G, A)$ and $H^1(G, A)$ are covariant functors in A. If $i\colon A \to A'$ is a morphism of G-sets (a morphism of G-groups) then the induced map(s) will be denoted by $i_*\colon H^0(G, A) \to H^0(G, A')$ (and $i_*\colon H^1(G, A) \to H^1(G, A')$).

iii) If A is abelian then the definitions above coincide with the usual group cohomology as one of the possible descriptions for $H^{\cdot}(G, A)$ is just the cohomology of the complex

$$0 \longrightarrow A \xrightarrow{d} \mathrm{Map}(G, A) \xrightarrow{d} \mathrm{Map}(G^2, A) \xrightarrow{d} \dots$$

with the differential

$$\begin{aligned}
d\varphi(g_1, \dots, g_{n+1}) &:= {}^{g_1}\varphi(g_2, \dots, g_{n+1}) \\
&+ \sum_{j=1}^{n} (-1)^j \varphi(g_1, \dots, g_j g_{j+1}, \dots, g_{n+1}) \\
&+ (-1)^{n+1} \varphi(g_1, \dots, g_n).
\end{aligned}$$

Proposition 1.4. — *Let G be a finite group.*

a) *If $A \subseteq B$ is a G-subgroup and B/A is the set of left cosets then there is a natural exact sequence of pointed sets*

$$1 \to H^0(G, A) \to H^0(G, B) \to H^0(G, B/A) \xrightarrow{\delta} H^1(G, A) \to H^1(G, B).$$

b) *If $A \subseteq B$ is even a normal G-subgroup then there is a natural exact sequence of pointed sets*

$$1 \to H^0(G, A) \to H^0(G, B) \to H^0(G, B/A) \xrightarrow{\delta} H^1(G, A) \to H^1(G, B) \to H^1(G, B/A).$$

c) *If $A \subseteq B$ is a G-module lying in the center of B then there is a natural exact sequence of pointed sets*

$$\begin{aligned} 1 \to H^0(G,A) \to H^0(G,B) \to H^0(G,B/A) \xrightarrow{\delta} H^1(G,A) \to \dots \\ \dots \to H^1(G,B) \to H^1(G,B/A) \xrightarrow{\delta'} H^2(G,A). \end{aligned}$$

Here the abelian group $H^2(G,A)$ is considered as a pointed set with the unit element.

We note that a sequence $(A,a) \xrightarrow{i} (B,b) \xrightarrow{j} (C,c)$ of pointed sets is said to be exact in (B,b) if $i(A) = j^{-1}(c)$.

Proof. δ is defined as follows. Let $x \in H^0(G,B/A)$. Take a representative $\overline{x} \in B$ for x and put $a_s := \overline{x}^{-1} \cdot {}^s\overline{x}$. That is a cocycle and its equivalence class is denoted by $\delta(x)$. This definition is independent of the $\overline{x}$ chosen.

In the situation of c) the map δ' is given similarly. Let $x \in H^1(G,B/A)$. Choose a cocycle $(x_g)_{g\in G}$ that represents x and lift each x_g to some $\overline{x}_g \in B$. Put $a(g_1,g_2) := {}^{g_1}\overline{x}_{g_2} \cdot \overline{x}_{g_1g_2}^{-1} \cdot \overline{x}_{g_1}$. That is a 2-cocyle with values in A and its equivalence class is denoted by $\delta'(x)$. This definition is independent of the choices.

Exactness has to be checked at each entry separately. This is not complicated at all but very tedious. We omit it here. □

Remark 1.5. — The question what non-abelian H^2 and H^3 might mean turns out to be substantially more difficult. The interested reader is referred to J. Giraud **[Gi]**.

Definition 1.6. — Let $h\colon G' \to G$ be a homomorphism of finite groups. Then for an arbitrary G-set E one has a natural *pull-back* map $h^*\colon H^0(G,E) \to H^0(G',E)$. If E is a G-group then the pull-back map is a group homomorphism. For an arbitrary G-group A there is the natural *pull-back* map $h^*\colon H^1(G,A) \to H^1(G',A)$ which is a morphism of pointed sets.

If h is the inclusion of a subgroup then the pull-back $\mathrm{res}^{G'}_G := h^*$ is usually called the *restriction* map. If h is the canonical projection on a quotient group then $\inf^{G'}_G := h^*$ is said to be the *inflation* map. The composition of $\mathrm{res}^{G'}_G$ or $\inf^{G'}_G$ with some extension of the G'-set E (the G'-group A) is usually called the restriction, respectively inflation, as well.

Remark 1.7. — Non-abelian group cohomology can easily be extended to the case where G is a profinite group and A is a discrete G-set (respectively G-group) on which G operates continuously. Indeed, put for $i = 0$ ($i = 1$)

$$H^i(G,A) := \varinjlim_{G'} H^i(G/G', A^{G'})$$

where the direct limit is taken over the inflation maps and G' runs through the normal open subgroups G' of G such that the quotient G/G' is finite.

2. Galois descent

Definition 2.1. — Let L be a field and $K \subset L$ be a subfield such that L/K is a finite Galois extension. Let $\pi_1\colon X_1 \to \operatorname{Spec} L$ and $\pi_2\colon X_2 \to \operatorname{Spec} L$ be two L-schemes.

Then, by a morphism from π_1 to π_2 that is *twisted by* $\sigma \in \mathrm{Gal}(L/K)$ we will mean a morphism $f\colon X_1 \to X_2$ of schemes such that the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\
\operatorname{Spec} L \arrow[r, "S(\sigma)"] & \operatorname{Spec} L
\end{tikzcd}$$

commutes. Here $S(\sigma)\colon \operatorname{Spec} L \to \operatorname{Spec} L$ denotes the morphism of affine schemes induced by $\sigma^{-1}\colon L \to L$.

Theorem 2.2. — *Let L/K be a finite Galois extension of fields and $G := \mathrm{Gal}(L/K)$ be its Galois group. Then*

a) *there are the following equivalences of categories,*

$$\begin{array}{ccl}
\{K\text{−vector spaces}\} & \longrightarrow & \left\{\begin{array}{l} L\text{−vector spaces} \\ \text{with a } G\text{−operation from the left} \\ \text{where each } \sigma \in G \text{ operates } \sigma\text{−linearly} \end{array}\right\}, \\
\{K\text{−algebras}\} & \longrightarrow & \left\{\begin{array}{l} L\text{−algebras} \\ \text{with a } G\text{−operation from the left} \\ \text{where each } \sigma \in G \text{ operates } \sigma\text{−linearly} \end{array}\right\}, \\
\left\{\begin{array}{l} \text{central simple} \\ \text{algebras} \\ \text{over } K \end{array}\right\} & \longrightarrow & \left\{\begin{array}{l} \text{central simple algebras over } L \\ \text{with a } G\text{−operation from the left} \\ \text{where each } \sigma \in G \text{ operates } \sigma\text{−linearly} \end{array}\right\}, \\
\left\{\begin{array}{l} \text{commutative} \\ K\text{−algebras} \end{array}\right\} & \longrightarrow & \left\{\begin{array}{l} \text{commutative } L\text{−algebras} \\ \text{with a } G\text{−operation from the left} \\ \text{where each } \sigma \in G \text{ operates } \sigma\text{−linearly} \end{array}\right\}, \\
\left\{\begin{array}{l} \text{commutative} \\ K\text{−algebras} \\ \text{with unit} \end{array}\right\} & \longrightarrow & \left\{\begin{array}{l} \text{commutative } L\text{−algebras with unit} \\ \text{with a } G\text{−operation from the left} \\ \text{where each } \sigma \in G \text{ operates } \sigma\text{−linearly} \end{array}\right\}, \\
A & \mapsto & A \otimes_K L,
\end{array}$$

b) *there is the following equivalence of categories,*

$$\begin{array}{ccl}
\left\{\begin{array}{l} \text{quasi} - \text{projective} \\ K\text{−schemes} \end{array}\right\} & \longrightarrow & \left\{\begin{array}{l} \text{quasi} - \text{projective } L\text{−schemes} \\ \text{with a } G\text{−operation from the left} \\ \text{by morphisms of } K\text{−schemes} \\ \text{where each } \sigma \in G \text{ operates} \\ \text{by a morphism twisted by } \sigma \end{array}\right\}, \\
X & \mapsto & X \times_{\operatorname{Spec} K} \operatorname{Spec} L.
\end{array}$$

c) *Let X be a K-scheme and r be a natural number. Then there are the following equivalences of categories,*

$$\begin{array}{ccc}
\left\{\begin{array}{l}\text{quasi}-\text{coherent}\\ \text{sheaves on } X\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{quasi}-\text{coherent sheaves } \mathscr{M}\\ \text{on } X\times_{\operatorname{Spec}K}\operatorname{Spec}L\\ \text{together with a system } (\iota_\sigma)_{\sigma\in G}\\ \text{of isomorphisms } \iota_\sigma\colon x_\sigma^*\mathscr{M}\to\mathscr{M}\\ \text{satisfying } \iota_\tau\circ x_\tau^*(\iota_\sigma)=\iota_{\sigma\tau}\\ \text{for every } \sigma,\tau\in G\end{array}\right\}, \\
\left\{\begin{array}{l}\text{locally free sheaves}\\ \text{of rank } r \text{ on } X\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{locally free sheaves } \mathscr{M} \text{ of rank } r\\ \text{on } X\times_{\operatorname{Spec}K}\operatorname{Spec}L\\ \text{together with a system } (\iota_\sigma)_{\sigma\in G}\\ \text{of isomorphisms } \iota_\sigma\colon x_\sigma^*\mathscr{M}\to\mathscr{M}\\ \text{satisfying } \iota_\tau\circ x_\tau^*(\iota_\sigma)=\iota_{\sigma\tau}\\ \text{for every } \sigma,\tau\in G\end{array}\right\}, \\
\mathscr{F} & \mapsto & \mathscr{M}:=\pi^*\mathscr{F}.
\end{array}$$

Here the morphisms in the categories are the obvious ones, i.e. those respecting all the extra structures. $\pi\colon X\times_{\operatorname{Spec}K}\operatorname{Spec}L\longrightarrow X$ *is the canonical morphism and* $x_\sigma\colon X\times_{\operatorname{Spec}K}\operatorname{Spec}L\longrightarrow X\times_{\operatorname{Spec}K}\operatorname{Spec}L$ *denotes the morphism that is induced by* $S(\sigma)\colon\operatorname{Spec}L\to\operatorname{Spec}L$.

Proof. In each case we have to prove that the functor given is fully faithful and essentially surjective. Full faithfulness is proven in Propositions 2.7, 2.8 and 2.9, respectively. Propositions 2.3, 2.5 and 2.6 show essential surjectivity. □

***Proposition 2.3* (*Galois descent-algebraic version*).** — *Let L/K be a finite Galois extension of fields and $G:=\operatorname{Gal}(L/K)$ be its Galois group. Further, let*

> *W be a vector space (an algebra, a central simple algebra, a commutative algebra, a commutative algebra with unit, ...) over L together with an operation $T\colon G\times W\to W$ of G from the left on W respecting all the extra structures such that for each $\sigma\in G$ the action of σ is a σ-linear map $T_\sigma\colon W\to W$*

Then there is a vector space V (an algebra, a central simple algebra, a commutative algebra, a commutative algebra with unit, ...) over K such that there is an isomorphism

$$V\otimes_K L\xrightarrow[\cong]{b}W$$

where $V\otimes_K L$ is equipped with the G-operation induced by the canonical one on L and b respects all the algebraic structures including the operation of G.

Proof. Define $V:=W^G$. This is clearly a K-vector space (a K-algebra, a commutative K-algebra, a commutative K-algebra with unit). If W is a central simple algebra over L then V is a central simple algebra over K. This can not be seen directly but it follows immediately from the formula $W^G\otimes_K L=W$ which will be proven below. For that let $\{l_1,\ldots,l_n\}$ be a K-basis of L. We have to show the following claim. □

Claim 2.4. — *There exist an index set A and a subset $\{x_\alpha\,|\,\alpha\in A\}\subset W^G$ such that $\{l_i x_\alpha\,|\,i\in\{1,\ldots,n\},\alpha\in A\}$ is a K-basis of W.*

Proof. By Zorn's Lemma there exists a maximal subset $\{x_\alpha\,|\,\alpha\in A\}\subset W^G$ such that $\{l_i x_\alpha \,|\, i \in \{1, \dots, n\},\ \alpha \in A\} \subset W$ is a system of K-linearly independent vectors. Assume that system is not a basis of W. Then $\langle l_i x_\alpha \,|\, i \in \{1, \dots, n\},\ \alpha \in A\rangle_K$ is a proper G-invariant L-sub-vector space of W and one can choose an element $x \in W \backslash \langle l_i x_\alpha \,|\, i \in \{1, \dots, n\}, \alpha \in A\rangle_K$. For every $l \in L$ the sum

$$\sum_{\sigma\in G} T_\sigma(lx) = \sum_{\sigma\in G} \sigma(l)\cdot T_\sigma(x)$$

is G-invariant. Further, by linear independence of characters, the matrix

$$\begin{pmatrix} \sigma_1(l_1) & \dots & \sigma_1(l_n) \\ \dots & \ddots & \dots \\ \sigma_n(l_1) & \dots & \sigma_n(l_n) \end{pmatrix}$$

is of maximal rank. In particular, there is some $l \in L$ such that the image of

$$x_\beta := \sum_{\sigma\in G} \sigma(l)\cdot\sigma(x)$$

in $W/\langle l_i x_\alpha \,|\, i \in \{1, \dots, n\},\ \alpha \in A\rangle_K$ is not equal to zero. Therefore,

$$\{l_i x_\alpha \,|\, i \in \{1, \dots, n\},\ \alpha \in A\} \cup \{l_i x_\beta \,|\, i \in \{1, \dots, n\}\}$$

is a K-linearly independent system of vectors contradicting the maximality of $\{x_\alpha \,|\, \alpha \in A\}$. □

***Proposition 2.5* (*Galois descent-geometric version*).** — *Let L/K be a finite Galois extension of fields and $G := \mathrm{Gal}\,(L/K)$ its Galois group. Further, let*

> *Y be a quasi-projective L-scheme together with an operation of G from the left by twisted morphisms, i.e. such that the diagrams*
>
> $$\begin{tikzcd} Y \arrow[r, "T_\sigma"] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} L \arrow[r, "S(\sigma)"] & \operatorname{Spec} L \end{tikzcd}$$
>
> *commute, where $S(\sigma)\colon \operatorname{Spec} L \to \operatorname{Spec} L$ is the morphism of schemes induced by $\sigma^{-1}\colon L \to L$.*

Then there exists a quasi-projective K-scheme X such that there is an isomorphism of L-schemes

$$X \times_{\operatorname{Spec} K} \operatorname{Spec} L \xrightarrow[\cong]{f} Y$$

where $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is equipped with the G-operation induced by the one on $\operatorname{Spec} L$ *and f is compatible with the operation of G.*

Proof. *Affine Case.* Let $Y = \operatorname{Spec} B$ be an affine scheme. The G-operation on $\operatorname{Spec} B$ corresponds to a G-operation from the right on B where each $\sigma \in G$ operates σ^{-1}-linearly. Define a G-operation from the left on B by $\sigma \cdot b := b \cdot \sigma^{-1}$. The assertion follows immediately from Proposition 2.3.

General Case. By Lemma 2.10 there exists an affine open covering $\{Y_1, \dots, Y_n\}$ of Y by G-invariant schemes. Galois descent yields affine K-schemes $X_1, \dots, X_n$ such that there are isomorphisms

$$X_i \times_{\operatorname{Spec} K} \operatorname{Spec} L \xrightarrow{\cong} Y_i$$

and affine K-schemes $X_{ij}, 1 \leq i < j \leq n$, such that there are isomorphisms

$$X_{ij} \times_{\operatorname{Spec} K} \operatorname{Spec} L \xrightarrow{\cong} Y_i \cap Y_j.$$

It remains to be shown that every X_{ij} admits canonical, open embeddings into X_i and X_j.
In each case on the level of rings we have a homomorphism $A \otimes_K L \longrightarrow B \otimes_K L$ and an isomorphism $B \otimes_K L \xrightarrow{\cong} (A \otimes_K L)_f$ such that their composition is the localization map. Clearly, f can be assumed to be G-invariant, i.e. we may suppose $f \in A$. Consequently, $B \otimes_K L \cong A_f \otimes_K L$ and, by consideration of the G-invariants on both sides, $B \cong A_f$.
The cocycle relations are clear. Therefore, we can glue the affine schemes $X_1, \ldots, X_n$ along the affine schemes $X_{ij}, 1 \leq i < j \leq n$, to obtain the scheme X desired. Lemma 2.12.ix) below completes the proof. □

***Proposition 2.6* (*Galois descent for quasi-coherent sheaves*)**
Let L/K be a finite Galois extension of fields and $G := \operatorname{Gal}(L/K)$ be its Galois group. Further, let X be a K-scheme, $\pi : X \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X$ the canonical morphism and $x_\sigma : X \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ be the morphism induced by $S(\sigma) \colon \operatorname{Spec} L \to \operatorname{Spec} L$. Let

> *$\mathscr{M}$ be a quasi-coherent sheaf over $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ together with a system $(\iota_\sigma)_{\sigma \in G}$ of isomorphisms $\iota_\sigma \colon x_\sigma^* \mathscr{M} \to \mathscr{M}$ that are compatible in the sense that for each $\sigma, \tau \in G$ there is the relation $\iota_\tau \circ x_\tau^*(\iota_\sigma) = \iota_{\sigma\tau}$.*

Then there exists a quasi-coherent sheaf $\mathscr{F}$ over X such that there is an isomorphism

$$\pi^* \mathscr{F} \xrightarrow[b]{\cong} \mathscr{M}$$

under which the canonical isomorphism $i_\sigma \colon x_\sigma^* \pi^* \mathscr{F} = (\pi x_\sigma)^* \mathscr{F} = \pi^* \mathscr{F} \xrightarrow{\mathrm{id}} \pi^* \mathscr{F}$ is identified with ι_σ for each σ, i.e. the diagrams

$$\begin{tikzcd}
x_\sigma^* \pi^* \mathscr{F} \arrow[r, "x_\sigma^*(b)"] \arrow[d, "i_\sigma"'] & x_\sigma^* \mathscr{M} \arrow[d, "\iota_\sigma"] \\
\pi^* \mathscr{F} \arrow[r, "b"] & \mathscr{M}
\end{tikzcd}$$

commute.

Proof. Assume $X \cong \operatorname{Spec} A$ to be an affine scheme first. Then $\mathscr{M} = \widetilde{M}$ for some $(A \otimes_K L)$-module M. We have

$$x_\sigma^* \mathscr{M} = \widetilde{M \otimes_{(A \otimes_K L)} (A \otimes_K L^{\sigma^{-1}})} = \widetilde{M \otimes_L L^{\sigma^{-1}}}.$$

Hence $x_\sigma^* \mathscr{M} = \widetilde{M^{\sigma^{-1}}}$ where $M^{\sigma^{-1}}$ coincides with M as an A-module, but its structure of an L-vector space is given by

$$l \cdot_{M^{\sigma^{-1}}} m := \sigma^{-1}(l) \cdot_M m.$$

Consequently, the isomorphism $\iota_\sigma : x_\sigma^* \mathscr{M} \to \mathscr{M}$ is induced by an A-module isomorphism $j_\sigma : M \to M$ being σ^{-1}-linear. The compatibility relations required above translate simply into the condition that the maps j_σ form a G-operation on M from the right. Define a G-operation from the left on M by $\sigma \cdot m := j_{\sigma^{-1}}(m)$.

By Galois descent for vector spaces the K-vector space M^G of G-invariants satisfies $M^G \otimes_K L \cong M$. This is also an isomorphism of A-modules as the G-operation on M is compatible with the A-operation. Putting $\mathscr{F} := \widetilde{M^G}$ we obtain a quasi-coherent sheaf over X such that $\pi^*\mathscr{F} \cong \mathscr{M}$. The commutativity of the diagram is a consequence of Proposition 2.3.

Now let X be a general scheme. Consider an affine open covering

$$\{X_\alpha \cong \operatorname{Spec} R_\alpha \mid \alpha \in A\}$$

of X where A is an arbitrary index set. For every intersection $X_{\alpha_1} \cap X_{\alpha_2}$ we again consider an affine open covering

$$\{X_{\alpha_1,\alpha_2,\beta} \cong \operatorname{Spec} R_{\alpha_1,\alpha_2,\beta} \mid \beta \in B_{\alpha_1,\alpha_2}\}.$$

By the affine case we are given R_α-modules M_α for each $\alpha \in A$ satisfying $\pi^*\widetilde{M_\alpha} \cong \mathscr{M}|_{X_\alpha \times_{\operatorname{Spec} K} \operatorname{Spec} L}$ and $R_{\alpha_1,\alpha_2,\beta}$-modules $M_{\alpha_1,\alpha_2,\beta}$ for each triple $(\alpha_1,\alpha_2,\beta)$ with $\alpha_1, \alpha_2 \in A$ and $\beta \in B_{\alpha_1,\alpha_2}$ satisfying $\pi^*\widetilde{M_{\alpha_1,\alpha_2,\beta}} \cong \mathscr{M}|_{X_{\alpha_1,\alpha_2,\beta} \times_{\operatorname{Spec} K} \operatorname{Spec} L}$. The construction of these modules is compatible with restriction to affine subschemes. Therefore, by Proposition 2.9 below, we get isomorphisms

$$i_{\alpha_1,\alpha_2,\beta} \colon M_{\alpha_1} \otimes_{R_{\alpha_1}} R_{\alpha_1,\alpha_2,\beta} \xrightarrow{\cong} M_{\alpha_2} \otimes_{R_{\alpha_2}} R_{\alpha_1,\alpha_2,\beta}.$$

It is clear that for every $\alpha_1, \alpha_2, \alpha_3 \in A$ and every $\beta_1 \in B_{\alpha_2,\alpha_3}, \beta_2 \in B_{\alpha_3,\alpha_1}, \beta_3 \in B_{\alpha_1,\alpha_2}$ these isomorphisms are compatible on the triple intersection $X_{\alpha_1,\alpha_2,\beta_3} \cap X_{\alpha_3,\alpha_1,\beta_2} \cap X_{\alpha_2,\alpha_3,\beta_1}$, i.e. we can glue the quasi-coherent sheaves $\widetilde{M_\alpha}$ along the $\widetilde{M_{\alpha_1,\alpha_2,\beta}}$ to obtain the quasi-coherent sheaf $\mathscr{M}$ desired. □

***Proposition 2.7* (*Galois descent for homomorphisms*).** — *Let L/K be a finite Galois extension of fields and $G := \operatorname{Gal}(L/K)$ be its Galois group. Then it is equivalent*

i) *to give a homomorphism $r \colon V \to V'$ of K-vector spaces (of algebras over K, of central simple algebras over K, of commutative K-algebras, of commutative K-algebras with unit, ...),*

ii) *to give a homomorphism $r_L \colon V \otimes_K L \to V' \otimes_K L$ of L-vector spaces (of algebras over L, of central simple algebras over L, of commutative L-algebras, of commutative L-algebras with unit, ...) which is compatible with the G-operations, i.e. such that for each $\sigma \in G$ the diagram*

$$\begin{tikzcd}
V \otimes_K L \arrow[r, "r_L"] \arrow[d, "\sigma"'] & V' \otimes_K L \arrow[d, "\sigma"] \\
V \otimes_K L \arrow[r, "r_L"] & V' \otimes_K L
\end{tikzcd}$$

commutes.

Proof. If r is given then one defines $r_L := r \otimes_K L$. Clearly, if r is a ring homomorphism then r_L is, too. Conversely, in order to construct r from r_L note that the commutativity of the diagrams above implies that r_L is compatible with the G-invariants on both sides. But we know $(V \otimes_K L)^G = V$ and $(V' \otimes_K L)^G = V'$, already, so we obtain a K-linear map $r \colon V \longrightarrow V'$. If r_L is a ring homomorphism then its restriction r is, too. It is clear that the two processes described are inverse to each other. □

Proposition 2.8 **(*Galois descent for morphisms of schemes*)**
Let L/K be a finite Galois extension of fields and $G := \mathrm{Gal}(L/K)$ be its Galois group. Then it is equivalent

i) *to give a morphism of K-schemes $f\colon X \to X'$,*

ii) *to give a morphism of L-schemes $f_L\colon X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \to X' \times_{\mathrm{Spec} K} \mathrm{Spec}\, L$ which is compatible with the G-operations, i.e. such that for each $\sigma \in G$ the diagram*

$$\begin{tikzcd}
X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \arrow[r, "f_L"] \arrow[d, "\sigma"'] & X' \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \arrow[d, "\sigma"] \\
X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \arrow[r, "f_L"] & X' \times_{\mathrm{Spec} K} \mathrm{Spec}\, L
\end{tikzcd}$$

commutes.

Proof. If f is given then one defines $f_L := f \times_{\mathrm{Spec} K} \mathrm{Spec}\, L$. Conversely, in order to construct f from f_L, the question is local in X' and X. So one has a homomorphism $r_L\colon A' \otimes_K L \longrightarrow A \otimes_K L$ of L-algebras with unit making the diagrams

$$\begin{tikzcd}
A' \otimes_K L \arrow[r, "r_L"] \arrow[d, "\sigma"] & A \otimes_K L \arrow[d, "\sigma"] \\
A' \otimes_K L \arrow[r, "r_L"] & A \otimes_K L
\end{tikzcd}$$

commute. That is exactly the situation covered by the proposition above. It is clear that the two processes described are inverse to each other. $\square$

Proposition 2.9 **(*Galois descent for morphisms of quasi-coherent sheaves*)**
Let L/K be a finite Galois extension of fields and $G := \mathrm{Gal}(L/K)$ be its Galois group. Further, let X be a K-scheme and $\pi\colon X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \to X$ be the canonical morphism. Then it is equivalent

i) *to give a morphism $r\colon \mathscr{F} \to \mathscr{G}$ of coherent sheaves over X,*

ii) *to give a morphism $r_L : \pi^*\mathscr{F} \to \pi^*\mathscr{G}$ of quasi-coherent sheaves over $X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L$ which is compatible with the G-operations, i.e. such that for each $\sigma \in G$ the diagram*

$$\begin{tikzcd}
\pi^*\mathscr{F} \arrow[r, "r_L"] \arrow[d, "x_\sigma^*"'] & \pi^*\mathscr{G} \arrow[d, "x_\sigma^*"] \\
\pi^*\mathscr{F} \arrow[r, "r_L"] & \pi^*\mathscr{G}
\end{tikzcd}$$

commutes.

Proof. If r is given then one defines $r_L := \pi^* r$. Conversely, if r_L is given, the question to construct r is local in X. So assume A is a commutative ring with unit and M and N are A-modules. We are given a homomorphism $r_L\colon M \otimes_K L \to N \otimes_K L$ of $A \otimes_K L$-modules such that the diagrams

$$\begin{tikzcd}
M \otimes_K L \arrow[r, "r_L"] \arrow[d, "\sigma"'] & N \otimes_K L \arrow[d, "\sigma"] \\
M \otimes_K L \arrow[r, "r_L"] & N \otimes_K L
\end{tikzcd}$$

commute for each $\sigma \in G$. We get a morphism $r\colon M \to N$ as M and N are the A-modules of G-invariants on the left and right hand side, respectively. The two procedures described above are inverse to each other. □

Lemma 2.10. — *Let L be a field and Y be a quasi-projective L-scheme equipped with an operation of some finite group G acting by morphisms of schemes. Then there exists a covering of Y by G-invariant affine open subsets.*

Proof. Let $y \in Y$ be an arbitrary closed point. Everything that is needed is an affine open G-invariant subset containing y. For that we choose an embedding $i\colon Y \hookrightarrow \mathbf{P}^N_L$. By Sublemma 2.11, there exists a hypersurface H_y such that $H_y \supset \overline{i(Y)} \backslash i(Y)$ and $i(\sigma(y)) \notin H_y$ for every $\sigma \in G$. Here $\overline{i(Y)}$ denotes the closure of $i(Y)$ in $\mathbf{P}^N_L$. By construction, the morphism

$$i|_{Y\backslash i^{-1}(H_y)}\colon Y\backslash i^{-1}(H_y) \longrightarrow \mathbf{P}^N_L \backslash H_y$$

is a closed embedding. As $\mathbf{P}^N_L \backslash H_y$ is an affine scheme, $Y\backslash i^{-1}(H_y)$ must be affine, too. Hence,

$$O_y := \bigcap_{\sigma \in G} \sigma^{-1}(Y\backslash i^{-1}(H_y)) \subset Y$$

is the intersection of finitely many affine open subsets in a quasi-projective, and therefore separated, scheme. Thus, it is an affine open subset. By construction, O_y is G-invariant and contains y. □

Sublemma 2.11. — *Let L be a field and $Z \subsetneqq \mathbf{P}^N_L$ be a closed subvariety of some projective space over L. Further, let $p_1, \ldots, p_n \in \mathbf{P}^N_L$ be finitely many closed points not contained in Z. Then there exists some hypersurface $H \subset \mathbf{P}^N_L$ that contains Z but does not contain any of the points $p_1, \ldots, p_n$.*

Proof. We will give two proofs, an elementary one and the natural one that uses cohomology of coherent sheaves.

1st proof. Let $S := L[X_0, \ldots, X_N]$ be the homogeneous coordinate ring for the projective space $\mathbf{P}^N_L$. It is a graded L-algebra. For $d \in \mathbb{N}$ we will denote by S_d the L-vector space of homogeneous elements of degree d.

We proceed by induction the case $n = 0$ being trivial. Assume the statement is proven for $n-1$ and consider the problem for n points $p_1, \ldots, p_n$. By induction hypothesis there exists a homogeneous element $s \in S$, i.e. some hypersurface $H := V(s)$, such that $H \supseteq Z$ and $p_1, \ldots, p_{n-1} \notin H$. Let d denote the degree of s, i.e. H is a hypersurface in $\mathbf{P}^N_L$ of degree d. We may assume $p_n \in H$ as, otherwise, the proof would be finished.

$Z \cup \{p_1\} \cup \ldots \cup \{p_{n-1}\}$ is a Zariski closed subset of $\mathbf{P}^N_L$ not containing p_n. Therefore, there exists some $d' \in \mathbb{N}$ and some homogeneous $s' \in S_{d'}$ such that $V(s') \supseteq Z \cup \{p_1\} \cup \ldots \cup \{p_{n-1}\}$ but $p_n \notin V(s')$. Any hypersurface $V(a \cdot s'^d + b \cdot s^{d'})$ for non-zero elements $a, b \in L$ contains Z but neither $p_1, \ldots, p_{n-1}$ nor p_n.

2nd proof. Tensoring the canonical exact sequence

$$0 \longrightarrow \mathscr{I}_{\{p_1,\ldots,p_n\}} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{\{p_1,\ldots,p_n\}} \longrightarrow 0$$

with the ideal sheaf $\mathscr{I}_Z$ yields an exact sequence

$$0 \longrightarrow \mathscr{I}_{\{p_1,\ldots,p_n\}\cup Z} \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{O}_{\{p_1,\ldots,p_n\}} \longrightarrow 0$$

of sheaves on $\mathbf{P}^N_L$ as one has $\mathscr{I}_{\{p_1,\dots,p_n\}} \otimes_{\mathscr{O}_X} \mathscr{I}_Z = \mathscr{I}_{\{p_1,\dots,p_n\}\cup Z}$ and $\mathscr{T}or_1^{\mathscr{O}_X}(\mathscr{O}_{\{p_1,\dots,p_n\}}, \mathscr{I}_Z) = 0$. For that note that locally at least one of two sheaves occurring in the products is free. For each $l \in \mathbb{Z}$ we tensor with the invertible sheaf $\mathscr{O}(l)$ and find a long cohomology exact sequence

$$\Gamma(\mathbf{P}^N_L, \mathscr{I}_Z(l)) \longrightarrow \Gamma(\mathbf{P}^N_L, \mathscr{O}_{\{p_1,\dots,p_n\}}(l)) \longrightarrow H^1(\mathbf{P}^N_L, \mathscr{I}_{\{p_1,\dots,p_n\}}(l)).$$

By Serre's vanishing theorem [**Ha**, Theorem III.5.2], $H^1(\mathbf{P}^N_L, \mathscr{I}_{\{p_1,\dots,p_n\}}(l)) = 0$ for $l \gg 0$. Hence, there is a surjection

$$\Gamma(\mathbf{P}^N_L, \mathscr{I}_Z(l)) \longrightarrow \Gamma(\mathbf{P}^N_L, \mathscr{O}_{\{p_1,\dots,p_n\}}(l)) \cong \kappa(p_1) \oplus \ldots \oplus \kappa(p_n).$$

That means that there exists a global section s of $\mathscr{I}_Z(l)$ that does not vanish in any of the points $p_1, \dots, p_n$. s defines a hypersurface of degree l in $\mathbf{P}^N_L$ that contains Z and does not contain any of the points $p_1, \dots, p_n$. □

***Lemma 2.12* (*A. Grothendieck and J. Dieudonné*).** — *Let L/K be a finite field extension and X be a K-scheme such that $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is*

i) *reduced,*

ii) *irreducible,*

iii) *quasi-compact,*

iv) *locally of finite type,*

v) *of finite type,*

vi) *locally Noetherian,*

vii) *Noetherian,*

viii) *proper,*

ix) *quasi-projective,*

x) *projective,*

xi) *affine,*

or

xii) *regular.*

Then X admits the same property.

Proof. Let $\pi\colon X \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X$ denote the canonical morphism. For iii) through xi) we may assume L/K to be Galois. Put $G := \operatorname{Gal}(L/K)$.

i) If $s \in \Gamma(U, \mathscr{O}_X)$ would be a nilpotent non-zero section of the structure sheaf $\mathscr{O}_X$ over some open subset $U \subseteq X$ then $\pi^\sharp(s) \in \Gamma(U \times_{\operatorname{Spec} K} \operatorname{Spec} L, \mathscr{O}_{X \times_{\operatorname{Spec} K} \operatorname{Spec} L})$ would be a nilpotent non-zero local section of the structure sheaf of $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$.

ii) If $X = X_1 \cup X_2$ would be a non-trivial decomposition into two closed subschemes then $X \times_{\operatorname{Spec} K} \operatorname{Spec} L = X_1 \times_{\operatorname{Spec} K} \operatorname{Spec} L \cup X_2 \times_{\operatorname{Spec} K} \operatorname{Spec} L$ would be the same for $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$.

iii) Let $\{U_\alpha \,|\, \alpha \in A\}$ be an arbitrary affine open covering of X. Then $\{U_\alpha \times_{\operatorname{Spec} K} \operatorname{Spec} L \,|\, \alpha \in A\}$ is an affine open covering of $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$. Quasi-compactness guarantees the existence of a finite sub-covering $\{U_\alpha \times_{\operatorname{Spec} K} \operatorname{Spec} L \,|\, \alpha \in A_0\}$. But then $\{U_\alpha \,|\, \alpha \in A_0\}$ is a finite affine open covering for X.

iv) We can assume $X = \operatorname{Spec} A$ to be affine. So $A \otimes_K L$ is a finitely generated L-algebra. Let $\{b_1, \ldots, b_n\}$ be a system of generators for $A \otimes_K L$. As one could decompose into elementary tensors, if necessary, we may assume without restriction that all $b_i = a_i \otimes l_i$ are elementary tensors. Consider the homomorphism of K-algebras

$$K[X_1, \ldots, X_n] \to A, \ X_i \mapsto a_i.$$

It induces a surjection after tensoring with L and, therefore, is surjective itself as $\otimes_K L$ is a faithful functor.

v) This is just the combination of iii) and iv).

vi) Let $X_\alpha \cong \operatorname{Spec} A_\alpha$ be an affine open subscheme of X. We have to show that A_α is Noetherian under the hypothesis that $A_\alpha \otimes_K L$ is. So assume $I_1 \subset I_2 \subset I_3 \subset \ldots$ is an ascending chain of ideals in A_α that does not stabilize. It induces an ascending chain $I_1 \otimes_K L \subset I_2 \otimes_K L \subset I_3 \otimes_K L \subset \ldots$ of ideals in $A_\alpha \otimes_K L$ that does not stabilize, either. This is a contradiction to our hypothesis.

vii) This is the combination of iii) and vi).

viii) That is v) together with a direct application of the valuation criterion for properness [**Ha**, Theorem 4.7]. So consider a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\
T \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

where $T = \operatorname{Spec} R$ is the spectrum of a valuation ring, $U = \operatorname{Spec} F$ is the spectrum of its quotient field and i is the canonical morphism. Taking the base change of the whole diagram to $\operatorname{Spec} L$ we find

$$\begin{tikzcd}
U \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[r] \arrow[d, "i"'] & X \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[d] \\
T \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[r] \arrow[ur, "\iota"] & \operatorname{Spec} L
\end{tikzcd}$$

where the diagonal morphism ι is the unique one making the diagram commute. Note that $U \times_{\operatorname{Spec} K} \operatorname{Spec} L = \operatorname{Spec}(F \otimes_K L)$ is no more the spectrum of a field but the union of finitely many spectra of fields. Similarly, $T \times_{\operatorname{Spec} K} \operatorname{Spec} L = \operatorname{Spec}(R \otimes_K L)$ is a finite union of spectra of valuation rings. Nevertheless, the valuation criterion implies existence and uniqueness of the diagonal arrow. Further, there is a canonical G-operation on the whole diagram without ι. As ι is uniquely determined by the condition that it makes the diagrams above commute, the diagrams

$$\begin{tikzcd}
T \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[r, "\iota"] \arrow[d, "\sigma"'] & X \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[d, "\sigma"] \\
T \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[r, "\iota"] & X \times_{\operatorname{Spec} K} \operatorname{Spec} L
\end{tikzcd}$$

must be commutative for each $\sigma \in G$. ι is the base change of a morphism $T \to X$ by Proposition 2.8.

ix) and x) Taking v) and viii) into account everything left to be shown is the existence an ample invertible sheaf on X. By assumption there is an ample invertible sheaf $\mathscr{M}$ on $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$. For $\sigma \in G$ let $x_\sigma\colon X \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ be the morphism of schemes induced by $S(\sigma)\colon \operatorname{Spec} L \to \operatorname{Spec} L$, i.e. by σ^{-1} on coordinate rings. The invertible sheaves $x_\sigma^*\mathscr{M}$ are ample, as well, and, therefore, $\overline{\mathscr{M}} := \bigotimes_{v\in G} x_v^*\mathscr{M}$ is an ample invertible sheaf. For each $\sigma \in G$ there are canonical identifications

$$\iota_\sigma\colon x_\sigma^*\overline{\mathscr{M}} = \bigotimes_{v\in G} x_\sigma^* x_v^*\mathscr{M} = \bigotimes_{v\in G} x_{v\sigma}^*\mathscr{M} \xrightarrow{\mathrm{id}} \bigotimes_{v\in G} x_v^*\mathscr{M} = \overline{\mathscr{M}}$$

and it is trivial that they are compatible in the sense that for each $\sigma, \tau \in G$ there is the relation $\iota_\tau \circ x_\tau^*(\iota_\sigma) = \iota_{\sigma\tau}$. By Galois descent for locally free sheaves there is an invertible sheaf $\mathscr{L} \in \operatorname{Pic}(X)$ such that $\pi^*\mathscr{L} \cong \overline{\mathscr{M}}$. By Lemma 2.13 below $\mathscr{L}$ is ample.

xi) If $X \times_{\operatorname{Spec} K} \operatorname{Spec} L \xrightarrow{\cong} \operatorname{Spec} B$ is affine the isomorphism defines a G-operation from the right on the L-algebra B where each $\sigma \in G$ operates σ^{-1}-linearly. Define a G-operation from the left on B by $\sigma \cdot b := b \cdot \sigma^{-1}$. By the algebraic version of descent, Proposition 2.3, we find a K-algebra A such that there is an isomorphism

$$X \times_{\operatorname{Spec} K} \operatorname{Spec} L \xrightarrow{\cong} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_K L) \xrightarrow{\cong} \operatorname{Spec} A \times_{\operatorname{Spec} K} \operatorname{Spec} L$$

being compatible with the G-operations. Proposition 2.5 implies $X \cong \operatorname{Spec} A$.

xii) Let $(A, \mathfrak{m})$ be the local ring at a point $p \in X$. By vi) we may assume $(A, \mathfrak{m})$ is Noetherian. We have to show it is regular. Let us give two proofs for that, the standard one using Serre's homological characterization of regularity and an elementary one.

1st proof. Assume $(A, \mathfrak{m})$ would not be regular. By [**Ma**, Theorem 19.2] one would have $\operatorname{gl.dim} A = \infty$. Then, [**Ma**, §19, Lemma 1] implies $\operatorname{proj.dim}_A A/\mathfrak{m} = \infty$ and

$$\sup\{i \mid \operatorname{Tor}_i^A(A/\mathfrak{m}, A/\mathfrak{m}) \neq 0\} = \infty.$$

On the other hand, we have

$$\begin{aligned} \operatorname{Tor}_i^A(A/\mathfrak{m}, A/\mathfrak{m}) \otimes_K L &= \operatorname{Tor}_i^A(A/\mathfrak{m}, A/\mathfrak{m}) \otimes_A (A \otimes_K L) \\ &= \operatorname{Tor}_i^A(A/\mathfrak{m}, (A \otimes_K L/\mathfrak{m} \otimes_K L)) \\ &= \operatorname{Tor}_i^{A \otimes_K L}((A \otimes_K L/\mathfrak{m} \otimes_K L), (A \otimes_K L/\mathfrak{m} \otimes_K L)) \\ &= 0 \end{aligned}$$

for $i > \dim A \otimes_K L$ by [**Ma**, Theorem 19.2] as $A \otimes_K L$ is regular. This is a contradiction. Note that everything we needed was that π is faithfully flat.

2nd proof. Put $d := \dim A$. The ring $A \otimes_K L$ is not local in general. But the quotient $(A \otimes_K L)/(\mathfrak{m} \otimes_K L) \cong A/\mathfrak{m} \otimes_K L$ is a direct product of finitely many fields since L/K is a finite, separable field extension. The quotients $(A \otimes_K L)/(\mathfrak{m}^n \otimes_K L)$ are Artin rings as they are flat of relative dimension zero over $A/\mathfrak{m}^n$. Consequently, they are direct products of Artin local rings,

$$(A \otimes_K L)/(\mathfrak{m}^n \otimes_K L) \xrightarrow{\cong} A_1^{(n)} \times \ldots \times A_l^{(n)}.$$

Under this isomorphism $(\mathfrak{m} \otimes_K L)/(\mathfrak{m}^n \otimes_K L)$ is mapped to the product of the maximal ideals $\mathfrak{m}_1^{(n)} \times \ldots \times \mathfrak{m}_l^{(n)}$ as it is a nilpotent ideal and the quotient has to be a direct product of fields. For each $i \in \{1, \ldots, l\}$ let $f_i \in A \otimes_K L$ be chosen such that its image has a non-zero component in $A_i^{(n)}$, only, and that be the unit element. Then

$$A_i^{(n)} \cong ((A \otimes_K L)/(\mathfrak{m}^n \otimes_K L))_{\overline{f_i}} \cong (A \otimes_K L)_{f_i}/\mathfrak{m}^n (A \otimes_K L)_{f_i}.$$

By our assumptions $(A \otimes_K L)_{f_i}$ is a regular local ring and $\mathfrak{m}(A \otimes_K L)_{f_i}$ is its maximal ideal. $(A \otimes_K L)_{f_i}$ is of dimension d since it is flat of relative dimension zero over A. By the standard computation of the Hilbert-Samuel function of a regular local ring we have

$$\begin{aligned} \operatorname{length}\left(A_i^{(n)}\right) &= \operatorname{length}((A \otimes_K L)_{f_i}/\mathfrak{m}^n (A \otimes_K L)_{f_i}) \\ &= \operatorname{length}((A \otimes_K L)_{f_i}/(\mathfrak{m}(A \otimes_K L)_{f_i})^n) \\ &= \binom{n+d-1}{d}. \end{aligned}$$

Consequently, for the function H_A with $H_A(n) := \dim_K (A/\mathfrak{m}^n)$ one gets

$$\begin{aligned} H_A(n) = \dim_L (A \otimes_K L/\mathfrak{m}^n \otimes_K L) &= \dim_L \left(A_1^{(n)} \times \ldots \times A_l^{(n)}\right) \\ &= \sum_{i=1}^{l} \dim_L \left(A_i^{(n)}\right) \\ &= \sum_{i=1}^{l} \left[\left(A_i^{(n)}/\mathfrak{m}_i^{(n)}\right) : L\right] \operatorname{length}\left(A_i^{(n)}\right), \end{aligned}$$

so H_A has to be some multiple of $n \mapsto \binom{n+d-1}{d}$ by a constant. Therefore, the Hilbert-Samuel function of A is exactly that of a regular local ring of dimension d. Note its value $H_A(1)$ is necessarily equal to 1, so there is no ambiguity about constant factors. A is regular. $\square$

Lemma 2.13. — *Let L/K be an arbitrary field extension, X a K-scheme of finite type, $\pi\colon X \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X$ the canonical morphism and $\mathscr{L} \in \operatorname{Pic}(X)$ be an invertible sheaf. If the pull-back $\pi^*\mathscr{L} \in \operatorname{Pic}(X \times_{\operatorname{Spec} K} \operatorname{Spec} L)$ is ample then $\mathscr{L}$ is ample.*

Proof. We have to prove that for every coherent sheaf $\mathscr{F}$ on X and each closed point $x \in X$ for $n \gg 0$ the canonical map $p_x^n\colon \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \to \mathscr{F}_x \otimes \mathscr{L}_x^{\otimes n}$ is surjective. For that it is obviously sufficient to prove surjectivity for $p_x^n \otimes_K L\colon \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \otimes_K L \to (\mathscr{F}_x \otimes \mathscr{L}_x^{\otimes n}) \otimes_K L$. But it is easy to see that

$$\Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \otimes_K L = \Gamma(X \times_{\operatorname{Spec} K} \operatorname{Spec} L, \pi^*\mathscr{F} \otimes \pi^*\mathscr{L}^{\otimes n})$$

while $(\mathscr{F}_x \otimes \mathscr{L}_x^n) \otimes_K L = \Gamma(\pi^{-1}(x), \pi^*\mathscr{F} \otimes \pi^*\mathscr{L}^{\otimes n})$. Here $\pi^{-1}(x)$ denotes the fiber of π above x. Note it may be a non-reduced scheme in the case that L/K is not separable. For $n \gg 0$ the map $p_x^n \otimes_K L$ is surjective as $\pi^*\mathscr{L}$ is ample. $\square$

Lemma 2.14. — *Let R be a ring, F a free R-module of finite rank and M an arbitrary R-module. If $M \otimes_R F$ is a locally free R-module of finite rank then M is locally free of finite rank, as well.*

Proof. M is a direct summand of the locally free R-module $M \otimes_R F$ being of finite rank. Therefore, there is some affine open covering $\{\operatorname{Spec} R_{f_1}, \ldots, \operatorname{Spec} R_{f_n}\}$

of $\operatorname{Spec} R$ such that each $(M \otimes_R F) \otimes_R R_{f_i} = (M \otimes_R R_{f_i}) \otimes_R F$ is a free R_{f_i}-module. $M \otimes_R R_{f_i} = M_{f_i}$ is a direct summand and, therefore, a projective R_{f_i}-module. There exists a surjection $(M \otimes_R F) \otimes_R R_{f_i} \twoheadrightarrow M_{f_i}$ whose kernel K has to be a direct summand of $(M \otimes_R F) \otimes_R R_{f_i}$, as well. In particular, K is a finitely generated R_{f_i}-module. Thus, M_{f_i} is finitely presented and, therefore, locally free by [**Ma**, Theorem 7.12 and Theorem 4.10]. □

Remark 2.15. — Galois descent is the central technique in André Weil's foundation of Algebraic Geometry. The Grothendieck school gave a far-reaching generalization of it, the so-called faithful flat descent. It turns out that we will not need faithful flat descent in its full generality later on here. For that reason we decided to present the more elementary Galois descent in detail. The reader who is interested in faithful flat descent can find detailed information in [**K/O**].

3. Central simple algebras and non-abelian H^1

We are going to make use of the following well-known facts about central simple algebras.

Lemma 3.1 (J. H. Maclagan-Wedderburn, R. Brauer). — *Let K be a field.*

a) *Let A be a central simple algebra over K. Then there exist a skew field D with center K and a natural number n such that $A \cong \mathrm{M}_n(D)$ is isomorphic to the full algebra of $n \times n$-matrices with entries in D.*

b) *Let L be a field extension of K and A be a central simple algebra over K. Then $A \otimes_K L$ is a central simple algebra over L.*

c) *Assume K to be separably closed. Let D be a skew field being finite dimensional over K whose center is equal to K. Then $D = K$.*

Proof. See the standard literature, for example S. Lang [**La**] or N. Bourbaki [**Bou**], or the book of I. Kersten [**Ke**]. □

Remarks 3.2. — a) Let A be a central simple algebra over a field K.

i) The proof of Lemma 3.1.a) shows that in the presentation $A \cong \mathrm{M}_n(D)$ the skew field D is unique up to isomorphism of K-algebras and the natural number n is unique.

ii) $A \otimes_K K^{\mathrm{sep}}$ is isomorphic to a full matrix algebra over K^{sep}. In particular, $\dim_K A$ is a perfect square. The natural number $\operatorname{ind}(A) := \sqrt{\dim_K(D)}$ is called the *index* of A.

b) Let A_1, A_2 be central simple algebras over a field K. Then $A_1 \otimes_K A_2$ can be shown to be a central simple algebra over K. Further, if A is a central simple algebra over a field K then $A \otimes_K A^{\mathrm{op}} \cong \mathrm{Aut}_{K-\mathrm{Vect}}(A)$, i.e. it is isomorphic to a matrix algebra.

c) Two central simple algebras $A_1 \cong \mathrm{M}_{n_1}(D_1), A_2 \cong \mathrm{M}_{n_2}(D_2)$ over a field K are said to be *similar* if the corresponding skew fields D_1 and D_2 are isomorphic as K-algebras. This is an equivalence relation on the set of all isomorphy classes of central simple algebras over K. The tensor product induces a group structure on the set of similarity classes of central simple algebras over K, this is the so-called *Brauer group* $\mathrm{Br}(K)$ of the field K.

***Definition* 3.3.** — Let K be a field and A be a central simple algebra over K. A field extension L of K admitting the property that $A \otimes_K L$ isomorphic to a full matrix algebra is said to be a *splitting field* for A. In this case one says A *splits* over L.

***Lemma 3.4* (*Theorem of Skolem-Noether*).** — *Let R be a commutative ring with unit. Then* $\mathrm{GL}_n(R)$ *operates on* $\mathrm{M}_n(R)$ *by conjugation,*

$$(g, m) \mapsto gmg^{-1}.$$

If $R = L$ *is a field then this defines an isomorphism*

$$\mathrm{PGL}_n(L) := \mathrm{GL}_n(L)/L^* \xrightarrow{\cong} \mathrm{Aut}_L(\mathrm{M}_n(L)).$$

Proof. One has $L = \mathrm{Zent}(\mathrm{M}_n(L))$. Therefore, the mapping is well-defined and injective.
Surjectivity. Let $j \colon \mathrm{M}_n(L) \to \mathrm{M}_n(L)$ be an automorphism. We consider the algebra $M := \mathrm{M}_n(L) \otimes_L \mathrm{M}_n(L)^{\mathrm{op}}$ ($\cong \mathrm{M}_{n^2}(L)$). $\mathrm{M}_n(L)$ gets equipped with the structure of a left M-module in two ways.

$$\begin{aligned} (A \otimes B) \bullet_1 C &:= A \cdot C \cdot B \\ (A \otimes B) \bullet_2 C &:= j(A) \cdot C \cdot B \end{aligned}$$

Two $\mathrm{M}_{n^2}(L)$-modules of the same L-dimension are isomorphic, as the n^2-dimensional standard L-vector space equipped with the canonical operation of $\mathrm{M}_{n^2}(L)$ is the only simple left $\mathrm{M}_{n^2}(L)$-module and there are no non-trivial extensions. Thus, there is an isomorphism $h \colon (\mathrm{M}_n(L), \bullet_1) \to (\mathrm{M}_n(L), \bullet_2)$. Let us put $I := h(E)$ to be the image of the identity matrix. For every $M \in \mathrm{M}_n(L)$ we have

$$h(M) = h((E \otimes M) \bullet_1 E) = (E \otimes M) \bullet_2 h(E) = h(E) \cdot M = I \cdot M.$$

In particular, $I \in \mathrm{GL}_n(L)$. Therefore,

$$I \cdot M = h(M) = h((M \otimes E) \bullet_1 E) = (M \otimes E) \bullet_2 h(E) = j(M) \cdot I$$

for each $M \in \mathrm{M}_n(L)$ and $j(M) = IMI^{-1}$. □

***Definition* 3.5.** — Let n be a natural number.

i) If K is a field then we will denote by Az_n^K the set of all isomorphy classes of central simple algebras A of dimension n^2 over K.

ii) Let L/K be a field extension. Then $\mathrm{Az}_n^{L/K}$ will denote the set of all isomorphy classes of central simple algebras A which are of dimension n^2 over K and split over L. Obviously, $\mathrm{Az}_n^K := \bigcup\limits_{L/K} \mathrm{Az}_n^{L/K}$.

***Theorem 3.6* (*cf. J.-P. Serre: Corps locaux* [*Se62, chap. X,* §5])**
Let L/K *be a finite Galois extension of fields,* $G := \mathrm{Gal}\,(L/K)$ *its Galois group and n be a natural number. Then there is a natural bijection of pointed sets*

$$\begin{aligned} a = a_n^{L/K} \colon \mathrm{Az}_n^{L/K} &\xrightarrow{\cong} H^1(G, \mathrm{PGL}_n(L)), \\ A &\mapsto a_A. \end{aligned}$$

Proof. Let A be a central simple algebra over K that splits over L,

$$A \otimes_K L \xrightarrow[f]{\cong} \mathrm{M}_n(L).$$

The diagrams

$$\begin{tikzcd}
A \otimes_K L \arrow[r, "f"] & \mathrm{M}_n(L) \\
A \otimes_K L \arrow[r, "f"] \arrow[u, "\sigma"] & \mathrm{M}_n(L) \arrow[u, "\sigma"']
\end{tikzcd}$$

for $\sigma \in G$ do not commute in general. Put $(f \circ \sigma) = a_\sigma \circ (\sigma \circ f)$ where $a_\sigma \in \mathrm{PGL}_n(L)$ for each σ. It turns out that

$$\begin{aligned}
f \circ \sigma\tau &= (f \circ \sigma) \circ \tau \\
&= a_\sigma \circ (\sigma \circ f) \circ \tau \\
&= a_\sigma \circ \sigma \circ (f \circ \tau) \\
&= a_\sigma \circ \sigma \circ (a_\tau \circ (\tau \circ f)) \\
&= a_\sigma \circ {}^\sigma a_\tau \circ (\sigma\tau \circ f),
\end{aligned}$$

i.e. $a_{\sigma\tau} = a_\sigma \cdot {}^\sigma a_\tau$ and $(a_\sigma)_{\sigma \in G}$ is a cocycle.
If one starts with another isomorphism $f' \colon A \otimes_K L \longrightarrow \mathrm{M}_n(L)$ then there exists some $b \in \mathrm{PGL}_n(L)$ such that $f = b \circ f'$. The equality $(f \circ \sigma) = a_\sigma \circ (\sigma \circ f)$ implies

$$f' \circ \sigma = b^{-1} \circ f \circ \sigma = b^{-1} \cdot a_\sigma \circ (\sigma \circ (b \circ f')) = b^{-1} \cdot a_\sigma \cdot {}^\sigma b \circ (\sigma \circ f').$$

Thus, the isomorphism f' yields a cocycle being cohomologous to $(a_\sigma)_{\sigma \in G}$. The mapping a is well-defined.

Injectivity. Assume A and A' are chosen such that the construction above yields the same cohomology class $a_A = a_{A'} \in H^1(G, \mathrm{PGL}_n(L))$. After the choice of suitable isomorphisms f and f' one has the formulas $(f \circ \sigma) = a_\sigma \circ (\sigma \circ f)$ and $(f' \circ \sigma) = a_\sigma \circ (\sigma \circ f')$ in the diagram

$$\begin{tikzcd}
A \otimes_K L \arrow[r, "f"] & \mathrm{M}_n(L) & A' \otimes_K L \arrow[l, "f'"'] \\
A \otimes_K L \arrow[r, "f"] \arrow[u, "\sigma"] & \mathrm{M}_n(L) \arrow[u, "\sigma"] & A' \otimes_K L. \arrow[l, "f'"'] \arrow[u, "\sigma"']
\end{tikzcd}$$

Consequently, $f \circ \sigma \circ f^{-1} \circ \sigma^{-1} = f' \circ \sigma \circ f'^{-1} \circ \sigma^{-1}$ and, therefore,

$$f \circ \sigma \circ f^{-1} \circ f' \circ \sigma^{-1} \circ f'^{-1} = \mathrm{id}.$$

The outer part of the diagram commutes. Taking the G-invariants on both sides gives $A \cong A'$.

Surjectivity. Let a cocycle $(a_\sigma)_{\sigma \in G}$ for $H^1(G, \mathrm{PGL}_n(L))$ be given. We define a new G-operation on $\mathrm{M}_n(L)$ by letting $\sigma \in G$ operate as

$$a_\sigma \circ \sigma \colon \mathrm{M}_n(L) \xrightarrow{\sigma} \mathrm{M}_n(L) \xrightarrow{a_\sigma} \mathrm{M}_n(L).$$

Note that this is a σ-linear mapping. Further, one has

$$(a_\sigma \circ \sigma) \circ (a_\tau \circ \tau) = a_\sigma \circ {}^\sigma a_\tau \circ \sigma\tau = a_{\sigma\tau} \circ \sigma\tau,$$

i.e. we constructed a group operation from the left. Galois descent yields the desired algebra. □

Lemma 3.7. — *Let L/K be a finite Galois extension of fields and n be a natural number.*

a) *Let L' be a field extension of L such that L'/K is Galois again. Then the following diagram of morphisms of pointed sets commutes,*

$$\begin{tikzcd}
\mathrm{Az}_n^{L/K} \arrow[r, "a_n^{L/K}"] \arrow[d, "\text{nat. incl.}"'] & H^1(\mathrm{Gal}\,(L/K), \mathrm{PGL}_n(L)) \arrow[d, "\inf_{\mathrm{Gal}\,(L/K)}^{\mathrm{Gal}\,(L'/K)}"] \\
\mathrm{Az}_n^{L'/K} \arrow[r, "a_n^{L'/K}"] & H^1(\mathrm{Gal}\,(L'/K), \mathrm{PGL}_n(L')).
\end{tikzcd}$$

b) *Let K' be an intermediate field of the extension L/K. Then the following diagram of morphisms of pointed sets commutes,*

$$\begin{tikzcd}
\mathrm{Az}_n^{L/K} \arrow[r, "a_n^{L/K}"] \arrow[d, "\otimes_K K'"'] & H^1(\mathrm{Gal}\,(L/K), \mathrm{PGL}_n(L)) \arrow[d, "\mathrm{res}_{\mathrm{Gal}\,(L/K)}^{\mathrm{Gal}\,(L/K')}"] \\
\mathrm{Az}_n^{L/K'} \arrow[r, "a_n^{L/K'}"] & H^1(\mathrm{Gal}\,(L/K'), \mathrm{PGL}_n(L)).
\end{tikzcd}$$

Proof. These are direct consequences of the construction of the bijections $a_n^{\cdot}$. □

Corollary 3.8. — *Let K be a field and n be a natural number. Then there is a unique natural bijection*

$$a = a_n^K \colon \mathrm{Az}_n^K \longrightarrow H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_n(K^{\mathrm{sep}}))$$

such that $a_n^K|_{\mathrm{Az}_n^{L/K}} = a_n^{L/K}$ for each finite Galois extension L/K in K^{sep}.

Proof. In order to get connected with the definition of cohomology of profinite groups the only technical point to prove is the formula

$$\mathrm{PGL}_n(K^{\mathrm{sep}})^{\mathrm{Gal}\,(K^{\mathrm{sep}}/K')} = \mathrm{PGL}_n(K')$$

for every intermediate field $K \subseteq K' \subseteq K^{\mathrm{sep}}$. For that the exact sequence

$$1 \longrightarrow (K^{\mathrm{sep}})^* \longrightarrow \mathrm{GL}_n(K^{\mathrm{sep}}) \longrightarrow \mathrm{PGL}_n(K^{\mathrm{sep}}) \longrightarrow 1$$

induces the cohomology exact sequence

$$1 \to (K')^* \to \mathrm{GL}_n(K') \to \mathrm{PGL}_n(K^{\mathrm{sep}})^{\mathrm{Gal}\,(K^{\mathrm{sep}}/K')} \to H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K'), (K^{\mathrm{sep}})^*)$$

whose right entry vanishes by Hilbert's Theorem 90 (cp. Lemma 4.10 below). □

Proposition 3.9. — *Let K be a field and m and n be natural numbers. Then the diagram*

$$\begin{tikzcd}
\mathrm{Az}_n^K \arrow[r, "a_n^K"] \arrow[d, "A \mapsto \mathrm{M}_m(A)"'] & H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_n(K^{\mathrm{sep}})) \arrow[d, "(i^n_{mn})_*"] \\
\mathrm{Az}_{mn}^K \arrow[r, "a_{mn}^K"] & H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_{mn}(K^{\mathrm{sep}}))
\end{tikzcd}$$

commutes where $(i^n_{mn})_*$ is the map induced by the block-diagonal embedding

$$\begin{aligned} i^n_{mn}\colon \mathrm{PGL}_n(K^{\mathrm{sep}}) &\longrightarrow \mathrm{PGL}_{mn}(K^{\mathrm{sep}}) \\ \overline{E} &\mapsto \overline{\begin{pmatrix} E & 0 & \cdots & 0 \\ 0 & E & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & E \end{pmatrix}}. \end{aligned}$$

Proof. Let $A \in \mathrm{Az}_n^K$. By the construction above, a cycle representing the cohomology class $a_n^K(A)$ is given as follows. Choose an isomorphism $f\colon A \otimes_K K^{\mathrm{sep}} \to \mathrm{M}_n(K^{\mathrm{sep}})$ and put $a_\sigma := (f \circ \sigma) \circ (\sigma \circ f)^{-1} \in \mathrm{Aut}(\mathrm{M}_n(K^{\mathrm{sep}}))$ for each $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$. On the other hand, for $\mathrm{M}_m(A) \in \mathrm{Az}_{mn}^K$ one may choose the isomorphism

$$\mathrm{M}_m(f)\colon \mathrm{M}_m(A) \otimes_K K^{\mathrm{sep}} = \mathrm{M}_m(A \otimes_K K^{\mathrm{sep}}) \longrightarrow \mathrm{M}_m(\mathrm{M}_n(K^{\mathrm{sep}})) \cong \mathrm{M}_{mn}(K^{\mathrm{sep}}).$$

For each $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ this yields the automorphism $\widetilde{a_\sigma}$ of $\mathrm{M}_m(\mathrm{M}_n(K^{\mathrm{sep}}))$ which operates as a_σ on each block. If a_σ is given by conjugation with a matrix A_σ then $\widetilde{a_\sigma}$ is given by conjugation with

$$\begin{pmatrix} A_\sigma & 0 & \cdots & 0 \\ 0 & A_\sigma & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & A_\sigma \end{pmatrix}.$$

This is exactly what was to be proven. □

Remark 3.10. — The proposition above shows

$$\mathrm{Br}(K) \cong \varinjlim_n H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_n(K^{\mathrm{sep}})).$$

Further, for each m and n there is a commutative diagram of exact sequences as follows,

$$\begin{tikzcd}
1 \arrow[r] & (K^{\mathrm{sep}})^* \arrow[r] \arrow[d, equal] & \mathrm{GL}_n(K^{\mathrm{sep}}) \arrow[r] \arrow[d] & \mathrm{PGL}_n(K^{\mathrm{sep}}) \arrow[r] \arrow[d, "j^n_{mn}"] & 1 \\
1 \arrow[r] & (K^{\mathrm{sep}})^* \arrow[r] & \mathrm{GL}_{mn}(K^{\mathrm{sep}}) \arrow[r] & \mathrm{PGL}_{mn}(K^{\mathrm{sep}}) \arrow[r] & 1.
\end{tikzcd}$$

We note that $(K^{\mathrm{sep}})^*$ is mapped into the centers of $\mathrm{GL}_n(K^{\mathrm{sep}})$ and $\mathrm{GL}_{mn}(K^{\mathrm{sep}})$, respectively. Therefore, there are boundary maps to the second group cohomology group and they all fit together to give a map

$$\varinjlim_{n} H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_n(K^{\mathrm{sep}})) \longrightarrow H^2(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), (K^{\mathrm{sep}})^*).$$

It is not complicated to show that this map is injective and surjective [**SGA4**$\frac{1}{2}$, Arcata, III.1], [**Se62**, chap. X, §5], [**Bou**, §10, Prop. 7].

4. Brauer-Severi varieties and non-abelian H^1

Definition 4.1. — Let K be a field. A scheme X over K is called a *Brauer-Severi variety* if there exists a finite, separable field extension L/K such that $X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L$ is isomorphic to a projective space $\mathbf{P}^N_L$. A field extension L of K admitting the property that $X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \cong \mathbf{P}^N_L$ for some $n \in \mathbb{N}$ is said to be a *splitting field* for X. In this case one says X *splits* over L.

Proposition 4.2. — *Let X be a Brauer-Severi variety over a field K. Then*

i) *X is a variety, i.e. a reduced and irreducible scheme.*

ii) *X is projective and regular.*

iii) *X is geometrically integral.*

iv) *One has $\Gamma(X, \mathscr{O}_X) = K$.*

v) *K is algebraically closed in the function field $X(K)$.*

Proof. We will denote the dimension of X by N.
i) and ii) are direct consequences of Lemma 2.12.i),ii), x) and xii).
iii) is clear from the definition.
iv) We have

$$\Gamma(X, \mathscr{O}_X) \otimes_K K^{\mathrm{sep}} = \Gamma(X \times_{\mathrm{Spec} K} \mathrm{Spec}\, K^{\mathrm{sep}}, \mathscr{O}_{X \times_{\mathrm{Spec} K} \mathrm{Spec}\, K^{\mathrm{sep}}}) = \Gamma(\mathbf{P}^N_{K^{\mathrm{sep}}}, \mathscr{O}_{\mathbf{P}^N_{K^{\mathrm{sep}}}}) = K^{\mathrm{sep}}.$$

Consequently, $\Gamma(X, \mathscr{O}_X) = K$.
v) Assume $g \in K(X)$ is some rational function being algebraic over K. We choose an algebraic closure $\overline{K}$ of K. The pull-back $g_{\overline{K}}$ is a rational function on $\mathbf{P}^N_{\overline{K}}$ being algebraic over $\overline{K}$. Thus, $g_{\overline{K}}$ is a constant function. Consequently, g itself has definitely no poles, i.e. g is a regular function on X. By iv) we have $g \in K$. □

Lemma 4.3. — *Let R be a commutative ring with unit.*

a) *Then $\mathrm{GL}_n(R)$ operates on $\mathbf{P}^{n-1}_R$ by morphisms of R-schemes as follows: $A \in \mathrm{GL}_n(R)$ gives rise to the morphism given by the graded automorphism*

$$\begin{array}{rcl} R[X_0, \ldots, X_{n-1}] & \longrightarrow & R[X_0, \ldots, X_{n-1}] \\ f(X_0, \ldots, X_{n-1}) & \mapsto & f((X_0, X_1, \ldots, X_{n-1}) \cdot A^{\mathrm{t}}) \end{array}$$

of the coordinate ring.

b) *If $R = L$ is a field then this induces an isomorphism*

$$\mathrm{PGL}_n(L) \xrightarrow{\cong} \mathrm{Aut}_{L-\mathrm{schemes}}(\mathbf{P}^{n-1}_L).$$

Proof. Note that in projective coordinates the definition above yields the naive operation

$$\begin{tikzcd}
\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ddots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \arrow[d, mapsto] \\
(x_0 : \ldots : x_{n-1}) \mapsto \big((a_{11}x_0 + \ldots + a_{1n}x_{n-1}) : \ldots : (a_{n1}x_0 : \ldots : a_{nn}x_{n-1})\big)
\end{tikzcd}$$

of $\mathrm{GL}_n(S)$ on the set $\mathbf{P}_R^{n-1}(S)$ of S-valued points on $\mathbf{P}_R^{n-1}$ for every commutative R-algebra S with unit.
a) is clear. For b) see [**Ha**, Example II.7.1.1]. The proof given there works equally well without the assumption on L to be algebraically closed. $\square$

Definition 4.4. — Let r be natural number.

i) If K is a field then we will denote by BS_r^K the set of all isomorphy classes of Brauer-Severi varieties X of dimension r over K.

ii) Let L/K be a field extension. Then $\mathrm{BS}_r^{L/K}$ will denote the set of all isomorphy classes of Brauer-Severi varieties X over K which are of dimension r and split over L. Obviously, $\mathrm{BS}_r^K := \bigcup\limits_{L/K} \mathrm{BS}_r^{L/K}$.

Theorem 4.5 (cf. J.-P. Serre: Corps locaux [Se62, chap. X, §6])
Let L/K be a finite Galois extension, $G := \mathrm{Gal}(L/K)$ its Galois group and n be a natural number. Then there exists a natural bijection of pointed sets

$$\begin{array}{rccc} \alpha = \alpha_{n-1}^{L/K}\colon & \mathrm{BS}_{n-1}^{L/K} & \xrightarrow{\cong} & H^1(G, \mathrm{PGL}_n(L)), \\ & X & \mapsto & \alpha_X. \end{array}$$

Proof. Let X be a Brauer-Severi variety over K that splits over L,

$$X \times_{\mathrm{Spec}K} \operatorname{Spec} L \xrightarrow[f]{\cong} \mathbf{P}_L^{n-1}.$$

On $X \times_{\mathrm{Spec}K} \operatorname{Spec} L$, as well as on $\mathbf{P}_L^{n-1}$, there are operations of G from the left by morphisms of K-schemes. The action of $\sigma \in G$ is induced by $S(\sigma)\colon \operatorname{Spec} L \to \operatorname{Spec} L$ in both cases. Unfortunately, the diagrams

$$\begin{tikzcd}
X \times_{\mathrm{Spec}K} \operatorname{Spec} L \arrow[r, "f"] & \mathbf{P}_L^{n-1} \\
X \times_{\mathrm{Spec}K} \operatorname{Spec} L \arrow[r, "f"] \arrow[u, "\sigma"] & \mathbf{P}_L^{n-1} \arrow[u, "\sigma"']
\end{tikzcd}$$

with $\sigma \in G$ do not commute in general. We put $(f \circ \sigma) = \alpha_\sigma \circ (\sigma \circ f)$ with $\alpha_\sigma \in \mathrm{PGL}_n(L)$. $(\alpha_\sigma)_{\sigma\in G}$ is a cocycle by the same calculation as above:

$$\begin{aligned} f \circ \sigma\tau &= (f \circ \sigma) \circ \tau \\ &= \alpha_\sigma \circ (\sigma \circ f) \circ \tau \\ &= \alpha_\sigma \circ \sigma \circ (f \circ \tau) \\ &= \alpha_\sigma \circ \sigma \circ (\alpha_\tau \circ (\tau \circ f)) \\ &= \alpha_\sigma \circ {}^\sigma\alpha_\tau \circ (\sigma\tau \circ f). \end{aligned}$$

If one starts with another isomorphism $f'\colon X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \longrightarrow \mathbf{P}_L^{n-1}$ then there exists some $b \in \mathrm{PGL}_n(L)$ such that $f = b \circ f'$. The equality $(f \circ \sigma) = \alpha_\sigma \circ (\sigma \circ f)$ implies

$$f' \circ \sigma = b^{-1} \circ f \circ \sigma = b^{-1} \cdot \alpha_\sigma \circ (\sigma \circ (b \circ f')) = b^{-1} \cdot \alpha_\sigma \cdot {}^\sigma b \circ (\sigma \circ f').$$

Thus, the isomorphism f' yields a cocycle being cohomologous to $(\alpha_\sigma)_{\sigma\in G}$. Consequently, the mapping α is well-defined.

Injectivity. Assume X and X' are chosen such that the same cohomology class $\alpha_X = \alpha_{X'} \in H^1(G, \mathrm{PGL}_n(L))$ arises. After the choice of suitable isomorphisms f and f' one has the formulas $(f \circ \sigma) = \alpha_\sigma \circ (\sigma \circ f)$ and $(f' \circ \sigma) = \alpha_\sigma \circ (\sigma \circ f')$ in the diagram

$$\begin{tikzcd} X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \arrow[r, "f"] & \mathbf{P}_L^{n-1} & X' \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \arrow[l, "f'"'] \\ X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \arrow[r, "f"] \arrow[u, "\sigma"] & \mathbf{P}_L^{n-1} \arrow[u, "\sigma"'] & X' \times_{\mathrm{Spec} K} \mathrm{Spec}\, L. \arrow[l, "f'"'] \arrow[u, "\sigma"'] \end{tikzcd}$$

Therefore, $f \circ \sigma \circ f^{-1} \circ \sigma^{-1} = f' \circ \sigma \circ f'^{-1} \circ \sigma^{-1}$ and, consequently,

$$f \circ \sigma \circ f^{-1} \circ f' \circ \sigma^{-1} \circ f'^{-1} = \mathrm{id}.$$

The outer part of the diagram commutes. Galois descent yields $X \cong X'$.

Surjectivity. Let a cocycle $(\alpha_\sigma)_{\sigma\in G}$ for $H^1(G, \mathrm{PGL}_n(L))$ be given. We define a new G-operation on $\mathbf{P}_L^{n-1}$ by letting $\sigma \in G$ operate as

$$\alpha_\sigma \circ \sigma\colon \mathbf{P}_L^{n-1} \xrightarrow{\sigma} \mathbf{P}_L^{n-1} \xrightarrow{\alpha_\sigma} \mathbf{P}_L^{n-1}.$$

This is a group operation as $(\alpha_\sigma)_{\sigma\in G}$ is a cocycle. The geometric version of Galois descent yields the desired variety. $\square$

Lemma 4.6. — *Let L/K be a finite Galois extension of fields and n be a natural number.*

a) *Let L' be a field extension of L such that L'/K is Galois again. Then the following diagram of morphisms of pointed sets commutes.*

$$\begin{tikzcd}
\mathrm{BS}^{L/K}_{n-1} \arrow[r, "\alpha^{L/K}_{n-1}"] \arrow[d, "\text{nat. incl.}"'] & H^1(\mathrm{Gal}\,(L/K), \mathrm{PGL}_n(L)) \arrow[d, "\mathrm{inf}^{\mathrm{Gal}\,(\mathrm{L}'/\mathrm{K})}_{\mathrm{Gal}\,(\mathrm{L}/\mathrm{K})}"] \\
\mathrm{BS}^{L'/K}_{n-1} \arrow[r, "\alpha^{L'/K}_{n-1}"] & H^1(\mathrm{Gal}\,(L'/K), \mathrm{PGL}_n(L'))
\end{tikzcd}$$

b) *Let K' be an intermediate field of the extension L/K. Then the following diagram of morphisms of pointed sets commutes.*

$$\begin{tikzcd}
\mathrm{BS}^{L/K}_{n-1} \arrow[r, "\alpha^{L/K}_{n-1}"] \arrow[d, "\times_{\mathrm{Spec} \mathrm{K}} \mathrm{Spec}\, \mathrm{K}'"'] & H^1(\mathrm{Gal}\,(L/K), \mathrm{PGL}_n(L)) \arrow[d, "\mathrm{res}^{\mathrm{Gal}\,(\mathrm{L}/\mathrm{K}')}_{\mathrm{Gal}\,(\mathrm{L}/\mathrm{K})}"] \\
\mathrm{BS}^{L/K'}_{n-1} \arrow[r, "\alpha^{L/K'}_{n-1}"] & H^1(\mathrm{Gal}\,(L/K'), \mathrm{PGL}_n(L))
\end{tikzcd}$$

Proof. These are direct consequences of the construction of the mappings $\alpha^{\cdot}_{n-1}$.□

Corollary 4.7. — *Let K be a field and n be a natural number. Then there is a natural bijection*

$$\alpha = \alpha^{K}_{n-1} \colon \mathrm{BS}^{K}_{n-1} \longrightarrow H^1(\mathrm{Gal}\,(K^{\mathrm{sep}}/K), \mathrm{PGL}_n(K^{\mathrm{sep}}))$$

such that $\alpha^{K}_{n-1}|_{\mathrm{BS}^{L/K}_{n-1}} = \alpha^{L/K}_{n-1}$ *for each finite Galois extension L/K in K^{sep}.*
Proof. The proof follows the same line as the proof of Corollary 3.8. □

Proposition 4.8 **(*F. Severi, cf. J.-P. Serre: Corps locaux* [*Se62, chap. X, §6, Exc. 1*])**
Let r be a natural number. If X is a Brauer-Severi variety of dimension r over a field K and $X(K) \neq \emptyset$ then, necessarily, $X \cong \mathbf{P}^r_K$.
Proof. Let L/K be a finite Galois extension being a splitting field for X. Denote its Galois group by G. Choose an isomorphism $X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L \xrightarrow{\cong} \mathbf{P}^r_L$. One may assume without restriction that the L-valued point $x_L \in X \times_{\mathrm{Spec} K} \mathrm{Spec}\, L\,(L)$ induced by the K-valued point $x \in X(K)$ is mapped to $(1\!:\!0\!:\ \ldots\ :\!0) \in \mathbf{P}^r_L(L)$. Therefore, the cohomology class $\alpha_X \in H^1\big(G, \mathrm{PGL}_{r+1}(L)\big)$ is given by a cocycle $(\alpha_\sigma)_{\sigma \in G}$ where every α_σ admits $(1\!:\!0\!:\ \ldots\ :\!0)$ as a fixed point.
Consequently, α_X belongs to the image of $H^1\big(G, F/L^*\big)$ under the natural homomorphism where

$$F := \left\{ \left(\begin{array}{ccc} a_{11} & \ldots & a_{1,r+1} \\ \vdots & \ddots & \vdots \\ a_{r+1,1} & \ldots & a_{r+1,r+1} \end{array} \right) \in \mathrm{GL}_{r+1}(L) \;\middle|\; a_{21} = a_{31} = \ \ldots\ = a_{r+1,1} = 0 \right\}.$$

But it is obvious that

$$F/L^* \cong F' := \left\{ \begin{pmatrix} 1 & a_{12} & \dots & a_{1,r+1} \\ 0 & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & a_{r+1,2} & \dots & a_{r+1,r+1} \end{pmatrix} \right\} \subseteq \mathrm{GL}_{r+1}(L)$$

and the natural homomorphism $H^1(G,F') \to H^1(G, \mathrm{PGL}_{r+1}(L))$ factors via $H^1(G, \mathrm{GL}_{r+1}(L))$. The assertion follows from Lemma 4.10 below. □

Remark 4.9. — One can easily show that even $H^1(G, F') = 0$. Indeed,

$$F_1 := \{(a_{ij})_{1 \leq i,j \leq r+1} \,|\, a_{ij} = 0 \text{ for } i \geq 2, i \neq j;\, a_{11} = a_{22} = \ \dots\ = a_{r+1,r+1} = 1\}$$

is a normal G-subgroup of F'. So F' admits a filtration by G-subgroups, each one being normal in the next one, such that all the subquotients occurring are isomorphic either to $\mathrm{GL}_r(L)$ or to $(L, +)$.

Lemma 4.10. — *Let L/K be a finite Galois extension of fields, $G := \mathrm{Gal}(L/K)$ its Galois group and $n \in \mathbb{N}$. Then $H^1(G, \mathrm{GL}_n(L)) = 0$.*

Proof. Let $(a_\sigma)_{\sigma \in G}$ be a cocycle with values in $\mathrm{GL}_n(L)$. We define a G-operation from the left on L^n by the declaration that $\sigma \in G$ acts as

$$a_\sigma \circ \sigma \colon L^n \xrightarrow{\sigma} L^n \xrightarrow{a_\sigma} L^n.$$

Clearly, $a_\sigma \circ \sigma$ is a σ-linear map. Galois descent yields a K-vector space V such that one has an isomorphism

$$V \otimes_K L \xrightarrow[b]{\cong} L^n$$

making the diagrams

$$\begin{tikzcd} V \otimes_K L \arrow[r, "b"] & L^n \\ V \otimes_K L \arrow[u, "\sigma"] \arrow[r, "b"] & L^n \arrow[u, "a_\sigma \circ \sigma"'] \end{tikzcd}$$

commute. In particular, one has $\dim_K V = n$. The choice of an isomorphism

$$V \xrightarrow{\cong} K^n$$

yields $b \in \mathrm{GL}_n(L)$ such that $b \circ \sigma = a_\sigma \circ \sigma \circ b$, i.e. $a_\sigma = b \circ \sigma \circ b^{-1} \circ \sigma^{-1} = b \circ {}^\sigma(b^{-1})$ for all $\sigma \in G$. $(a_\sigma)_{\sigma \in G}$ is cohomologous to the trivial cocycle. □

Definition 4.11. — Let K be a field, r a natural number and X be a Brauer-Severi variety of dimension r. Then *a linear subspace* of X is a closed subvariety $Y \subset X$ such that $Y \times_{\mathrm{Spec} K} \mathrm{Spec}\, K^{\mathrm{sep}} \subset X \times_{\mathrm{Spec} K} \mathrm{Spec}\, K^{\mathrm{sep}} \cong \mathbf{P}^r_{K^{\mathrm{sep}}}$ is a linear subspace of the projective space. This property is independent of the isomorphism chosen.

Remark 4.12. — A K-valued point would be a zero-dimensional linear subspace but except for the trivial case $X \cong \mathbf{P}^r_K$ there are none of them. Nevertheless, it may happen that there exist linear subspaces $Y \subsetneq X$ of higher dimension. They can be investigated by cohomological methods generalizing the argument given above.

***Proposition 4.13* (*F. Châtelet, M. Artin*).** — *Let K be a field, r and d be natural numbers, X be a Brauer-Severi variety of dimension r and Y a linear subspace of dimension d. Then the natural boundary maps send the cohomology classes $\alpha_r^K(X)\in H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K),\mathrm{PGL}_{r+1}(K^{\mathrm{sep}}))$ and $\alpha_d^K(Y)\in H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K),\mathrm{PGL}_{d+1}(K^{\mathrm{sep}}))$ to one the same class in the cohomological Brauer group $H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K),(K^{\mathrm{sep}})^*)$.*

Proof. Let L/K be a finite Galois extension being a common splitting field for X and Y. Denote its Galois group by G. Choose an isomorphism $X\times_{\mathrm{Spec}K}\mathrm{Spec}\,L\xrightarrow{\cong}\mathbf{P}^r_L$. One may assume without restriction that the linear subspace $Y\times_{\mathrm{Spec}K}\mathrm{Spec}\,L\subset X\times_{\mathrm{Spec}K}\mathrm{Spec}\,L\cong\mathbf{P}^r_L$ is given by the homogeneous equations $X_{d+1}=\ldots=X_r=0$. Therefore, the cohomology class $\alpha_r^{L/K}(X)\in H^1\big(G,\mathrm{PGL}_{r+1}(L)\big)$ is given by a cocycle $(\alpha_\sigma)_{\sigma\in G}$ where every α_σ fixes that linear subspace.

Consequently, $\alpha_r^{L/K}(X)$ belongs to the image of $H^1\big(G,F/L^*\big)$ under the natural homomorphism where

$$F:=\left\{\left(\begin{array}{c|c} E_1 & H \\ \hline 0 & E_2\end{array}\right)\in\mathrm{GL}_{r+1}(L)\;\middle|\;E_1\in\mathrm{GL}_{d+1}(L),E_2\in\mathrm{GL}_{r-d}(L),H\in\mathrm{M}_{(d+1)\times(r-d)}(L)\right\}.$$

F comes equipped with the homomorphism of G-groups

$$\begin{array}{rccc} p\colon & F & \longrightarrow & \mathrm{GL}_{d+1}(L), \\ & \left(\begin{array}{c|c} E_1 & H \\ \hline 0 & E_2\end{array}\right) & \mapsto & E_1. \end{array}$$

Thus, we obtain a commutative diagram that connects the three central inclusions

$$\begin{tikzcd}
L^* \arrow[r] \arrow[d, equal] & \mathrm{GL}_{d+1}(L) \\
L^* \arrow[r] \arrow[d, equal] & F \arrow[u, "p"'] \arrow[d, "i"] \\
L^* \arrow[r] & \mathrm{GL}_{r+1}(L),
\end{tikzcd}$$

and, therefore, the following commutative diagram that unites the boundary maps,

$$\begin{tikzcd}
H^1(G,\mathrm{PGL}_{d+1}(L)) \arrow[r] & H^2(G,L^*) \\
H^1(G,F/L^*) \arrow[r] \arrow[u, "p_*"] \arrow[d, "i_*"'] & H^2(G,L^*) \arrow[u, equal] \arrow[d, equal] \\
H^1(G,\mathrm{PGL}_{r+1}(L)) \arrow[r] & H^2(G,L^*).
\end{tikzcd}$$

But we know there exists $\alpha\in H^1(G,F/L^*)$ such that $i_*(\alpha)=\alpha_r^{L/K}(X)$ and $p_*(\alpha)=\alpha_d^{L/K}(Y)$. □

5. Central simple algebras and Brauer-Severi varieties

Theorem 5.1. — *Let n be a natural number, K a field and A a central simple algebra over K of dimension n^2.*

a) *Then there exists a Brauer-Severi variety X_A of dimension $n-1$ over K satisfying condition* (+) *below.* (+) *determines X_A uniquely up to isomorphism of K-schemes.*

(+) *If L/K is a finite Galois extension being a splitting field for A then is a splitting field for X_A, too, and there is one and the same cohomology class*

$$a_A = \alpha_{X_A} \in H^1(\mathrm{Gal}\,(L/K), \mathrm{PGL}_n(L))$$

associated with A and X_A.

b) *The assignment $X\colon A \mapsto X_A$ admits the following properties.*

i) *It is compatible with extensions K'/K of the base field, i.e.*

$$X_{A \otimes_K K'} \cong X_A \times_{\mathrm{Spec}K} \mathrm{Spec}\, K'.$$

ii) *L/K is a splitting field for A if and only if L/K is a splitting field for X_A.*

Proof. a) *Uniqueness* is clear from the results of the preceding sections.
Existence. Choose a finite Galois extension L/K being a splitting field for A and take condition (+) as a definition for X_A. This is independent of the choice of L by Lemmas 3.7.a) and 4.6.a).

b) i) Let K'/K be an arbitrary field extension. We have the obvious diagram of field extensions

$$\begin{tikzcd}
& LK' & \\
K' \arrow[ur] & & L \arrow[ul] \\
& K \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and the canonical inclusion $\mathrm{Gal}\,(LK'/K') \subseteq \mathrm{Gal}\,(L/K)$. Under the constructions given above one assigns to the central simple algebra $A \otimes_K K'$ and the Brauer-Severi variety $X \times_{\mathrm{Spec}K} \mathrm{Spec}\, K'$ the restrictions of the cohomology classes assigned to A, respectively X,

$$\begin{aligned} a_{(A \otimes_K K')} &= \mathrm{res}^{\mathrm{Gal}\,(LK'/K')}_{\mathrm{Gal}\,(L/K)}(a_A), \\ \alpha_{(X \times_{\mathrm{Spec}K} \mathrm{Spec}\, K')} &= \mathrm{res}^{\mathrm{Gal}\,(LK'/K')}_{\mathrm{Gal}\,(L/K)}(\alpha_X). \end{aligned}$$

ii) The two statements above are equivalent to

$$\mathrm{res}^{\mathrm{Gal}\,(LK'/K')}_{\mathrm{Gal}\,(L/K)}(a_A) \in H^1\big(\mathrm{Gal}\,(LK'/K'), \mathrm{PGL}_n(LK')\big) = 0$$

and

$$\mathrm{res}^{\mathrm{Gal}\,(LK'/K')}_{\mathrm{Gal}\,(L/K)}(\alpha_{X_A}) \in H^1\big(\mathrm{Gal}\,(LK'/K'), \mathrm{PGL}_n(LK')\big) = 0,$$

respectively. As we have $a_A = \alpha_{X_A}$, the assertion follows. □

Corollary 5.2. — *Let K be a field and A a central simple algebra over K. Then K' is a splitting field for A if and only if $X_A(K') \neq \emptyset$.*

Proof. "$\Longrightarrow$" is trivial and "$\Longleftarrow$" is an easy consequence of Proposition 4.8. □

Corollary 5.3. — i) *Let K be a field and n a natural number. Then X induces a bijection*

$$X_n^K \colon \mathrm{Az}_n^K \to \mathrm{BS}_{n-1}^K.$$

ii) *Let L/K be a field extension. Then X induces a bijection*

$$X_n^{L/K} \colon \mathrm{Az}_n^{L/K} \to \mathrm{BS}_{n-1}^{L/K}.$$

iii) *These mappings are compatible with extensions of the base field, i.e. the diagram*

$$\begin{tikzcd} \mathrm{Az}_n^K \arrow[r, "X_n^K"] \arrow[d, "\otimes_K K'"'] & \mathrm{BS}_{n-1}^K \arrow[d, "\times_{\mathrm{Spec} K}\mathrm{Spec}\, K'"] \\ \mathrm{Az}_n^{K'} \arrow[r, "X_n^{K'}"] & \mathrm{BS}_{n-1}^{K'} \end{tikzcd}$$

commutes for every field extension K'/K.

Proof. i) follows immediately from Theorem 5.1.a). ii) is a consequence of Theorem 5.1.a) together with Theorem 5.1.b.ii). iii) is simply a reformulation of Theorem 5.1.b.i). □

Remark 5.4. — It may happen that two Brauer-Severi varieties X_1, X_2 over some field K are birationally equivalent but not isomorphic. S. A. Amitsur [**Am55**] proved that in this case the corresponding central simple algebras A_1 and A_2 generate the same subgroup of the Brauer group $\mathrm{Br}(K)$. It is still an open question whether the converse is true although interesting partial results have been obtained by P. Roquette [**Roq64**] and S. L. Tregub [**Tr**].

Proposition 5.5. — *Let K be a field, n be a natural number and A a central simple algebra of dimension n^2 over K. Then there is an isomorphism*

$$x_A \colon \mathrm{Aut}_K(A) \xrightarrow{\cong} \mathrm{Aut}_{K-\mathrm{schemes}}(X_A).$$

Proof. Let L/K be a finite Galois extension such that L is a splitting field for A and put $G := \mathrm{Gal}\,(L/K)$. Choose an isomorphism $A \otimes_K L \xrightarrow{f} \mathrm{M}_n(L)$. Then there are commutative diagrams

$$\begin{tikzcd} A \otimes_K L \arrow[r, "f"] \arrow[d, "\sigma"'] & \mathrm{M}_n(L) \arrow[d, "a_\sigma \circ \sigma"] \\ A \otimes_K L \arrow[r, "f"] & \mathrm{M}_n(L) \end{tikzcd}$$

for each $\sigma \in G$ where $(a_\sigma)_{\sigma \in G}$ is a cocycle for $a_A \in H^1(G, \mathrm{PGL}_n(L))$. By Galois descent, it is equivalent to give an element of $\mathrm{Aut}_K(A)$ or to give an element of $\mathrm{PGL}_n(L)$ being invariant under $a_\sigma \circ \sigma$ for every $\sigma \in G$.

As $\alpha_{X_A} = a_A$ we find an isomorphism $X \times_{\operatorname{Spec} K} \operatorname{Spec} L \xrightarrow{f'} \mathbf{P}_L^{n-1}$ such that the diagrams

$$\begin{tikzcd}
X \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[r, "f'"] \arrow[d, "\sigma"'] & \mathbf{P}_L^{n-1} \arrow[d, "a_\sigma \circ \sigma"] \\
X \times_{\operatorname{Spec} K} \operatorname{Spec} L \arrow[r, "f'"] & \mathbf{P}_L^{n-1}
\end{tikzcd}$$

commute. Therefore, by Galois descent for morphisms of schemes, it is equivalent to give an element of $\operatorname{Aut}_{K-\text{schemes}}(X_A)$ or to give an element of $\operatorname{PGL}_n(L)$ being invariant under $a_\sigma \circ \sigma$ for every $\sigma \in G$. □

***Proposition 5.6* (F. Châtelet, M. Artin).** — *Let K be a field, n and d be natural numbers, and A be a central simple algebra of dimension n^2 over K. Then the Brauer-Severi variety X_A associated with A admits a linear subspace of dimension d if and only if $d \leq n-1$ and*

$$d \equiv -1 \,(\operatorname{mod} \operatorname{ind}(A)).$$

Proof. We write $A = \mathrm{M}_m(D)$ with a skew field D and put $e := \operatorname{ind}(A)$. Clearly, $n = me$.

"$\Longrightarrow$" Let $H \subset X_A$ be a linear subspace of dimension d. By Proposition 4.13 we know $H \cong X_{A'}$ for some central simple algebra A' which is similar to A, i.e. $A' \cong \mathrm{M}_k(D)$ for a certain $k \in \mathbb{N}$. It follows that $\dim A' = k^2 \cdot \dim D = k^2 \cdot e^2$ and $\dim H = \dim X_{A'} = k \cdot e - 1$. This implies the congruence desired. Further, we have $\dim X_A = n-1$. Consequently, $d = \dim H \leq \dim X_A = n-1$.

"$\Longleftarrow$" Let L/K be a finite Galois extension such that L is a splitting field for A and denote its Galois group by G. Further, let k be the natural number such that $d = k \cdot e - 1$. By assumption, $k \cdot e - 1 = d \leq n - 1 = m \cdot e - 1$, hence $k \leq m$.

We consider the cohomology class $a_D \in H^1(G, \operatorname{PGL}_e(L))$ associated with D. By Proposition 3.9, $a_A = (i^e_{me})_*(a_D)$ where $i^e_{me} \colon \operatorname{PGL}_e(L) \to \operatorname{PGL}_{me}(L)$ is the block-diagonal embedding. Let $(a_\sigma)_{\sigma \in G}$ be a cocycle representing the cohomology class a_D. Then $(i^e_{me}(a_\sigma))_{\sigma \in G}$ is a cocycle that represents a_A.

We define a G-operation on $\mathbf{P}_L^{me-1}$ by letting $\sigma \in G$ operate as

$$i^e_{me}(a_\sigma) \circ \sigma \colon \mathbf{P}_L^{me-1} \xrightarrow{\quad\sigma\quad} \mathbf{P}_L^{me-1} \xrightarrow{i^e_{me}(a_\sigma)} \mathbf{P}_L^{me-1}.$$

This is a group operation as $(i^e_{me}(a_\sigma))_{\sigma \in G}$ is a cocycle. The geometric version of Galois descent yields the Brauer-Severi variety X_A. Further, the G-operation fixes the linear subspace defined by the homogeneous equations $X_{ke} = X_{ke+1} = \ldots = X_{me-1} = 0$. So Galois descent can be applied to this subspace, as well. It gives a variety Y of dimension $ke - 1 = d$. Galois descent for morphisms of schemes, applied to the canonical embedding, yields a morphism $Y \longrightarrow X_A$. This is a closed immersion as that property descends under faithful flat base change. Consequently, Y is a linear subvariety of X_A of the dimension desired. □

6. Functoriality

Remark 6.1. — The preceding results suggest that $X\colon A \mapsto X_A$ should somehow be a functor. This is indeed the case but for that the construction of the Brauer-Severi variety associated with a central simple algebra given above is not sufficient. The problem is that X_A is determined by its associated class in group cohomology only up to isomorphism and not up to canonical isomorphism. Thus, in order to make X into a functor, it would still be necessary to make choices. For that reason we aim at a more natural description of X_A.

***Lemma 6.2* (*A. Grothendieck*).** — *Let n be a natural number and R a commutative ring with unit. Then there is a bijection*

$$\kappa_R\colon \mathbf{P}_R^{n-1}(R) \longrightarrow G(R) := \left\{ \begin{array}{l} \text{submodules } M \text{ in } R^n \text{ such that } R^n/M \\ \text{is a locally free } R\text{-module of rank } n-1 \end{array} \right\}$$

subject to the conditions given below.

i) *κ_R is natural in R, i.e. for every ring homomorphism $i\colon R \to R'$ the diagram*

$$\begin{tikzcd} \mathbf{P}_R^{n-1}(R) \arrow[r, "\kappa_R"] \arrow[d, "\mathbf{P}^{n-1}(i)"'] & G(R) \arrow[d, "G(i)\colon M \mapsto M \otimes_R R'"] \\ \mathbf{P}_{R'}^{n-1}(R') \arrow[r, "\kappa_{R'}"] & G(R') \end{tikzcd}$$

commutes.

ii) *For every $a \in \mathrm{PGL}_n(R)$ the canonical actions of a on $\mathbf{P}_R^{n-1}(R)$ and $G(R)$ are compatible with κ_R. That means that the diagrams*

$$\begin{tikzcd} \mathbf{P}_R^{n-1}(R) \arrow[r, "\kappa_R"] \arrow[d, "a"'] & G(R) \arrow[d, "a"] \\ \mathbf{P}_R^{n-1}(R) \arrow[r, "\kappa_R"] & G(R) \end{tikzcd}$$

commute where $a \in \mathrm{PGL}_n(R)$ acts on a submodule $M \subset R^n$ by matrix multiplication from the left, i.e. by $M \mapsto \overline{a}\cdot M$. Here $\overline{a} \in \mathrm{GL}_n(R)$ is a representative of a.

Proof. A submodule $M \subseteq R^n$ such that the quotient R^n/M is locally free of rank $n-1$ defines an R-valued point in $\mathbf{P}_R^{n-1}$. To see this, let first $\mathfrak{m} \subseteq R$ be an arbitrary maximal ideal. Then $R^n_\mathfrak{m}/M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module of rank $n-1$. In particular, it is projective. Hence, $M_\mathfrak{m}$ is a direct summand of $R^n_\mathfrak{m}$ and, therefore, projective and of finite presentation. Consequently, by [**Ma**, Theorem 7.12], $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module of rank one. As R^n/M is R-flat, there is an exact sequence

$$0 \longrightarrow M \otimes_R (R/\mathfrak{m}) \longrightarrow R^n/\mathfrak{m}R^n \longrightarrow R^n/(M + \mathfrak{m}R^n) \longrightarrow 0.$$

So the canonical map $M/\mathfrak{m}M \to R^n/\mathfrak{m}R^n$ is injective, i.e. $\mathfrak{m}M = \mathfrak{m}R^n \cap M$. In particular, M can not be contained in $\mathfrak{m}R^n$ and the free rank-1 $R_\mathfrak{m}$-module $M_\mathfrak{m}$ is not contained in $\mathfrak{m}R^n_\mathfrak{m}$. Consequently, if $M_\mathfrak{m} = \langle (r_0, \ldots, r_{n-1}) \rangle$ then there is some $\alpha \in \{0, \ldots, n-1\}$ such that $r_\alpha \in R_\mathfrak{m}$ is a unit. This is equivalent to

$$R^n_\mathfrak{m}/(M_\mathfrak{m} + e_0 \cdot R_\mathfrak{m} + \ldots + e_{\alpha-1} \cdot R_\mathfrak{m} + e_{\alpha+1} \cdot R_\mathfrak{m} + \ldots + e_{n-1} \cdot R_\mathfrak{m}) = 0$$

where $e_0, \ldots, e_{n-1} \in R^n_\mathfrak{m}$ denote the standard elements with an index shift by -1. The latter is an open condition on $\mathfrak{m}$ by [**Ma**, Theorem 4.10]. Therefore, there exists some $f \in R \backslash \mathfrak{m}$ such that

$$R^n_f = M_f + e_1 \cdot R_f + \ldots + e_{\alpha-1} \cdot R_f + e_{\alpha+1} \cdot R_f + \ldots + e_n \cdot R_f,$$

i.e. such that the $(\alpha+1)$-th projection $M_f \to R_f$ is surjective. As there are locally free modules of rank 1 on both sides, that must be an isomorphism. Consequently, M_f is free and there is a generator of type $(r_0, \ldots, r_{\alpha-1}, 1, r_{\alpha+1}, \ldots, r_{n-1})$. Taking the entries as homogeneous coordinates we get a morphism $\operatorname{Spec} R_f \to \mathbf{P}^{n-1}_R$. It is easy to see that all these can be glued together to give a morphism of R-schemes $\operatorname{Spec} R \to \mathbf{P}^{n-1}_R$.

Conversely, an R-valued point $\operatorname{Spec} R \to \mathbf{P}^{n-1}_R$ defines a quotient module of R^n of rank $n-1$ as follows. Cover $\operatorname{Spec} R$ by affine, open subsets

$$\operatorname{Spec} R = \bigcup_{\alpha=0}^{n-1} \operatorname{Spec} R_\alpha$$

such that for each α the image $i(\operatorname{Spec} R_\alpha)$ is contained in the standard affine set "$X_\alpha \neq 0$". Then $i|_{\operatorname{Spec} R_{\alpha_0}}$ can be given in the form

$$(r_0 : r_1 : \ldots : r_{\alpha-1} : 1 : r_{\alpha+1} : \ldots : r_{n-1}).$$

But $R^n_\alpha/(r_0, r_1, \ldots, r_{\alpha-1}, 1, r_{\alpha+1}, \ldots, r_{n-1})$ is a free R_α-module of rank $n-1$ for trivial reasons.

The compatibility stated as i) follows directly from the construction given above. ii) is clear. □

Definition 6.3. — Let R be a commutative ring with unit.

i) Let X be a R-scheme. Then by

$$P_X \colon \{R - \text{schemes}\}^{\text{op}} \to \{\text{sets}\}$$

we will denote the functor defined by

$$T \mapsto \operatorname{Mor}_{R\text{-schemes}}(T, X)$$

on objects and by composition on morphisms. The various functors P_X depend on X in a natural manner, i.e. if $p \colon X \to X'$ is a morphism of R-schemes then there is a morphism of functors $i_* \colon P_X \to P_{X'}$ given by composition with i. Therefore, there is a covariant functor

$$P \colon \{R-\text{schemes}\} \longrightarrow \operatorname{Fun}(\{R-\text{schemes}\}^{\text{op}}, \{\text{sets}\})$$

described by $X \mapsto P_X$ on objects.

The functor P_X induced by a R-scheme X is exactly what in category theory is usually called the Hom-functor. Nevertheless, we will follow the standards in Algebraic Geometry and refer to it as the *functor of points*. Note that $P_X(T)$ is the set of T-valued points in X.

ii) Let $F \colon \{R - \text{schemes}\}^{\text{op}} \to \{\text{sets}\}$ be any functor. If for some R-scheme X there is an isomorphism $F \xrightarrow{\cong} P_X$ then we will say F is *represented* by X. Having fixed the isomorphism of functors, then, by Yoneda's Lemma, the K-scheme representing a functor is determined up to unique isomorphism.

***Corollary 6.4* (*A. Grothendieck*).** — *Let n be a natural number and R a commutative ring with unit. Then the functor*

$$\begin{array}{rcl} \underline{\mathbf{P}}^{n-1}_R\colon \{R\text{−schemes}\} & \longrightarrow & \{\text{sets}\} \\ T & \mapsto & \underline{\mathbf{P}}^{n-1}_R(T) := \left\{\begin{array}{l} \text{subsheaves } \mathscr{M} \text{ in } \mathscr{O}_T^n \text{ such that} \\ \mathscr{O}_T^n/\mathscr{M} \text{ is a locally free} \\ \mathscr{O}_T\text{−module of rank } n-1 \end{array}\right\} \end{array}$$

is represented by the R-scheme $\mathbf{P}^{n-1}_R$.

Proof. It is clear that both functors satisfy the sheaf axiom for Zariski coverings. Therefore it is sufficient to construct the isomorphism on the full subcategory of affine R-schemes. This is exactly what is done in Lemma 6.2. □

***Corollary-Definition 6.5*.** — Let L be a field, n a natural number and F an L-vector space of dimension n. Put $\mathscr{F} := \widetilde{F}$ to be the coherent sheaf being associated with F on $\operatorname{Spec} L$. Then the functor

$$\begin{array}{rcl} \underline{\mathbf{P}}(F)\colon \{L\text{−schemes}\} & \longrightarrow & \{\text{sets}\} \\ (\pi\colon T \to \operatorname{Spec} L) & \mapsto & \underline{\mathbf{P}}(F)(\pi) := \left\{\begin{array}{l} \text{subsheaves } \mathscr{M} \text{ in } \pi^*\mathscr{F} \text{ such that} \\ \pi^*\mathscr{F}/\mathscr{M} \text{ is a locally free} \\ \mathscr{O}_T\text{−module of rank } n-1 \end{array}\right\} \end{array}$$

is representable by an L-scheme being isomorphic to $\mathbf{P}^{n-1}_L$. We will denote that scheme by $\mathbf{P}(F)$ and call it *the projective space of lines in F*.

***Remark 6.6*.** — If L' is a field containing L then $\underline{\mathbf{P}}(F \otimes_L L')$ is naturally isomorphic to the functor $\underline{\mathbf{P}}(F)|_{L'\text{−schemes}}$. Therefore, there is a canonical isomorphism

$$\mathbf{P}(F \otimes_L L') \xrightarrow{\cong} \mathbf{P}(F) \times_{\operatorname{Spec} L} \operatorname{Spec} L'$$

between the representing objects such that for each L'-scheme U the diagram of natural mappings

$$\begin{tikzcd} \operatorname{Mor}_{L'\text{−schemes}}(U, \mathbf{P}(F)) \arrow[r] \arrow[d, "\cong"'] & \{\mathscr{M} \subset \pi_U^*\mathscr{F}\} \arrow[d, equal] \\ \operatorname{Mor}_{L'\text{−schemes}}(U, \mathbf{P}(F \otimes_L L')) \arrow[r] & \{\mathscr{M} \subset \pi_U^*\mathscr{F}\} \end{tikzcd}$$

commutes. Consequently, for each L-scheme T there is a commutative diagram

$$\begin{tikzcd} \operatorname{Mor}_{L\text{−schemes}}(T, \mathbf{P}(F)) \arrow[rr] \arrow[d, "\text{comp. with projection}"'] & & \{\mathscr{M}' \subset \pi_T^*\mathscr{F}\} \arrow[dd, "\text{pull−back}"] \\ \operatorname{Mor}_{L'\text{−schemes}}(T \times_{\operatorname{Spec} L} \operatorname{Spec} L', \mathbf{P}(F)) \arrow[d, "\cong"'] & & \\ \operatorname{Mor}_{L'\text{−schemes}}(T \times_{\operatorname{Spec} L} \operatorname{Spec} L', \mathbf{P}(F \otimes_L L')) \arrow[rr] & & \{\mathscr{M} \subset \pi^*_{T \times_{\operatorname{Spec} L} \operatorname{Spec} L'}\mathscr{F}\}. \end{tikzcd}$$

We note explicitly that in the column on the left, there is, up to the canonical isomorphism obtained above, exactly the base extension map from $\operatorname{Spec} L$ to $\operatorname{Spec} L'$.

Definition 6.7. — Let L be a field and $K \subset L$ be a subfield such that L/K is a finite Galois extension. Denote its Galois group by G.

i) By $L-\underline{\mathrm{Vect}}^K$ we will denote the category of all finite dimensional L-vector spaces where as morphisms there are allowed all injections being σ-linear for a certain $\sigma \in G$.

ii) $L-\underline{\mathrm{Vect}}_0^K$ is the subcategory of $L-\underline{\mathrm{Vect}}^K$ that consists of the same class of objects but allows only bijections as morphisms.

Definition 6.8. — i) Let H be a group. Then, by $\underline{H}$ we will denote the category consisting of exactly one object $*$ with $\mathrm{Mor}(*, *) = H$.

ii) Let L be a field and $K \subset L$ be a subfield such that L/K is a finite Galois extension. Denote its Galois group by G. By $\underline{\mathrm{Sch}}^{L/K}$ we will denote the category of all L-schemes with morphisms twisted by any element of G. We note that $\underline{\mathrm{Sch}}^{L/K}$ has a canonical structure of a fibered category over $\underline{G}$. The pull-back under $\sigma \in G$ is given by $X \mapsto X \times_{\mathrm{Spec} L} \mathrm{Spec}\, L^{\sigma^{-1}}$.

Lemma 6.9. — *Let L be a field and $K \subset L$ a subfield such that the extension L/K is finite and Galois. Denote the Galois group* $\mathrm{Gal}(L/K)$ *by G.*

i) *Then* $\mathbf{P}$ *is a covariant functor from $L-\underline{\mathrm{Vect}}^K$ to $\underline{\mathrm{Sch}}^{L/K}$. Here a σ-linear monomorphism $i\colon F \to F'$ with $\sigma \in G$ induces a morphism*

$$i_* = \mathbf{P}(i)\colon \mathbf{P}(F) \longrightarrow \mathbf{P}(F')$$

of schemes that is twisted by σ.

ii) *Let F be an L-vector space and $i\colon F \to F$ be the multiplication map with an element of L. Then $i_*\colon \mathbf{P}(F) \to \mathbf{P}(F)$ is equal to the identity morphism.*

Proof. i) Let $i\colon F \to F'$ be a σ-linear monomorphism. We have to consider the diagram

$$\begin{tikzcd}
\mathbf{P}(F) \arrow[d] & \mathbf{P}(F') \arrow[d] \\
\mathrm{Spec}\, L \arrow[r, "S(\sigma)"] & \mathrm{Spec}\, L.
\end{tikzcd}$$

By Yoneda's Lemma, there has to be constructed a natural transformation

$$i_+\colon \underline{\mathbf{P}}(F) \to \underline{\mathbf{P}}(F')(S(\sigma)\circ\cdot),$$

i.e. for each L-scheme $\pi\colon T \to \mathrm{Spec}\, L$ there is a mapping

$$i_+(\pi)\colon \underline{\mathbf{P}}(F)(\pi) \to \underline{\mathbf{P}}(F')(S(\sigma)\circ\pi)$$

to be given such that for each morphism $p\colon \pi_1 \to \pi_2$ of L-schemes the diagram

$$\begin{tikzcd}
\underline{\mathbf{P}}(F)(\pi_2) \arrow[r, "i_+(\pi_2)"] \arrow[d, "\underline{\mathbf{P}}(F)(p)"'] & \underline{\mathbf{P}}(F')(S(\sigma)\circ\pi_2) \arrow[d, "\underline{\mathbf{P}}(F')(p)"] \\
\underline{\mathbf{P}}(F)(\pi_1) \arrow[r, "i_+(\pi_1)"] & \underline{\mathbf{P}}(F')(S(\sigma)\circ\pi_1)
\end{tikzcd}$$

commutes. We construct $i_+(\pi)$ as follows: There is the description

$$\begin{aligned}\underline{\mathbf{P}}(F)(\pi) &= \{\mathscr{M} \subset \pi^*\mathscr{F} \mid \pi^*\mathscr{F}/\mathscr{M} \text{ is locally free of rank } \dim F - 1\} \\ &= \left\{\begin{array}{c} \mathscr{M} \subset (S(\sigma)\circ\pi)^* S(\sigma^{-1})^*\mathscr{F} \mid \\ (S(\sigma)\circ\pi)^* S(\sigma^{-1})^*\mathscr{F}/\mathscr{M} \text{ is locally free of rank } \dim F - 1 \end{array}\right\}\end{aligned}$$

where $S(\sigma^{-1})^*\mathscr{F} = \widetilde{F^\sigma}$. Here F^σ is nothing but F equipped with the ordinary structure of an abelian group and the multiplication by a scalar given by $l \cdot f := \sigma(l)f$. Therefore, i gives rise to an L-linear monomorphism $\underline{i}\colon F^\sigma \to F'$ and to a morphism $\widetilde{\underline{i}}\colon S(\sigma^{-1})^*\mathscr{F} \to \mathscr{F}'$ of sheaves over $\operatorname{Spec} L$. If $\mathscr{M}$ is a subsheaf of $(S(\sigma)\circ\pi)^* S(\sigma^{-1})^*\mathscr{F}$ such that $(S(\sigma)\circ\pi)^* S(\sigma^{-1})^*\mathscr{F}/\mathscr{M}$ is locally free of rank $\dim F - 1$ then $\big((S(\sigma)\circ\pi)^*(\widetilde{\underline{i}})\big)(\mathscr{M})$ is a subsheaf of $(S(\sigma)\circ\pi)^*\mathscr{F}'$ such that the quotient is locally free of rank $\dim F' - 1$. This gives rise to a morphism of functors $i_+\colon \underline{\mathbf{P}}(F) \longrightarrow \underline{\mathbf{P}}(F')(S(\sigma)\circ\,\cdot\,)$ as desired. Consequently, there is a morphism $i_*\colon \mathbf{P}(F) \longrightarrow \mathbf{P}(F')$ of schemes that makes commute the diagram considered above.
ii) This is clear from the construction. □

Corollary 6.10. — *Let L be a field and $K \subset L$ a subfield such that the extension L/K is finite and Galois. Denote the Galois group* $\operatorname{Gal}(L/K)$ *by G.*

i) *Then* $\mathbf{P}$ *can as well be made into a contravariant functor from* $L-\underline{\mathrm{Vect}}_0^K$ *to* $\underline{\mathrm{Sch}}^{L/K}$*. Here a σ-linear isomorphism $i\colon F \to F'$ with $\sigma \in G$ induces a morphism*

$$i^*\colon \mathbf{P}(F') \to \mathbf{P}(F)$$

of L-schemes being twisted by σ^{-1}.

ii) *Let F be an L-vector space and $i\colon F \to F$ be the multiplication with an element of L. Then $i^*\colon \mathbf{P}(F) \to \mathbf{P}(F)$ is equal the identity morphism.*
Proof. For an isomorphism $i\colon F \to F'$ put $i^* := i_*^{-1}$. □

Remark 6.11. — The morphisms i^* can also be constructed directly, in a manner being completely analogous to the construction of i_* given above. Indeed, the task is to give a natural transformation $i^+\colon \underline{\mathbf{P}}(F') \to \underline{\mathbf{P}}(F)(S(\sigma^{-1})\circ\,\cdot\,)$. There are the descriptions

$$\underline{\mathbf{P}}(F')(\pi) = \{\mathscr{M} \subset \pi^*\mathscr{F}' \mid \pi^*\mathscr{F}'/\mathscr{M} \text{ is locally free of rank } \dim F' - 1\}$$

and

$$\underline{\mathbf{P}}(F)(S(\sigma^{-1})\circ\pi) = \left\{\begin{array}{c} \mathscr{M} \subset \pi^* S(\sigma^{-1})^*\mathscr{F} \mid \\ \pi^* S(\sigma^{-1})^*\mathscr{F}/\mathscr{M} \text{ is locally free of rank } \dim F - 1 \end{array}\right\}.$$

If $\mathscr{M}$ is a subsheaf of $\pi^*\mathscr{F}'$ such that $\pi^*\mathscr{F}'/\mathscr{M}$ is locally free of rank $\dim F' - 1 = \dim F - 1$ then $\pi^*(\widetilde{i})^{-1}(\mathscr{M})$ is a subsheaf of $\pi^* S(\sigma^{-1})^*\mathscr{F}$ such that the quotient is locally free of rank $\dim F - 1$. This gives rise to a morphism of functors $i^+\colon \underline{\mathbf{P}}(F') \longrightarrow \underline{\mathbf{P}}(F)(S(\sigma^{-1})\circ\,\cdot\,)$ as desired. Consequently, there is a morphism $i^*\colon \mathbf{P}(F') \to \mathbf{P}(F)$ of schemes making commute the diagram above.

Remark 6.12. — Let us mention the following observation explicitly. If L is a field and A is a central simple algebra of dimension n^2 over L then all the non-zero, simple, left A-modules are isomorphic to each other. Further, if $\mathfrak{l}$ is a non-zero, simple, left A-module then each automorphism of $\mathfrak{l}$ is given by multiplication with an element from the *center* of A. Hence it induces the identity morphism on $\mathbf{P}(\mathfrak{l})$. Therefore, two arbitrary isomorphisms $i_1, i_2 \colon \mathfrak{l} \to \mathfrak{l}'$ between non-zero, simple, left A-modules induce one and the same isomorphism $i_1^* = i_2^* \colon \mathbf{P}(\mathfrak{l}') \to \mathbf{P}(\mathfrak{l})$. Consequently, A determines $\mathbf{P}(\mathfrak{l})$ not only up to isomorphism, but up to unique isomorphism. We will formulate this in a more sophisticated manner below.

Definition 6.13. — Let L/K be a finite Galois extension of fields and r be a natural number. Denote the Galois group $\mathrm{Gal}(L/K)$ by G.

i) By $\underline{\mathrm{Mat}}_r^{L/K}$ we will denote the category of all split central simple algebras of dimension r^2 over L, i.e. of all algebras being isomorphic to $\mathrm{M}_r(L)$, where as morphisms we take all homomorphisms of K-algebras which are σ-linear for a certain $\sigma \in G$ and preserve the unit element. Note that in $\underline{\mathrm{Mat}}_r^K$ every morphism is an isomorphism and every two objects are isomorphic.

ii) $\underline{\mathrm{P}}_r^{L/K}$ will denote the subcategory of $\underline{\mathrm{Sch}}^{L/K}$ which consists of all L-schemes isomorphic to the projective space $\mathbf{P}_L^r$ and which allows all isomorphisms as morphisms.

Proposition 6.14. — *Let L/K be a finite Galois extension of fields and n be a natural number. Denote the Galois group* $\mathrm{Gal}(L/K)$ *by G.*

i) *There is an equivalence of categories $\Xi_n^{L/K} \colon \underline{\mathrm{Mat}}_n^{L/K} \to \underline{\mathrm{P}}_{n-1}^{L/K}$. On objects it is given by $A \mapsto \mathbf{P}(\mathfrak{l})$ where $\mathfrak{l} \subset A$ is a non-zero, simple, left A-module. If $i \colon A \to A'$ is a morphism in $\underline{\mathrm{Mat}}_n^{L/K}$ then $\Xi_n^{L/K}(i)$ is the morphism of schemes being induced by the canonical homomorphism $\mathfrak{l} \to A' \otimes_A \mathfrak{l}$.*

ii) *If the morphism $i \colon A \to A'$ is σ-linear for $\sigma \in G$ then $\Xi_n^{L/K}(i)$ is a morphism twisted by σ.*

Proof. If $\mathfrak{l}$ is a non-zero, simple, left A-module then $A' \otimes_A \mathfrak{l}$ is a non-zero, simple, left A'-module. Both are n-dimensional L-vector spaces. Up to isomorphism, these modules are known to be unique. Therefore, the morphism of schemes

$$i_{\mathfrak{l}*} := \Xi_n^{L/K}(i) \colon \Xi_n^{L/K}(A) = \mathbf{P}(\mathfrak{l}) \longrightarrow \mathbf{P}(A' \otimes_A \mathfrak{l}) = \Xi_n^{L/K}(A')$$

induced by the homomorphism

$$i_{\mathfrak{l}} \colon \mathfrak{l} \to A' \otimes_A \mathfrak{l}, \qquad x \mapsto 1 \otimes x$$

is well-defined. If $i \colon A \to A'$ is a σ-linear homomorphism then $\mathfrak{l} \to A' \otimes_A \mathfrak{l}$ is a σ-linear homomorphism of vector spaces. By Lemma 6.9 above, the morphism $\Xi_n^{L/K}(i)$ is twisted by σ. As i is automatically invertible, $\Xi_n^{L/K}(i)$ must be an isomorphism of schemes. Consequently, $\Xi_n^{L/K}$ is a functor between the categories described.

To prove $\Xi_n^{L/K}$ is an equivalence of categories we have to show it is full, faithful and essentially surjective. As in $\underline{P}_{n-1}^{L/K}$ every two objects are isomorphic to each other, essential surjectivity is clear. For full faithfulness it will suffice to prove that

$$\Xi_n^{L/K}|_{\mathrm{Aut}(A)} \colon \mathrm{Aut}(A) \longrightarrow \mathrm{Aut}(\mathrm{X_A})$$

is an isomorphism of groups in the case $A = \mathrm{M}_n(L)$. Then $\mathrm{Aut}(A) = \mathrm{PGL}_n(L) \rtimes G$ and $\mathrm{Aut}(\Xi_n^{L/K}(A)) = \mathrm{Aut}_{K-\mathrm{schemes}}(\mathbf{P}_L^{n-1}) = \mathrm{PGL}_n(L) \rtimes G$. Thus, the functor $\Xi_n^{L/K}$ induces on $\mathrm{Aut}(A)$ a group homomorphism

$$\Xi \colon \mathrm{PGL}_n(L) \rtimes G \to \mathrm{PGL}_n(L) \rtimes G.$$

By Lemma 6.2.ii), the restriction of Ξ to $\mathrm{PGL}_n(L)$ is the identity. By statement ii) proven above, the quotient map $G \to G$ is the identity, as well. Consequently, $\Xi_n^{L/K}|_{\mathrm{Aut}(A)}$ is an isomorphism. $\square$

Corollary 6.15. — *The categories $\underline{\mathrm{Mat}}_n^{L/K}$ and $\underline{\mathrm{P}}_{n-1}^{L/K}$ are also anti-equivalent, i.e. there is an equivalence of categories $\check{\Xi}_n^{L/K}\colon \underline{\mathrm{Mat}}_n^{L/K} \to (\underline{\mathrm{P}}_{n-1}^{L/K})^{\mathrm{op}}$ given on objects by $A \mapsto \mathbf{P}(\mathfrak{l})$ where $\mathfrak{l} \subset A$ is a non-zero, simple, left A-module. If the morphism $i\colon A \to A'$ is σ-linear for $\sigma \in G$ then $\check{\Xi}_n^{L/K}(i)$ is twisted by σ^{-1}.*
Proof. Put simply $\check{\Xi}_n^{L/K}(i) := (\Xi_n^{L/K}(i))^{-1}$. $\square$

Proposition 6.16. — *Let n be a natural number, K be a field and A be a central simple algebra of dimension n^2 over K. Then the Brauer-Severi variety X_A being associated with A can be described as follows:*

> *Let L/K be a finite Galois extension such that L is a splitting field for A. Denote the Galois group* $\mathrm{Gal}(L/K)$ *by G. By covariant functoriality, the canonical G-operation on $A \otimes_K L$ induces an operation of G on the projective space $\Xi_n^{L/K}(A \otimes_K L)$ where the morphism induced by $\sigma \in G$ is a morphism of L-schemes twisted by σ. The geometric version of Galois descent yields the K-scheme X_A.*

Proof. Let $f\colon A \otimes_K L \to \mathrm{M}_n(L)$ be an isomorphism. Then there is a cocycle $(a_\sigma)_{\sigma \in G}$ from G to $\mathrm{PGL}_n(L)$ such that for each $\sigma \in G$ the diagram

$$\begin{tikzcd}
A \otimes_K L \arrow[r, "\sigma"] \arrow[d, "f"'] & A \otimes_K L \arrow[d, "f"] \\
\mathrm{M}_n(L) \arrow[r, "a_\sigma \circ \sigma"] & \mathrm{M}_n(L)
\end{tikzcd}$$

commutes. Applying the functor $\Xi_n^{L/K}$ to the whole situation we obtain commutative diagrams

$$\begin{tikzcd}
\Xi_n^{L/K}(A \otimes_K L) \arrow[r, "\sigma"] \arrow[d, "\Xi_n^{L/K}(f)"'] & \Xi_n^{L/K}(A \otimes_K L) \arrow[d, "\Xi_n^{L/K}(f)"] \\
\mathbf{P}_L^{n-1} \arrow[r, "a_\sigma \circ \sigma"] & \mathbf{P}_L^{n-1}
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Galois descent on the lower half of the diagram is the description of X_A given in the last section. Galois descent on the upper half of the diagram is the description claimed. $\square$

Corollary 6.17. — *Let L/K be a field extension and A be a central simple algebra over K.*

i) *Then there is a canonical isomorphism of L-schemes*

$$\xi_A^{L/K}\colon X_{A \otimes_K L} \longrightarrow X_A \times_{\operatorname{Spec} K} \operatorname{Spec} L.$$

ii) *If L/K and L'/L are field extensions then the isomorphisms $\xi_A^{L'/K}, \xi_A^{L'/L}$ and $\xi_A^{L/K}$ are* *compatible, i.e. the diagram*

$$\begin{tikzcd}
X_{A\otimes_K L'} \arrow[dr, "\xi_A^{L'/K}"] \arrow[dd, "\xi_A^{L'/L}"'] & \\
& X_A \times_{\mathrm{Spec} K} \mathrm{Spec}\, L' \\
X_{A\otimes_K L} \times_{\mathrm{Spec} L} \mathrm{Spec}\, L' \arrow[ur, "\xi_A^{L/K} \times_{\mathrm{Spec} L} \mathrm{Spec}\, L'"'] &
\end{tikzcd}$$

commutes.

Proof. Let n^2 be the dimension of A. We choose a splitting field L'' for A which contains L'. Then X_A, $X_{A\otimes_K L}$ and $X_{A\otimes_K L'}$ are constructed from $\Xi_n^{L''/K}(A\otimes_K L'') = \Xi_n^{L''/L}(A\otimes_K L'') = \Xi_n^{L''/L'}(A\otimes_K L'')$ by Galois descent using compatible descent data. □

Definition 6.18. — Let K be a field and r a natural number.

i) By $\underline{\mathrm{Az}}_r^K$ we will denote the category of all central simple algebras of dimension r^2 over K where as morphisms we take all homomorphisms of K-algebras that preserve the unit element. Note that in $\underline{\mathrm{Az}}_r^K$ every morphism is an isomorphism.

ii) $\underline{\mathrm{Az}}_r^{/K}$ will denote the category of all central simple algebras of dimension r^2 over any field extension of K where as morphisms we take all K-algebra homomorphisms that preserve the unit element.

iii) By $\underline{\mathrm{BS}}_r^K$ we will denote the category of all Brauer-Severi varieties of dimension r over K where the isomorphisms of K-schemes are taken as morphisms.

iv) $\underline{\mathrm{BS}}_r^{/K}$ will denote the category of all Brauer-Severi varieties of dimension r over any field extension of K. As morphisms one takes all compositions of an isomorphism of K-schemes with a morphism of type $\mathrm{id}\times\mathrm{Spec}\, a\colon X\times_{\mathrm{Spec} L}\mathrm{Spec}\, L' \to X$ where $a\colon L\to L'$ is a homomorphism of fields containing K.

Theorem 6.19. — *Let K be a field and n be a natural number.*

i) *There is an equivalence of categories*

$$\underline{X}_n^{/K}\colon \underline{\mathrm{Az}}_n^{/K} \longrightarrow (\underline{\mathrm{BS}}_{n-1}^{/K})^{\mathrm{op}},$$

that induces for each field extension L/K the bijection $X_n^L\colon \mathrm{Az}_n^L \to \mathrm{BS}_{n-1}^L$ on isomorphy classes found in Corollary 5.3.i).

ii) *In particular, for each field extension L/K the functor $\underline{X}_n^{/K}$ induces an equivalence of categories $\underline{X}_n^L\colon \underline{\mathrm{Az}}_n^L \longrightarrow (\underline{\mathrm{BS}}_{n-1}^L)^{\mathrm{op}}$.*

Proof. *1st step.* Construction of the functor.

For $A\in\mathrm{Ob}(\underline{\mathrm{Az}}_n^{/K})$ put $\underline{X}_n^{/K}(A) := X_A$. Let us remark at this place that we are going to make use of the intrinsic description of X_A given in Proposition 6.16.

If $i\colon A\to A'$ is a morphism in $\underline{\mathrm{Az}}_n^{/K}$ then, by restriction to the centers, i induces a homomorphism of fields $i|_{Z(A)}\colon Z(A)\to Z(A')$ containing K. Therefore, there is a unique factorization

$$A \xrightarrow{c_A^{Z(A')}} A\otimes_{Z(A)} Z(A') \xrightarrow{j} A'$$

of i via the canonical inclusion $c_A^{Z(A')}$. By Corollary 6.17, one has the canonical isomorphism

$$\xi_A^{Z(A')/Z(A)}\colon X_{A\otimes_{Z(A)}Z(A')} \longrightarrow X_A \times_{\operatorname{Spec} Z(A)} \operatorname{Spec} Z(A').$$

Put $\underline{X}_n^{/K}(c_A^{Z(A')})$ to be the morphism of schemes being induced from $\xi_A^{Z(A')/Z(A)}$ by projection to the first factor.
In order to construct $\underline{X}_n^{/K}(i)$ as a functor on the category $\underline{\mathrm{Az}}_n^{/K}$ it remains to describe it as a functor on the full subcategories $\underline{\mathrm{Az}}_n^{K_1}$ where K_1/K is any field extension. That means we are left with the case that $Z(A) = Z(A')$ and $i|_{Z(A)}$ is the identity. In order to make sure the functoriality of $\underline{X}_n^{/K}$ on the category $\underline{\mathrm{Az}}_n^{/K}$, that construction has to be done in a way compatible with field extensions, i.e. such that the diagram

$$\begin{tikzcd}
X_{A'\otimes_{K_1}K_2} \arrow[r, "\underline{X}_n^{/K}(i\otimes_{K_1}K_2)"] \arrow[d, "\underline{X}_n^{/K}(c_{A'}^{K_2})"'] & X_{A\otimes_{K_1}K_2} \arrow[d, "\underline{X}_n^{/K}(c_A^{K_2})"] \\
X_{A'} \arrow[r, "\underline{X}_n^{/K}(i)"] & X_A
\end{tikzcd}$$

commutes for every field K_1 containing K, every morphism $i\colon A \to A'$ in $\underline{\mathrm{Az}}_n^{K_1}$ and every extension K_2/K_1 of fields containing K.
For that choose a finite Galois extension L/K_1 such that L is a splitting field for A. As i is automatically an isomorphism, L is a splitting field for A', too. Thus, we obtain a Galois invariant homomorphism $i\otimes_{K_1}L\colon A\otimes_{K_1}L \to A'\otimes_{K_1}L$. Applying the functor Ξ_n^{L/K_1} one gets a Galois invariant morphism of schemes

$$\begin{aligned}
\Xi_n^{L/K_1}(i\otimes_L M)\colon \qquad & X_{A'} \times_{\operatorname{Spec} K_1} \operatorname{Spec} L = \Xi_n^{L/K_1}(A'\otimes_{K_1}L) \\
\longrightarrow \;\; & \Xi_n^{L/K_1}(A\otimes_{K_1}L) \qquad = X_A \times_{\operatorname{Spec} K_1} \operatorname{Spec} L.
\end{aligned}$$

Galois descent for morphisms of schemes yields the morphism $\underline{X}_n^{/K}(i)\colon X_{A'} \to X_A$ desired. This is even an isomorphism of schemes as i is an isomorphism. Consequently, $\underline{X}_n^{/K}$ is a functor between the categories stated.
By construction, $\underline{X}_n^{/K}$ is essentially surjective. It remains to show it is full and faithful.
We note explicitly that $\underline{X}_n^{/K}$ induces a functor $\underline{X}_n^{K'}\colon \underline{\mathrm{Az}}_n^{K'} \longrightarrow (\underline{\mathrm{BS}}_{n-1}^{K'})^{\mathrm{op}}$ for each field extension K'/K. $\underline{X}_n^{K'}$ is automatically an equivalence of categories as soon as $\underline{X}_n^{/K}$ is. By its definition on objects the functor $\underline{X}_n^{K'}$ induces the bijection on isomorphy classes found above.

2nd step. Full faithfulness on automorphisms.
Let us first deal with the following statement which is a special case of full faithfulness of $\underline{X}_n^{/K}$.

(*) Let K' be a field containing K and A be a central simple algebra of dimension n^2 over K'. Then the functor $\underline{X}_n^{K'}$ induces an isomorphism of groups $x_A := \underline{X}_n^{K'}|_{\mathrm{Aut}_{K'}(A)}\colon \mathrm{Aut}_{K'}(A) \longrightarrow \mathrm{Aut}_{K'\text{-schemes}}(X_A)$.

This was proven in the case $A = \mathrm{M}_n(K')$ in Proposition 6.14. Let A be a general central simple algebra of dimension n^2 over K'. Let L/K' be a finite Galois extension of fields such that L is a splitting field for A.

Assume $a \neq a' \in \operatorname{Aut}_{K'}(A)$ induce one and the same morphism

$$a^* = a'^* \in \operatorname{Aut}_{K'-\text{schemes}}(X_A).$$

By the universal property of the tensor product there are uniquely determined homomorphisms $a_L, a'_L\colon A \otimes_K L \to A \otimes_K L$ of central simple algebras over L making the diagrams

$$\begin{tikzcd}
A \otimes_K L \arrow[r, "a_L"] & A \otimes_K L & & A \otimes_K L \arrow[r, "a'_L"] & A \otimes_K L \\
A \arrow[r, "a"] \arrow[u, "c_A^L"] & A, \arrow[u, "c_A^L"'] & & A \arrow[r, "a'"] \arrow[u, "c_A^L"] & A \arrow[u, "c_A^L"']
\end{tikzcd}$$

commute. As c_A^L is an injection, $a_L \neq a'_L$. When one applies the functor $\underline{X}_n^{K'}$ to the whole situation one obtains the commutative diagrams below:

$$\begin{tikzcd}
X_{A \otimes_K L} \arrow[d, "(c_A^L)^*"'] & X_{A \otimes_K L} \arrow[l, "a_L^*"'] \arrow[d, "(c_A^L)^*"] & & X_{A \otimes_K L} \arrow[d, "(c_A^L)^*"'] & X_{A \otimes_K L} \arrow[l, "a_L'^*"'] \arrow[d, "(c_A^L)^*"] \\
X_A & X_A, \arrow[l, "a^* = a'^*"'] & & X_A & X_A. \arrow[l, "a^* = a'^*"']
\end{tikzcd}$$

As $A \otimes_K L \cong \mathrm{M}_n(L)$, we have $a_L^* \neq a_L'^*$. On the other hand, $(c_A^L)^*$ is, up to isomorphism, the canonical morphism $X_A \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X_A$ from the fiber product to the first factor. Therefore, the morphism $a^* \circ (c_A^L)^* = a'^* \circ (c_A^L)^*$ admits a unique factorization as a morphism of L-schemes composed with $(c_A^L)^*$. This implies $a_L^* = a_L'^*$ being a contradiction. Consequently, the homomorphism x_A is injective.

Let $p\colon X_A \to X_A$ be an automorphism of K'-schemes. As $(c_A^L)^*$ is, up to isomorphism, the canonical morphism $X_A \times_{\operatorname{Spec} K} \operatorname{Spec} L \to X_A$ from the fiber product to the first factor, the composition $p \circ (c_A^L)^*$ factors uniquely via $(c_A^L)^*$, i.e. there exists a unique morphism $p_L\colon X_{A \otimes_K L} \to X_{A \otimes_K L}$ of L-schemes such that there is a commutative diagram

$$\begin{tikzcd}
X_{A \otimes_K L} \arrow[d, "(c_A^L)^*"'] & X_{A \otimes_K L} \arrow[l, "p_L"'] \arrow[d, "(c_A^L)^*"] \\
X_A & X_A. \arrow[l, "p"']
\end{tikzcd}$$

As $A \otimes_K L \cong \mathrm{M}_n(L)$, by Proposition 6.14, there exists some $b \in \operatorname{Aut}(A \otimes_K L)$ such that $p_L = b^*$. For $\sigma \in \operatorname{Gal}(L/K')$ let $\mathrm{id} \times \sigma\colon A \otimes_K L \to A \otimes_K L$ the corresponding automorphism of $A \otimes_K L$. Clearly, $(\mathrm{id} \times \sigma) \circ c_A^L = c_A^L$. Therefore, the diagram

$$\begin{tikzcd}
X_{A \otimes_K L} \arrow[d, "(\mathrm{id}\times\sigma)^*"'] & X_{A \otimes_K L} \arrow[l, "b^* = p_L"'] \arrow[d, "(\mathrm{id}\times\sigma)^*"] \\
X_{A \otimes_K L} \arrow[d, "(c_A^L)^*"'] & X_{A \otimes_K L} \arrow[d, "(c_A^L)^*"] \\
X_A & X_A. \arrow[l, "p"']
\end{tikzcd}$$

commutes, as well. Hence, $b^* = (\mathrm{id}\times\sigma)^* \circ b^* \circ (\mathrm{id}\times\sigma^{-1})^* = ((\mathrm{id}\times\sigma^{-1}) \circ b \circ (\mathrm{id}\times\sigma))^*$. Using the injectivity of the homomorphism (*) one gets $b = (\mathrm{id}\times\sigma^{-1}) \circ b \circ (\mathrm{id}\times\sigma)$, i.e. b is invariant with respect to the operation of $\mathrm{Gal}(L/K')$. By Galois descent for homomorphisms, there exists a homomorphism $a\colon A \to A$ of central simple algebras such that the diagram

$$\begin{tikzcd}
A\otimes_K L \arrow[r, "b"] & A\otimes_K L \\
A \arrow[u, "c_A^L"] \arrow[r, "a"] & A \arrow[u, "c_A^L"']
\end{tikzcd}$$

commutes. Consequently, there is a commutative diagram on the level of Brauer-Severi varieties as follows:

$$\begin{tikzcd}
X_{A\otimes_K L} \arrow[d, "(c_A^L)^*"'] & X_{A\otimes_K L} \arrow[l, "b^*=p_L"'] \arrow[d, "(c_A^L)^*"] \\
X_A & X_A . \arrow[l, "a^*"']
\end{tikzcd}$$

A direct comparison with the original definition of p_L shows $p\circ(c_A^L)^* = a^*\circ(c_A^L)^*$ as both these compositions are equal to $(c_A^L)^* \circ p_L$. Since $(c_A^L)^*$ is dominant this implies that $p = a^*$. The homomorphism x_A is surjective, too.

3rd step. Isomorphisms.
We note the following consequence of the results of the last step.

(**) Let K' be a field containing K and A and A' be central simple algebras of dimension n^2 over K' being isomorphic to each other. Then $\underline{X}_n^{K'}$ induces a bijection $\mathrm{Iso}_{K'}(A, A') \longrightarrow \mathrm{Iso}_{K'-\text{schemes}}(X_{A'}, X_A)$.

4th step. On faithfulness.
Assume the homomorphisms $i_1, i_2\colon A \to A'$ induce one and the same morphism of schemes $i_1^* = i_2^*\colon X_{A'} \to X_A$. Our first observation is that, necessarily, i_1 and i_2 give rise to the same homomorphism $i_1|_{Z(A)} = i_2|_{Z(A)}\colon Z(A) \to Z(A')$ on the centers. Indeed, X_A and $X_{A'}$ are Brauer-Severi varieties over $Z(A)$ and $Z(A')$, respectively. Further, by Proposition 4.2.iv), $\Gamma(X_A, \mathscr{O}_{X_A}) = Z(A)$ and $\Gamma(X_{A'}, \mathscr{O}_{X_{A'}}) = Z(A')$, i.e. one can recover the centers of A and A' from the Brauer-Severi varieties being associated with them. By the construction of $\underline{X}_n^{/K}$ on morphisms, the pull-back on the level of global sections

$$(i_1^*)^\sharp = (i_2^*)^\sharp\colon Z(A) = \Gamma(X_A, \mathscr{O}_{X_A}) \longrightarrow \Gamma(X_{A'}, \mathscr{O}_{X_{A'}}) = Z(A')$$

is equal to the homomorphism $Z(A) \to Z(A')$ given by i_1, respectively i_2, by restriction to the centers.

Consequently, i_1 and i_2 can both be factorized via the canonical homomorphism $c_A^{Z(A')}\colon A \to A\otimes_{Z(A)} Z(A')$. Let $j_1, j_2\colon A\otimes_{Z(A)} Z(A') \to A'$ be homomorphisms of central simple algebras over $Z(A')$ such that $j_1\circ c_A^{Z(A')} = i_1$ and $j_2\circ c_A^{Z(A')} = i_2$. We note that j_1 and j_2 are both isomorphisms as they are homomorphisms of central simple algebras over the same base field. Furthermore, $j_1 \neq j_2$. On the other hand, the morphisms

$$j_1^*, j_2^*\colon X_{A'} \longrightarrow X_{A\otimes_{Z(A)} Z(A')} = X_A \times_{\mathrm{Spec}\, Z(A)} \mathrm{Spec}\, Z(A')$$

coincide as their projections to $\operatorname{Spec} Z(A')$ do, both being the structural morphism, and their projections to X_A, being equal to $i_1^* = i_2^*$, coincide by assumption. This is a contradiction.

5th step. On fullness.
Let A, $A' \in \operatorname{Ob}(\underline{\mathrm{Az}}_n^{/K})$ and $f\colon X_{A'} \to X_A$ be a morphism in the category $\underline{\mathrm{BS}}_{n-1}^{/K}$. Considering the associated map of global sections we obtain a homomorphism $Z(A) \to Z(A')$ of fields. The canonical inclusion $c_A^{Z(A')}\colon A \to A \otimes_{Z(A)} Z(A')$ induces the canonical morphism

$$(c_A^{Z(A')})^*\colon X_{A\otimes_{Z(A)} Z(A')} = X_A \times_{\operatorname{Spec} Z(A)} \operatorname{Spec} Z(A') \longrightarrow X_A.$$

Remembering the definition of what is a morphism in the category $\underline{\mathrm{BS}}_{n-1}^{/K}$ we see that f gives rise to an isomorphism

$$\overline{f}\colon X_{A'} \longrightarrow X_A \times_{\operatorname{Spec} Z(A)} \operatorname{Spec} Z(A') = X_{A\otimes_{Z(A)} Z(A')}$$

such that $f = (c_A^{Z(A')})^* \circ \overline{f}$. As we adjusted the structural morphism in the right way $\overline{f}$ is even an isomorphism of $Z(A')$-schemes. By Corollary 5.3 the central simple algebras A' and $A \otimes_{Z(A)} Z(A')$ over $Z(A')$ are isomorphic. Therefore, by the result of the 3rd step, there is some homomorphism

$$a\colon A \otimes_{Z(A)} Z(A') \longrightarrow A'$$

such that $\overline{f} = a^*$. Consequently, $f = (c_A^{Z(A')})^* \circ a^* = (a \circ c_A^{Z(A')})^*$ is in the image of $\underline{X}_n^{/K}$ on morphisms. $\underline{X}_n^{/K}$ is full. $\square$

Corollary 6.20. — *Let K be a field and n be a natural number. Then $\underline{\mathrm{Az}}_n^K$ and $\underline{\mathrm{BS}}_{n-1}^K$ are also equivalent to each other, i.e. there is an equivalence of categories*

$$\underline{\check{X}}_n^K\colon \underline{\mathrm{Az}}_n^K \longrightarrow \underline{\mathrm{BS}}_{n-1}^K.$$

Proof. Compose $\underline{X}_n^K$ with the equivalence of categories $\iota\colon \underline{\mathrm{BS}}_{n-1}^K \to (\underline{\mathrm{BS}}_{n-1}^K)^{\mathrm{op}}$ given by the identity on objects and by $\iota(g) := g^{-1}$ on morphisms. $\square$

7. The functor of points

Remark 7.1. — This section deals with the contravariant functor on the category of all K-schemes defined by the Brauer-Severi variety X_A associated with a central simple algebra A. It turns out that, as for projective spaces, Graßmannians and flag varieties, this functor can be described completely explicitly. Thus, it is clear that there is a different method to introduce X_A. One can start with the functor and has to prove its representability by a scheme. It seems, that method is closer to A. Grothendieck's style in Algebraic Geometry than the approach presented here. But, as one might expect, a direct proof of representability, avoiding all descent arguments, is not trivial at all. It is presented in detail in [**Hen**] or [**Ke**].

Definition 7.2. — Let K be a field, n be a natural number and $A \in \operatorname{Ob}(\underline{\mathrm{Az}}_n^K)$. Then by

$$I_A\colon \{K-\text{schemes}\}^{\mathrm{op}} \to \{\text{sets}\}$$

we will denote the functor given on objects by

$$T \mapsto \left\{ \begin{matrix} \text{sheaves of right ideals } \mathscr{J} \text{ in } \pi_T^*\mathscr{A} \text{ such that} \\ \pi_T^*\mathscr{A}/\mathscr{J} \text{ is a locally free } \mathscr{O}_T\text{-module of rank } n^2 - n \end{matrix} \right\}$$

and on morphisms by the pull-back. Here $\mathscr{A} := \widetilde{A}$ denotes the sheaf of $\mathscr{O}_K$-algebras associated with A on $\operatorname{Spec} K$ and $\pi = \pi_T \colon T \to \operatorname{Spec} K$ is the structural morphism. The various functors I_A depend on A in a natural manner, i.e. if $i \colon A \to A'$ is a morphism in $\underline{\mathrm{Az}}_n^K$ then there is a morphism of functors $i^* \colon I_{A'} \to I_A$ given by the inverse image $\mathscr{J} \mapsto \pi_T^*(\widetilde{i})^{-1}(\mathscr{J})$. Thus, there exists a contravariant functor

$$I \colon \underline{\mathrm{Az}}_n^K \longrightarrow \operatorname{Fun}(\{K\text{-schemes}\}^{\mathrm{op}}, \{\text{sets}\})$$

given on objects by $A \mapsto I_A$.

Theorem 7.3. — *Let K be a field and n be a natural number. Then there is an isomorphism*

$$\iota \colon I \longrightarrow P \circ \underline{X}_n^K$$

between the contravariant functors

$$I,\, P \circ \underline{X}_n^K \colon \underline{\mathrm{Az}}_n^K \longrightarrow \operatorname{Fun}(\{K\text{-schemes}\}^{\mathrm{op}}, \{\text{sets}\}).$$

Remark 7.4. — For a fixed central simple algebra A of dimension n^2 over K the statement of Theorem 7.3 says that there is an isomorphism

$$\iota_A \colon I_A \longrightarrow P_{X_A}$$

between the functors $I_A, P_{X_A} \colon \{K - \text{schemes}\}^{\mathrm{op}} \to \{\text{sets}\}$ described above. This means that the T-valued points in X_A are in a natural bijection with the sheaves of right ideals $\mathscr{J} \subset \pi_T^*\mathscr{A}$ such that $\pi_T^*\mathscr{A}/\mathscr{J}$ is a locally free $\mathscr{O}_T$-module of rank $n^2 - n$. Further, if $i \colon A \to A'$ is a morphism of n^2-dimensional central simple algebras over K then the diagram

$$\begin{tikzcd}
I_{A'} \arrow[r, "\iota_{A'}"] \arrow[d, "i^*"'] & P_{X_{A'}} \arrow[d, "{[\underline{X}_n^K(i)]_*}"] \\
I_A \arrow[r, "\iota_A"] & P_{X_A}
\end{tikzcd}$$

commutes.

Remark 7.5. — We note that I_A is canonically a subfunctor of the Graßmann functor $\underline{\mathrm{Grass}}_{n^2-n}^{n^2}$ parametrizing $(n^2 - n)$-dimensional quotients of an n^2-dimensional vector space. Thus, there is an embedding $X_A \hookrightarrow \mathrm{Grass}_{n^2-n}^{n^2}$ into the Graßmann scheme. We note that this is the key observation for a direct proof of representability for I_A.

Definition 7.6. — Let G be any group. By $\underline{\mathrm{Sets}}_G$ we will denote the category of all mappings $\mathfrak{M} \to G$ where $\mathfrak{M}$ is any set. As morphisms between $f_1 \colon \mathfrak{M}_1 \to G$

and $f_2\colon \mathfrak{M}_2 \to G$ we allow all mappings which make the diagram

$$\begin{tikzcd}
\mathfrak{M}_1 \arrow[r] \arrow[d, "f_1"'] & \mathfrak{M}_2 \arrow[d, "f_2"] \\
G \arrow[r, "\cdot g"] & G
\end{tikzcd}$$

commutative for a certain $g \in G$ where $\cdot g\colon G \to G$ denotes the multiplication by g from the right.

***Definition* 7.7.** — Let L/K be a finite Galois extension of fields and denote its Galois group $\mathrm{Gal}(L/K)$ by G.

i) By $\underline{\mathrm{Sch}}_+^{L/K}$ we will denote the full subcategory of $\underline{\mathrm{Sch}}^{L/K}$ consisting of all L-schemes being non-empty.

ii) Let $X \in \mathrm{Ob}\,(\underline{\mathrm{Sch}}^{L/K})$. Then the contravariant Hom-functor

$$\mathrm{Hom}_{\underline{\mathrm{Sch}}^{L/K}}(\,\cdot\,,\, X)\colon \underline{\mathrm{Sch}}_+^{L/K} \longrightarrow \{\text{sets}\}$$

factors canonically via the category $\underline{\mathrm{Sets}}_G$. Indeed, for each morphism $T \to X$ in $\underline{\mathrm{Sch}}_+^{L/K}$ the $\sigma \in G$ being associated with it is uniquely determined. Note for that we need the assumption $T \neq \emptyset$ while $X = \emptyset$ may be allowed as in that case there are no morphisms. The functor just constructed will be denoted by

$$P_X^{L/K}\colon \underline{\mathrm{Sch}}_+^{L/K} \longrightarrow \underline{\mathrm{Sets}}_G$$

and called the *functor of points* of X. There is a covariant functor

$$P^{L/K}\colon \underline{\mathrm{Sch}}^{L/K} \to \mathrm{Fun}\,((\underline{\mathrm{Sch}}_+^{L/K})^{\mathrm{op}},\, \underline{\mathrm{Sets}}_G)$$

given on objects by $X \mapsto P_X^{L/K}$.

***Remark* 7.8.** — The Hom-functor $\mathrm{Hom}_{\underline{\mathrm{Sch}}_+^{L/K}}(\,\cdot\,,\, X)$ remembers all information X as an object of the category $\underline{\mathrm{Sch}}_+^{L/K}$. The functor of points

$$P_X^{L/K}\colon \underline{\mathrm{Sch}}_+^{L/K} \longrightarrow \underline{\mathrm{Sets}}_G$$

carries the additional information coming from the structure of $\underline{\mathrm{Sch}}_+^{L/K}$ as a fibered category.

***Lemma* 7.9.** — *Let L/K be a finite Galois extension of fields and n be a natural number. Denote the Galois group* $\mathrm{Gal}(L/K)$ *by G.*

i) *Then there is an isomorphism of functors $j_n^{L/K}\colon P_n^{L/K} \longrightarrow I_n^{L/K}$ between the composition*

$$P_n^{L/K}\colon \underline{\mathrm{Mat}}_n^{L/K} \xrightarrow{\Xi_n^{L/K}} \underline{P}_{n-1}^{L/K} \xrightarrow{\text{embedding}} \underline{\mathrm{Sch}}_+^{L/K} \xrightarrow{P^{L/K}} \mathrm{Fun}\,((\underline{\mathrm{Sch}}_+^{L/K})^{\mathrm{op}},\, \underline{\mathrm{Sets}}_G)$$

and the functor $I_n^{L/K}$ given by

$$A \mapsto \left(T \mapsto \coprod_{\sigma \in G} \left\{ \begin{array}{l} \text{sheaves of right ideals } \mathscr{J} \text{ in } (S(\sigma)\circ\pi_T)^*\mathscr{A} \text{ such that} \\ (S(\sigma)\circ\pi_T)^*\mathscr{A}/\mathscr{J} \text{ is a locally free } \mathscr{O}_T\text{–module of rank } n^2-n \end{array} \right\}\right)$$

on objects and by the pull-back on morphisms. Here $\mathscr{A} := \widetilde{A}$ *denotes the sheaf of* $\mathscr{O}_L$*-algebras associated with* A *on* $\operatorname{Spec} L$ *and* $\pi_T\colon T \to \operatorname{Spec} L$ *is the structural morphism. The set given on the right hand side is equipped with a map to* G *in the obvious way.*

ii) *Let* L' *be a finite field extension such that* L'/K *is Galois again. Then the isomorphism* $j_n^{L/K}$ *is compatible with the base extension from* L *to* L'*, i.e. for every* $A \in \operatorname{Ob}(\underline{\mathrm{Mat}}_n^{L/K})$ *and every* $T \in \operatorname{Ob}(\underline{\mathrm{Sch}}_+^{L/K})$ *the diagram*

$$\begin{tikzcd}
P_n^{L/K}(A)(T) \arrow[d, equal] & \\
\operatorname{Mor}_{\underline{\mathrm{Sch}}^{L/K}}(T, \Xi_n^{L/K}(A)) \arrow[r, "j_n^{L/K}(A)(T)"] \arrow[d, "\times_{\operatorname{Spec} L}\operatorname{Spec} L'"'] & I_n^{L/K}(A)(T) \arrow[d, "\text{pull-back}"] \\
\operatorname{Mor}_{\underline{\mathrm{Sch}}^{L'/K}}(T \times_{\operatorname{Spec} L} \operatorname{Spec} L', \Xi_n^{L'/K}(A \otimes_L L')) \arrow[r, "j"] \arrow[d, equal] & I_n^{L'/K}(A \otimes_L L')(T \times_{\operatorname{Spec} L} \operatorname{Spec} L') \\
P_n^{L'/L}(A \otimes_L L')(T \times_{\operatorname{Spec} L} \operatorname{Spec} L') &
\end{tikzcd}$$

commutes. Here j *is used as an abbreviation for* $j_n^{L'/K}(A \otimes_L L')(T \times_{\operatorname{Spec} L} \operatorname{Spec} L')$.

Remark 7.10. — The lemma states in particular that the ordinary functor of points of the L-scheme $\Xi_n^{L/K}(A)$ for $A \in \operatorname{Ob}(\underline{\mathrm{Mat}}_n^{L/K})$ is isomorphic to

$$T \mapsto \left\{ \begin{matrix} \text{sheaves of right ideals } \mathscr{J} \text{ in } \pi_T^*\mathscr{A} \text{ such that} \\ \pi_T^*\mathscr{A}/\mathscr{J} \text{ is a locally free } \mathscr{O}_T\text{–module of rank } n^2 - n \end{matrix} \right\},$$

i.e. for every L-algebra R the set $\Xi_n^{L/K}(A)(R)$ of R-valued points in $\Xi_n^{L/K}(A)$ is naturally isomorphic to the set of all right ideals I in $A \otimes_L R$ such that $A \otimes_L R/I$ is a locally free R-module of rank $n^2 - n$.

7.11 (Proof of the lemma). — i) For each $A \in \operatorname{Ob}(\underline{\mathrm{Mat}}_n^{L/K})$ both of the two functors under consideration satisfy the sheaf axiom for Zariski coverings. Therefore, it will suffice to work with affine schemes as test objects.

Let R be a commutative L-algebra with unit. By construction we have $\Xi_n^{L/K}(A) = \mathbf{P}(\mathfrak{l})$ where $\mathfrak{l}$ is a non-zero, simple, left A-module. By Corollary-Definition 6.5 the set of R-valued points in $\mathbf{P}(\mathfrak{l})$ is in a natural bijection with the set of all submodules $M \subset \mathfrak{l} \otimes_L R$ such that the quotient $\mathfrak{l} \otimes_L R/M$ is a locally free R-module of rank $n-1$.

As an R-module $A \otimes_L R$ is isomorphic to the direct sum of n copies of $\mathfrak{l} \otimes_L R$. Under this isomorphism an R-submodule $M \subseteq \mathfrak{l} \otimes_L R$ determines a right ideal $\overline{M} \subseteq A \otimes_L R$ by the definition

$$\overline{M} := \bigoplus_{i=1}^{n} M.$$

This gives a natural bijection between the set of all right ideals in $A \otimes_L R$ and the set of all R-submodules $M \subseteq \mathfrak{l}$. Obviously,

$$A \otimes_L R/\overline{M} \cong (\mathfrak{l} \otimes_L R/M)^n \cong (\mathfrak{l} \otimes_L R/M) \otimes_R R^n$$

as R-modules. So, if $\mathfrak{l} \otimes_L R/M$ is locally free of rank $n-1$ then $A \otimes_L R/\overline{M}$ is locally free of rank $n^2 - n$. Conversely, if $A \otimes_L R/\overline{M}$ is locally free of rank $n^2 - n$ then, by Lemma 2.14, $\mathfrak{l} \otimes_L R/M$ is necessarily locally free. Clearly, it is of rank $n-1$.
We obtained the description of the ordinary functor of points of $\Xi_n^{L/K}(A)$ given in the remark above. Giving a morphism of L-schemes $f\colon T \to \Xi_n^{L/K}(A)$ that is twisted by some $\sigma \in G$ is equivalent to giving a commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_T"'] & T \arrow[r, "f"] \arrow[d, "S(\sigma) \circ \pi_T"'] & \Xi_n^{L/K}(A) \arrow[d, "\pi_{\Xi_n^{L/K}(A)}"] \\
\operatorname{Spec} L \arrow[r, "S(\sigma)"] & \operatorname{Spec} L \arrow[r, "\mathrm{id}"] & \operatorname{Spec} L.
\end{tikzcd}$$

Therefore, f becomes a morphism of L-schemes in the ordinary sense if one simply changes the structural morphism of T from π_T into $S(\sigma) \circ \pi_T$. Consequently, the functor of points $P^{L/K}_{\Xi_R^{L/K}(A)}$ is isomorphic to functor given in the claim.
The construction just made is compatible with K-linear ring homomorphisms $i\colon A \to A'$ by the description of $\Xi_n^{L/K}$ as a covariant functor given above.
ii) Again we need the concrete description of $\Xi_n^{\cdot/K}$. We have $\Xi_n^{L/K}(A) = \mathbf{P}(\mathfrak{l})$ where $\mathfrak{l}$ is a non-zero, simple, left A-module and $\Xi_n^{L'/K}(A \otimes_L L') = \mathbf{P}(\mathfrak{l} \otimes_L L')$. The claim is a direct consequence of Remark 6.6 together with the connection of submodules in $\mathfrak{l} \otimes_L R$ and $\mathfrak{l} \otimes_L R \otimes_L L'$ with right ideals in $A \otimes_K R$ and $A \otimes_L R \otimes_L L'$, respectively, that was established in the proof of part i). $\square$

***7.12* (*Proof of Theorem 7.3*).** — We have to construct an isomorphism $\iota_A\colon I_A \to P_{X_A}$ of functors for every $A \in \mathrm{Ob}\,(\underline{\mathrm{Az}}_n^K)$ in a way being natural in A. We will proceed in three steps.

1st step. The case $A \cong \mathrm{M}_n(K)$.
We have $A \otimes_K R \cong \mathrm{M}_n(R)$ and $X_A = \mathbf{P}(\mathfrak{l})$ by the intrinsic description of X_A given in Proposition 6.16 above. Here $\mathfrak{l}$ denotes a non-zero, simple, left A-module. The assertion is a special case of Lemma 7.9 above. We note explicitly that $\mathbf{P}(\mathfrak{l}) = \Xi_n^{K/K'}(A)$ for every subfield $K' \subset K$ such that K/K' is a finite Galois extension. In particular, the isomorphism $\iota_A\colon I_A \to P_{X_A}$ is compatible with the action of automorphisms of A which are only K'-linear.

2nd step. Reduction to T-valued points for affine schemes T.
Let again A be an arbitrary central simple algebra over K of dimension n^2. Then, the functors I_A and P_{X_A} satisfy the sheaf axioms for Zariski coverings of T. Hence, it is sufficient to consider the affine case: There is a natural isomorphism

$\iota_A\colon I_A \to P_{X_A}$ of functors on the full subcategory of affine schemes to be constructed, i.e. an isomorphism

$$\begin{array}{c} I_A(R) := \left\{\begin{array}{l} \text{right ideals } J \text{ in } A\otimes_K R \text{ such that } A\otimes_K R/J \\ \text{is a locally free } R\text{-module of rank } n^2 - n \end{array}\right\} \\ \downarrow \iota_A(R) \\ P_{X_A}(R) := \mathrm{Mor}_{K\text{-schemes}}(\operatorname{Spec} R,\, X_A) \end{array}$$

for each commutative ring R with unit such that for homomorphisms $r\colon R \to R'$ of K-algebras the corresponding diagram

$$\begin{tikzcd} I_A(R) \arrow[r, "I_A(r)"] \arrow[d, "\iota_A(R)"'] & I_A(R') \arrow[d, "\iota_A(R')"] \\ P_{X_A}(R) \arrow[r, "P_{X_A}(r)"] & P_{X_A}(R') \end{tikzcd}$$

commutes.

3rd step. Galois descent.
Let L/K be some finite Galois extension such that L is a splitting field for A. Put $G := \mathrm{Gal}\,(L/K)$. By Theorem 2.2 in the version for schemes, there is a bijection

$$\begin{array}{rl} P_{X_A}(R) = & \left\{\begin{array}{l} p\colon \operatorname{Spec} R \to X_A \mid \\ p \text{ morphism of } K\text{-schemes} \end{array}\right\} \\ \cong & \left\{\begin{array}{l} \overline{p}\colon \operatorname{Spec} R\otimes_K L \to X_{A\otimes_K L} \mid \\ \overline{p} \text{ morphism of } L\text{-schemes}, \\ \overline{p} \text{ compatible with the } G\text{-operations on both sides} \end{array}\right\} \end{array}$$

being natural in the ring R. As $A\otimes_K L$ is isomorphic to the matrix algebra, we have

$$X_{A\otimes_K L} = \Xi_n^{L/K}(A\otimes_K L) = \mathbf{P}(\mathfrak{l})$$

where $\mathfrak{l}$ is a non-zero, simple, right $A\otimes_K L$-module. Hence, there is a second natural bijection

$$\begin{array}{rl} & \left\{\begin{array}{l} \overline{p}\colon \operatorname{Spec} R\otimes_K L \to X_{A\otimes_K L} \mid \\ \overline{p} \text{ morphism of } L\text{-schemes}, \\ \overline{p} \text{ compatible with the } G\text{-operations on both sides} \end{array}\right\} \\ \cong & \left\{\begin{array}{l} \overline{I} \subset (A\otimes_K L)\otimes_L (R\otimes_K L) \mid \\ \overline{I} \text{ right ideal}, \\ (A\otimes_K L)\otimes_L (R\otimes_K L)/\overline{I} \text{ locally free } R\otimes_K L\text{-module}, \\ \mathrm{rk}_{R\otimes_K L}\, (A\otimes_K L)\otimes_L (R\otimes_K L)/\overline{I} = n^2 - n, \\ \overline{I} \text{ invariant with respect to the } G\text{-operation} \end{array}\right\}. \end{array}$$

Indeed, this is exactly the result of the first step when we note that the isomorphism of functors $P_{X_{A \otimes_K L}} \longleftarrow I_{A \otimes_K L}$ is compatible with K-linear ring automorphisms of $A \otimes_K L$. Finally, there is a natural bijection

$$\left\{\begin{array}{l} \overline{I} \subset (A \otimes_K L) \otimes_L (R \otimes_K L) \,| \\ \overline{I} \text{ right ideal,} \\ (A \otimes_K L) \otimes_L (R \otimes_K L)/\overline{I} \text{ locally free } R \otimes_K L\text{-module,} \\ \mathrm{rk}_{R \otimes_K L}\,(A \otimes_K L) \otimes_L (R \otimes_K L)/\overline{I} = n^2 - n, \\ \overline{I} \text{ invariant with respect to the } G\text{-operation} \end{array}\right\}$$
$$\cong \left\{\begin{array}{l} I \subset A \otimes_K R \,| \\ I \text{ right ideal,} \\ (A \otimes_K R)/I \text{ locally free } R\text{-module,} \\ \mathrm{rk}_R\,(A \otimes_K R)/I = n^2 - n \end{array}\right\}$$

by Galois descent for right ideals, i.e. by Lemma 7.14. Indeed, everything would be clear if there would be no assumption on the ranks of the quotients. But if $A \otimes_K R/I$ is a locally free R-module of rank $n^2 - n$ then

$$(A \otimes_K L) \otimes_L (R \otimes_K L)/\overline{I} \cong (A \otimes_K R/I) \otimes_R (R \otimes_K L)$$

is a locally free $R \otimes_K L$-module of the same rank. On the other hand, if $(A \otimes_K L) \otimes_L (R \otimes_K L)/\overline{I} \cong (A \otimes_K R/I) \otimes_R (R \otimes_K L)$ is a locally free $(R \otimes_K L)$-module of rank $n^2 - n$ then it is a locally free R-module, as well. By Lemma 2.14 $A \otimes_K R/I$ is a locally free R-module, too. Clearly, it is of rank $n^2 - n$.

We note finally that the construction of ι_A given above can easily be extended to morphisms and gives a functor in A. For that we first choose a splitting field $L_A \supset K$ for each $A \in \mathrm{Ob}\,(\underline{\mathrm{Az}}_n^K)$ in such a way that L_A depends only on the isomorphy class of A in $\underline{\mathrm{Az}}_n^K$. If $i \colon A \to A'$ is a morphism in $\underline{\mathrm{Az}}_n^K$ then A and A' are automatically isomorphic and we execute the construction of ι_A and $\iota_{A'}$ via the splitting field chosen. The first and the third natural bijection constructed above are applications of descent and, therefore, compatible with the morphisms induced by i. For the second one that was proven in Lemma 7.9.i) above. The proof is complete. □

Remark 7.13. — The isomorphism of functors ι_A is independent of the choice of the splitting field made in the proof. Indeed, let $L'_A \supset L_A$ be splitting fields for A. Going through the proof given above, one sees that the two constructions for ι_A yield the same result. The main ingredient for that is Lemma 7.9.ii).

***Lemma 7.14* (*Galois descent for right ideals*).** — *Let L/K be a finite Galois extension of fields and $G := \mathrm{Gal}(L/K)$ be its Galois group. Further, let A be a K-algebra and $\overline{I} \subset A \otimes_K L$ a right ideal being invariant under the canonical operation of G on $A \otimes_K L$. Then there is a unique right ideal $I \subset A$ such that $\overline{I} = I \otimes_K L$.*

Proof. $\overline{I}$ inherits from A a structure of an L-algebra and an operation of the group G by homomorphisms of K-algebras where $\sigma \in G$ acts σ-linearly. By the algebraic version of Galois descent, there exists some K-algebra I such that $\overline{I} = I \otimes_K L$. Clearly, the canonical homomorphism of L-algebras $\overline{I} = I \otimes_K L \hookrightarrow A \otimes_K L$ is compatible with the G-operations on both sides. Therefore, by Galois descent for homomorphisms, we get a homomorphism of K-algebras $I \to A$ that induces the ideal $\overline{I}$. As the functor $\otimes_K L$ is exact and faithful, that morphism is necessarily injective. Consider $I \subset A$ as a subring. The multiplication $\cdot a\colon \overline{I} \to \overline{I}$ with some element $a \in A$ from the right is compatible with the operation of G. Hence, it descends to a homomorphism $I \to I$. That homomorphism is compatible with the multiplication by a from the right on A. Consequently, I is a right ideal. Uniqueness is clear. □

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