DEL PEZZO SURFACES OF DEGREE FOUR VIOLATING THE HASSE PRINCIPLE ARE ZARISKI DENSE IN THE MODULI SCHEME

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ABSTRACT. We show that, over every number field, the degree four del Pezzo surfaces that violate the Hasse principle are Zariski dense in the moduli scheme.

RÉSUMÉ. Nous montrons que, sur chaque corps de nombres, les surfaces de del Pezzo de degré quatre qui violent le principe de Hasse sont denses pour la topologie de Zariski dans le schéma de modules.

1. Introduction

A del Pezzo surface is a smooth, proper algebraic surface S over a field K with an ample anticanonical sheaf $\mathscr{K}^{-1}$. Over an algebraically closed field, every del Pezzo surface of degree $d \leqslant 7$ is isomorphic to $\mathbf{P}^2$, blown up in $(9-d)$ points in general position [14, Theorem 24.4.iii)].

According to the adjunction formula, a smooth complete intersection of two quadrics in $\mathbf{P}^4$ is del Pezzo. The converse is true, as well. For every del Pezzo surface of degree four, its anticanonical image is the complete intersection of two quadrics in $\mathbf{P}^4$ [8, Theorem 8.6.2].

Although del Pezzo surfaces over number fields are generally expected to have many rational points, they do not always fulfil the Hasse principle. The first example of a degree four del Pezzo surface for which the Hasse principle is violated was conceived by B. Birch and Sir Peter Swinnerton-Dyer [3, Theorem 3]. It is given in $\mathbf{P}^4_{\mathbb{Q}}$ by the equations

$$\begin{aligned} T_0T_1 &= T_2^2 - 5T_3^2\,, \\ (T_0+T_1)(T_0+2T_1) &= T_2^2 - 5T_4^2\,. \end{aligned}$$

Meanwhile, more counterexamples to the Hasse principle have been constructed, see, e.g., [5, Examples 15 and 16]. Quite recently, N. D. Q. Nguyen [18, Theorem 1.1] proved that the degree four del Pezzo surface, given by

$$\begin{aligned} T_0T_1 &= T_2^2 - (64k^2+40k+5)T_3^2\,, \\ (T_0+(8k+1)T_1)(T_0+(8k+2)T_1) &= T_2^2 - (64k^2+40k+5)T_4^2 \end{aligned}$$

is a counterexample to the Hasse principle if k is an integer such that $64k^2+40k+5$ is a prime number. In particular, under the assumption of Schinzel's hypothesis, this family contains infinitely many members violating the Hasse principle.

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In this article, we prove that del Pezzo surfaces of degree four that fail the Hasse principle are Zariski dense in the moduli scheme. In particular, we establish, for the first time unconditionally, that their number up to isomorphism is infinite. We show, in addition, that these results hold over an arbitrary number field K.

Before we can state our main results, we need to recall some notation and facts about the coarse moduli scheme of degree four del Pezzo surfaces.

For this we consider a del Pezzo surface X of degree four given as the zero set of two quinary quadrics

$$Q_1(T_0, \ldots, T_4) = Q_2(T_0, \ldots, T_4) = 0\,.$$

The pencil $(uQ_1 + vQ_2)_{(u:v)\in\mathbf{P}^1}$ of quadrics defined by the forms Q_1 and Q_2 contains exactly five degenerate elements. The corresponding five values $t_1, \ldots, t_5 \in \mathbf{P}^1(\overline{K})$ of $t := (u : v)$ are uniquely determined by the surface X, up to permutation and the natural operation of $\mathrm{Aut}(\mathbf{P}^1) \cong \mathrm{PGL}_2(\overline{K})$.

Let $\mathscr{U} \subset (\mathbf{P}^1_K)^5$ be the Zariski open subset given by the condition that no two of the five components coincide. Then there is a K-isomorphism

$$j\colon \mathscr{U}/(S_5 \times \mathrm{PGL}_2) \xrightarrow{\cong} \mathscr{M}$$

to the coarse moduli scheme $\mathscr{M}$ of degree four del Pezzo surfaces [12, Section 5].

The quotient of $\mathscr{U}$ modulo S_5 alone is the space of all binary quintics without multiple roots, up to multiplication by constants. This is part of the stable locus in the sense of Geometric Invariant Theory, which is formed by all quintics without roots of multiplicity $\geqslant 3$ [16, Proposition 4.1].

Furthermore, classical invariant theory teaches that, for binary quintics, there are three fundamental invariants I_4, I_8, and I_{12} of degrees 4, 8, and 12, respectively, that define an open embedding

$$\iota\colon \mathscr{U}/(S_5 \times \mathrm{PGL}_2) \hookrightarrow \mathbf{P}(1, 2, 3)_K$$

into a weighted projective plane. This result is originally due to Ch. Hermite [13, Section VI], cf. [21, Paragraphs 224–228]. A more recent treatment from a computational point of view is due to A. Abdesselam [1].

Altogether, this yields an open embedding $I\colon \mathscr{M} \hookrightarrow \mathbf{P}(1, 2, 3)_K$. More generally, every family $\pi\colon \mathscr{S} \to B$ of degree four del Pezzo surfaces over a K-scheme B induces a morphism

$$I_\pi = I\colon B \to \mathbf{P}(1, 2, 3)_K\,,$$

which we call the *invariant map* associated with π.

Remark 1.1. — There cannot be a fine moduli scheme for degree four del Pezzo surfaces, as geometrically every such surface X has at least 16 automorphisms [8, Theorem 8.6.8]. (The statement of Theorem 8.6.8 in [8] contains a misprint, but it is clear from the proof that the described quotient group may either be isomorphic to one of the listed groups or be trivial).

Let us, for simplicity of notation, identify the space $S^2((K^5)^*)$ of all quinary quadratic forms with coefficients in K with K^{15}. This is clearly a non-canonical isomorphism. To give an intersection of two quadrics in $\mathbf{P}^4_K$ is then equivalent to giving a K-rational plane through the origin of K^{15}, i.e. a K-rational point on the Graßmann scheme $\mathrm{Gr}(2, 15)_K$. The open subset $U_\mathrm{reg} \subset \mathrm{Gr}(2, 15)_K$ that parametrises non-singular surfaces is isomorphic to the Hilbert scheme [10] of del Pezzo surfaces of degree four in $\mathbf{P}^4_K$. We will not go into the details as they are not necessary for our purposes. Using this identification, we can now state our main result in the following form.

THEOREM 1.1. — *Let K be a number field, $U_{\mathrm{reg}} \subset \mathrm{Gr}(2,15)_K$ the open subset of the Graßmann scheme that parametrises degree four del Pezzo surfaces, and $\mathscr{HC}_K \subset U_{\mathrm{reg}}(K)$ be the set of all degree four del Pezzo surfaces over K that are counterexamples to the Hasse principle.*
Then $\mathscr{HC}_K$ is Zariski dense in $\mathrm{Gr}(2,15)_K$.

Remark 1.2. — An analogous result for cubic surfaces has recently been established by A.-S. Elsenhans together with the first author [9]. Our approach is partly inspired by the methods applied in the cubic surface case. The concrete construction of del Pezzo surfaces of degree four that violate the Hasse principle is motivated by the work [18] of N. D. Q. Nguyen. In particular, all the failures of the Hasse principle we consider below are due to the Brauer-Manin obstruction.

The following result could be seen as a corollary of Theorem 1.1, but it is, in fact, more or less equivalent. Our strategy will be to prove Theorem 1.2 first and then to deduce Theorem 1.1 from it.

THEOREM 1.2. — *Let K be a number field, $U_{\mathrm{reg}} \subset \mathrm{Gr}(2,15)_K$ the open subset of the Graßmann scheme that parametrises degree four del Pezzo surfaces, and $\mathscr{HC}_K \subset U_{\mathrm{reg}}(K)$ be the set of all degree four del Pezzo surfaces over K that are counterexamples to the Hasse principle.*
Then the image of $\mathscr{HC}_K$ under the invariant map

$$I\colon U_{\mathrm{reg}} \longrightarrow \mathbf{P}(1,2,3)_K$$

is Zariski dense.

Remark 1.3 (Particular $K3$ surfaces that fail the Hasse principle). — In their article [23], A. Várilly-Alvarado and B. Viray provide, among other things, families of $K3$ surfaces of degree eight that violate the Hasse principle. These $K3$ surfaces allow a morphism $p\colon Y \to X$ being generically 2:1 down to a degree four del Pezzo surface X that fails the Hasse principle. Since $X(K) = \emptyset$, the existence of the morphism alone ensures that $Y(K) = \emptyset$. Examples of the same kind also appear in [18].

The construction of these families easily generalises to our setting. One has to intersect the cone $CX \subset \mathbf{P}^5$ over the del Pezzo surface with a quadric that avoids the vertex. The intersection Y is then a degree eight $K3$ surface, provided it is smooth, which it is generically according to Bertini's theorem. Thus, Y is a counterexample to the Hasse principle provided it has an adelic point.

For Y, the failure of the Hasse principle may be explained by the Brauer-Manin obstruction (cf. Section 3 for details). If $\alpha \in \mathrm{Br}(X)$ explains the failure for X then $p^*\alpha$ does so for Y.

However, the $K3$ surfaces obtained in this way do clearly not dominate the moduli space of degree eight $K3$ surfaces. Indeed, the pull-back homomorphism $p^*\colon \mathrm{Pic}(X_{\overline{K}}) \to \mathrm{Pic}(Y_{\overline{K}})$ doubles the intersection numbers and is, in particular, injective. This means that Y has geometric Picard rank at least six, while a general degree eight $K3$ surface is of geometric Picard rank one.

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2. A family of degree four del Pezzo surfaces

We consider the surface $S := S^{(D;A,B)}$ over a field K, given by the equations

$$T_0T_1 = T_2^2 - DT_3^2\,, \tag{2.1}$$

$$(T_0 + AT_1)(T_0 + BT_1) = T_2^2 - DT_4^2 \tag{2.2}$$

for $A, B, D \in K$. We will typically assume that D is not a square in K and that S is non-singular. If S is non-singular then S is a del Pezzo surface of degree four.

PROPOSITION 2.1. — *Let K be a field of characteristic $\neq 2$ and $A, B, D \in K$.*

a) *Then the surface $S^{(D;A,B)}$ is non-singular if and only if $ABD \neq 0$, $A \neq B$, and $A^2 - 2AB + B^2 - 2A - 2B + 1 \neq 0$.*

b) *If $D \neq 0$ then $S^{(D;A,B)}$ is not a cone and has at worst isolated singularities.*

Proof. a) The surface $S^{(D;A,B)}$ is defined by the two quadrics Q_1 and Q_2 that are given by the symmetric matrices

$$M_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & \frac{A+B}{2} & 0 & 0 & 0 \\ \frac{A+B}{2} & AB & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D \end{pmatrix},$$

respectively. Therefore,

$$\begin{aligned} \det(uM_1 + vM_2) &= -D^2[ABv^2 - \tfrac{(Av+Bv+u)^2}{4}](u+v)uv \\ &= \tfrac{1}{4}D^2[u^2 + 2(A+B)uv + (A^2 - 2AB + B^2)v^2](u+v)uv\,. \end{aligned}$$

It is well-known [20, Proposition 2.1] that S is non-singular if and only if $\det(uM_1 + vM_2)$ has five distinct roots in $\mathbf{P}^1(\overline{K})$.

In particular, S is clearly singular for $D = 0$. Otherwise, the roots are $u/v = -(A+B) \pm 2\sqrt{AB}$, 0, -1, and ∞. The first two coincide exactly when $AB = 0$. It therefore remains to investigate the cases that $-(A+B) \pm 2\sqrt{AB} = 0$ or $-(A+B) \pm 2\sqrt{AB} = -1$.

Clearly, the first equality is equivalent to $\pm 2\sqrt{AB} = (A+B)$, hence to $4AB = (A+B)^2$, and $A = B$. On the other hand, $-(A+B) \pm 2\sqrt{AB} = -1$ means nothing but $\pm 2\sqrt{AB} = A+B-1$, hence $4AB = A^2 + 2AB + B^2 + 1 - 2A - 2B$, and $A^2 - 2AB + B^2 - 2A - 2B + 1 = 0$.

b) First of all, the binary quintic form $\det(uM_1 + vM_2)$ does not entirely vanish. Therefore, the pencil of quadrics defining S contains one of full rank, which is enough to show that S is not a cone.

On the other hand, a point $(t_0 : \ldots : t_4) \in S(K)$ is singular if and only if the Jacobian matrix

$$\begin{pmatrix} t_1 & t_0 & -2t_2 & 2Dt_3 & 0 \\ 2t_0 + (A+B)t_1 & (A+B)t_0 + 2ABt_1 & -2t_2 & 0 & 2Dt_4 \end{pmatrix}$$

is not of full rank. In particular, this means that $t_0^2 = ABt_1^2$ and that at least two of the coordinates t_2, t_3, and t_4 must vanish. Together these conditions define six lines in $\mathbf{P}^4$, which collapse to three in the case that $AB = 0$.

If there were infinitely many singular points then at least one of these lines would be entirely contained in S. But this is not the case, as, on each of the six lines, one equation of the form

$$F(T_1) = T_2^2, \quad F(T_1) = -DT_3^2, \quad \text{or} \quad F(T_1) = -DT_4^2$$

remains from the equations of S. $\square$

Remark 2.1. — Assume that $D \in K$ is a non-square and that $S^{(D;A,B)}$ is non-singular. Then there is neither a K-rational point $(t_0 : t_1 : t_2 : t_3 : t_4) \in S(K)$ such that $t_0 = t_1 = 0$, nor one such that $t_0 + At_1 = t_0 + Bt_1 = 0$. Indeed, in view of $A \neq B$ either condition implies that $t_0 = t_1 = 0$, so $t_2^2 = Dt_3^2 = Dt_4^2$. Since D is a non-square, there is no K-rational point satisfying these conditions.

3. A class in the Grothendieck-Brauer group

It is a discovery of Yu. I. Manin [14, §47] that a non-trivial element $\alpha \in \operatorname{Br}(S)$ of the Grothendieck-Brauer group [11], [15, Chapter IV] of a variety S may cause a failure of the Hasse principle. Today, this phenomenon is called the Brauer-Manin obstruction. Its mechanism works as follows.

Let K be a number field, $\mathfrak{l} \subset \mathscr{O}_K$ a prime ideal, and $K_\mathfrak{l}$ be the corresponding completion. The Grothendieck-Brauer group is a contravariant functor from the category of schemes to the category of abelian groups. In particular, for an arbitrary scheme S and a $K_\mathfrak{l}$-rational point $x\colon \operatorname{Spec} K_\mathfrak{l} \to S$, there is a restriction homomorphism $x^*\colon \operatorname{Br}(S) \to \operatorname{Br}(\operatorname{Spec} K_\mathfrak{l}) \cong \mathbb{Q}/\mathbb{Z}$. For a Brauer class $\alpha \in \operatorname{Br}(X)$, we call

$$\operatorname{ev}_{\alpha,\mathfrak{l}}\colon S(K_\mathfrak{l}) \longrightarrow \mathbb{Q}/\mathbb{Z}\,, \quad x \mapsto x^*(\alpha)$$

the local evaluation map, associated to α. Analogously, for $\sigma\colon K \hookrightarrow \mathbb{R}$ a real prime, there is the local evaluation map $\operatorname{ev}_{\alpha,\sigma}\colon S(K_\sigma) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

PROPOSITION 3.1 (The Brauer-Manin obstruction to the Hasse principle). — *Let S be a projective variety over a number field K and $\alpha \in \operatorname{Br}(S)$ be a Brauer class.*

For every prime ideal $\mathfrak{l} \subset \mathscr{O}_K$, suppose that $S(K_\mathfrak{l}) \neq \emptyset$ and that the local evaluation map $\operatorname{ev}_{\alpha,\mathfrak{l}}$ is constant. Analogously, assume that, for every real prime $\sigma\colon K \hookrightarrow \mathbb{R}$, one has $S(K_\sigma) \neq \emptyset$ and that the local evaluation map $\operatorname{ev}_{\alpha,\sigma}$ is constant. Denote the values of $\operatorname{ev}_{\alpha,\mathfrak{l}}$ and $\operatorname{ev}_{\alpha,\sigma}$ by $e_\mathfrak{l}$ and e_σ, respectively. If, in this situation,

$$\sum_{\mathfrak{l} \subset \mathscr{O}_K} e_\mathfrak{l} + \sum_{\sigma\colon K \hookrightarrow \mathbb{R}} e_\sigma \neq 0 \in \mathbb{Q}/\mathbb{Z}$$

then S is a counterexample to the Hasse principle.

Proof. The assumptions imply, in particular, that S is not the empty scheme. Consequently, there are K_τ-rational points on S for every complex prime $\tau\colon K \hookrightarrow \mathbb{C}$. The Hasse principle would assert that $S(K) \neq \emptyset$.

On the other hand, by global class field theory [22, Section 10, Theorem B] one has a short exact sequence

$$0 \to \operatorname{Br}(K) \to \bigoplus_\nu \operatorname{Br}(K_\nu) \to \mathbb{Q}/\mathbb{Z} \to 0\,,$$

where the direct sum is taken over all places ν of the number field K. Assume that there is a point $x\colon \operatorname{Spec} K \to S$. Then $x^*(\alpha) \in \operatorname{Br}(\operatorname{Spec} K)$ is a Brauer class that naturally maps to an element of $\bigoplus_\mathfrak{l} \operatorname{Br}(K_\mathfrak{l}) \oplus \bigoplus_\sigma \operatorname{Br}(K_\sigma) \cong \bigoplus_\mathfrak{l} \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_\sigma \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ of a non-zero sum, which is a contradiction to the exactness of the above sequence. $\square$

PROPOSITION 3.2. — *Let K be a field of characteristic $\neq 2$ and $A, B, D \in K \setminus \{0\}$ be arbitrary elements. Suppose that D is a non-square and set $L := K(\sqrt{D})$. Assume that $S := S^{(D;A,B)}$ is non-singular.*

a) *Then the quaternion algebra (see* [19, Section 15.1] *for the notation)*

$$\mathscr{A} := \big(L(S), \tau, \tfrac{T_0 + AT_1}{T_0}\big)$$

over the function field $K(S)$ extends to an Azumaya algebra over the whole of S. Here, by $\tau \in \operatorname{Gal}(L(S)/K(S))$, we denote the nontrivial element.

b) *Assume that K is a number field and denote by $\alpha \in \operatorname{Br}(S)$ the Brauer class, defined by the extension of $\mathscr{A}$. Let $\mathfrak{l}$ be any prime of K.*

i) *Let* $(t_0 : t_1 : t_2 : t_3 : t_4) \in S(K_{\mathfrak{l}})$ *be a point and assume that at least one of the quotients* $(t_0 + At_1)/t_0$, $(t_0 + At_1)/t_1$, $(t_0 + Bt_1)/t_0$, *and* $(t_0 + Bt_1)/t_1$ *is properly defined and non-zero. Denote that by* q. *Then*

$$\mathrm{ev}_{\alpha,\mathfrak{l}}(t_0 \!:\! t_1 \!:\! t_2 \!:\! t_3 \!:\! t_4) = \begin{cases} 0 & \textit{if } (q,D)_{\mathfrak{l}} = 1\,, \\ \frac{1}{2} & \textit{if } (q,D)_{\mathfrak{l}} = -1\,, \end{cases}$$

for $(q,D)_{\mathfrak{l}}$ *the Hilbert symbol.*

ii) *If* $\mathfrak{l}$ *is split in* L *then the local evaluation map* $\mathrm{ev}_{\alpha,\mathfrak{l}}$ *is constantly zero.*

Proof. a) (Cf. [24, Lemma 3.2].) First of all, $\mathscr{A}$ is, by construction, a cyclic algebra of degree two. In particular, $\mathscr{A}$ is simple [19, Section 15.1, Corollary d]. Furthermore, $\mathscr{A}$ is obviously a central $K(S)$-algebra.

To prove the extendability assertion, it suffices to show that $\mathscr{A}$ extends as an Azumaya algebra over each valuation ring that corresponds to a prime divisor on S. Indeed, this is the classical Theorem of Auslander-Goldman for non-singular surfaces [2, Proposition 7.4], cf. [15, Chapter IV, Theorem 2.16].

For this, we observe that the principal divisor $\mathrm{div}((T_0 + AT_1)/T_0) \in \mathrm{Div}(S)$ is the norm of a divisor on S_L. In fact, it is the norm of the difference of two prime divisors, the conic, given by $T_0 + AT_1 = T_2 - \sqrt{D}T_4 = 0$, and the conic, given by $T_0 = T_2 - \sqrt{D}T_3 = 0$. In particular, $\mathscr{A}$ defines the zero element in $H^2(\langle\sigma\rangle, \mathrm{Div}(S_L))$. Under such circumstances, the extendability of $\mathscr{A}$ over the valuation ring corresponding to an arbitrary prime divisor on S is worked out in [14, Paragraph 42.2].

b.i) The quotients

$$\tfrac{T_0+AT_1}{T_0}\Big/\tfrac{T_0+AT_1}{T_1} = \tfrac{T_2^2-DT_3^2}{T_0^2}\,, \quad \tfrac{T_0+BT_1}{T_0}\Big/\tfrac{T_0+BT_1}{T_1} = \tfrac{T_2^2-DT_3^2}{T_0^2}\,, \text{ and } \tfrac{T_0+AT_1}{T_0}\Big/\tfrac{T_0+BT_1}{T_0} = \tfrac{T_2^2-DT_4^2}{(T_0+BT_1)^2}$$

are norms of rational functions. Thus, each defines the zero class in $H^2(\langle\sigma\rangle, K(S_L)^*) \subseteq \mathrm{Br}\,K(S)$, and hence in $\mathrm{Br}\,S$. In particular, the four expressions $(T_0+AT_1)/T_0$, $(T_0+AT_1)/T_1$, $(T_0+BT_1)/T_0$, and $(T_0 + BT_1)/T_1$ define the same Brauer class.

The general description of the evaluation map, given in [14, Paragraph 45.2] shows that $\mathrm{ev}_{\alpha,\mathfrak{l}}(t_0 : t_1 : t_2 : t_3 : t_4)$ is equal to 0 or $\frac{1}{2}$ depending on whether q is in the image of the norm map $N_{L_{\mathfrak{L}}/K_{\mathfrak{l}}}\colon L_{\mathfrak{L}}^* \to K_{\mathfrak{l}}^*$, or not, for $\mathfrak{L}$ a prime of L lying above $\mathfrak{l}$. This is exactly what is tested by the Hilbert symbol $(q,D)_{\mathfrak{l}}$.

ii) If $\mathfrak{l}$ is split in L then the norm map $N\colon K(S_{L_{\mathfrak{L}}})^* \to K(S_{K_{\mathfrak{l}}})^*$ is surjective. In particular, $\frac{T_0+AT_1}{T_0} \in K(S_{K_{\mathfrak{l}}})^*$ is the norm of a rational function on $S_{L_{\mathfrak{L}}}$. Therefore, it defines the zero class in $H^2(\langle\sigma\rangle, K(S_{L_{\mathfrak{L}}})^*) \subseteq \mathrm{Br}\,K(S_{K_{\mathfrak{l}}})$, and thus in $\mathrm{Br}\,S_{K_{\mathfrak{l}}}$. Finally, we observe that every $K_{\mathfrak{l}}$-rational point $x\colon \mathrm{Spec}\,K_{\mathfrak{l}} \to S$ factors via $S_{K_{\mathfrak{l}}}$. $\square$

Geometrically, on a rank four quadric in $\mathbf{P}^4$, there are two pencils of planes. In our situation, these are conjugate to each other under the operation of $\mathrm{Gal}(K(\sqrt{D})/K)$. The equation $T_0 = 0$ cuts two conjugate planes out of the quadric (2.1) and the same is true for $T_1 = 0$. The equations $T_0 + AT_1 = 0$ and $T_0 + BT_1 = 0$ each cut two conjugate planes out of (2.2).

Remark 3.1. — A. Várilly-Alvarado and B. Viray [24, Theorem 5.3] prove for a certain class of degree four del Pezzo surfaces that the Brauer-Manin obstruction is the only obstruction to the Hasse principle and to weak approximation. Their result is conditional under the assumption of Schinzel's hypothesis and the finiteness of Tate-Shafarevich groups of elliptic curves and based on ideas of O. Wittenberg [26, Théorème 1.1]. The class considered in [24] includes our family (2.1, 2.2).

One might formulate our strategy to prove $S^{(D;A,B)}(K) = \emptyset$ for K a number field and particular choices of A, B, and D in a more elementary way as follows.

Suppose that there is a point $(t_0 : t_1 : t_2 : t_3 : t_4) \in S(K)$. Then $(t_0, t_1) \neq (0, 0)$. Among $(t_0+At_1)/t_0$, $(t_0 + At_1)/t_1$, $(t_0 + Bt_1)/t_0$, and $(t_0 + Bt_1)/t_1$, consider an expression q that is properly defined and non-zero. Then show that, for every prime $\mathfrak{l}$ of K including the Archimedean ones, but with the exception of exactly an odd number, the Hilbert symbol $(q, D)_\mathfrak{l}$ is equal to 1. Finally, observe that such a behaviour contradicts the Hilbert reciprocity law [17, Chapter VI, Theorem 8.1].

In other words, the element $q \in K_\mathfrak{l}$ belongs to the image of the norm map $N\colon L_\mathfrak{L} \to K_\mathfrak{l}$, for $L := K(\sqrt{D})$ and $\mathfrak{L}$ a prime of L lying above $\mathfrak{l}$, for all but an odd number of primes. Which is incompatible with [17, Chapter VI, Corollary 5.7] or [22, Theorem 5.1 together with 6.3].

4. Unramified primes

Lemma 4.1. — *Let K be a field of characteristic $\neq 2$ and $A, B, D \in K$ be elements such that $D \neq 0$. Then the minimal resolution of singularities $\widetilde{S}$ of $S := S^{(D;A,B)}$ is geometrically isomorphic to $\mathbf{P}^2$, blown up in five points (some of which may be infinitely near points).*

Proof. By Proposition 2.1.b), we know that $S_{\overline{K}}$ is not a cone and has at worst isolated singularities. In this situation, it is well-known that all the singularities of $S_{\overline{K}}$ are of ADE-type. The usual argument for this is based on the classification of singularities of cubic surfaces (e.g. [8, Section 9.2]). Cf. [6, Section 5, particularly Proposition 5.1] for details.

Consequently, according to [6, Example 0.7.b)], $S_{\overline{K}}$ is either a del Pezzo surface of degree 4 or a singular del Pezzo surface of degree 4 in the sense of [6]. That is, its minimal resolution of singularities $\widetilde{S}_{\overline{K}}$ is a generalised del Pezzo surface of degree 4 [7]. But those are isomorphic to $\mathbf{P}^2_{\overline{K}}$, blown up in five points [6, Proposition 0.4]. $\square$

Corollary 4.1. — *Let $\mathbb{F}_\ell$ be a finite field of characteristic $\neq 2$ and $A, B, D \in \mathbb{F}_\ell$ such that $D \neq 0$. Then $S := S^{(D;A,B)}$ has a regular $\mathbb{F}_\ell$-rational point.*

Proof. By Lemma 4.1, the minimal resolution of singularities $\widetilde{S}$ of S is geometrically isomorphic to $\mathbf{P}^2$, blown up in five points. In such a situation, the Weil conjectures have been established by A. Weil himself [25, page 557], cf. [14, Theorem 27.1].

At least one of the eigenvalues of Frobenius on $\mathrm{Pic}(\widetilde{S}_{\overline{\mathbb{F}}_\ell})$ is equal to $(+1)$. Say, the number of eigenvalues $(+1)$ is exactly $n \geqslant 1$. The remaining $(6 - n)$ eigenvalues are of real part $\geqslant (-1)$. Hence, $\#\widetilde{S}(\mathbb{F}_\ell) \geqslant \ell^2 + (2n - 6)\ell + 1$.

Among these, at most $(n - 1)(\ell + 1)$ points may have originated from blowing up the singular points of $S_\mathfrak{l}$. Indeed, each time an $\mathbb{F}_\ell$-rational point is blown up, a $(+1)$-eigenspace is added to the Picard group. Therefore,

$$\#S_{\mathrm{reg}}(\mathbb{F}_\ell) \geqslant \ell^2 + (2n - 6)\ell + 1 - (n - 1)(\ell + 1) = \ell^2 - 5\ell + 2 + n(\ell - 1) \geqslant \ell^2 - 4\ell + 1\,.$$

For $\ell \geqslant 5$, this is positive.

Thus, it only remains to consider the case that $\ell = 3$. Then S is the closed subvariety of $\mathbf{P}^4_{\mathbb{F}_3}$, given by

$$\begin{aligned} T_0T_1 &= T_2^2 - DT_3^2\,, \\ (T_0 + aT_1)(T_0 + bT_1) &= T_2^2 - DT_4^2 \end{aligned}$$

for $D = \pm 1$ and certain $a, b \in \mathbb{F}_3$. Independently of the values of a and b, S has the regular $\mathbb{F}_3$-rational point $(1\!:\!0\!:\!1\!:\!1\!:\!0)$ in the case that $D = 1$ and $(1\!:\!0\!:\!0\!:\!0\!:\!1)$ in the case that $D = -1$. $\square$

Proposition 4.2 (Unramified primes). — *Let K be a number field, $A, B, D \in \mathscr{O}_K$, and $\mathfrak{l} \subset \mathscr{O}_K$ be a prime ideal that is unramified under the field extension $K(\sqrt{D})/K$. Consider the surface $S := S^{(D;A,B)}$.*

a) *If $\#\mathscr{O}_K/\mathfrak{l}$ is not a power of 2 then $S(K_\mathfrak{l}) \neq \emptyset$.*

b) *Assume that $A \not\equiv B \pmod{\mathfrak{l}}$, that S is non-singular, and that $S(K_\mathfrak{l}) \neq \emptyset$. Let $\alpha \in \mathrm{Br}(S)$ be the Brauer class, described in Proposition 3.2.a). Then the local evaluation map $\mathrm{ev}_{\alpha,\mathfrak{l}}\colon S(K_\mathfrak{l}) \to \mathbb{Q}/\mathbb{Z}$ is constantly zero.*

Proof. We put $\ell := \#\mathscr{O}_K/\mathfrak{l}$. Furthermore, we normalise D to be a unit in $\mathscr{O}_{K_\mathfrak{l}}$. This is possible because $\mathfrak{l}$ is unramified.

a) It suffices to verify the existence of a regular $\mathbb{F}_\ell$-rational point on the reduction $S_\mathfrak{l}$ of S. For this, we observe that $(D \bmod \mathfrak{l}\mathscr{O}_{K_\mathfrak{l}}) \neq 0$, which shows that Corollary 4.1 applies.

b) If $\mathfrak{l}$ is split then this is Proposition 3.2.b.ii). Otherwise, let $(t_0 : t_1 : t_2 : t_3 : t_4) \in S(K_\mathfrak{l})$ be an arbitrary point. Normalise the coordinates such that $t_0, \ldots, t_4 \in \mathscr{O}_{K_\mathfrak{l}}$ and at least one is a unit.

We first observe that one of t_0 and t_1 must be a unit. Indeed, otherwise one has $\mathfrak{l}|t_0, t_1$. According to equation (2.1), this implies that $\mathfrak{l}|N_{K_\mathfrak{l}(\sqrt{D})/K_\mathfrak{l}}(t_2 + t_3\sqrt{D})$. Such a divisibility is possible only when $\mathfrak{l}|t_2, t_3$, since $K_\mathfrak{l}(\sqrt{D})/K_\mathfrak{l}$ is an unramified, proper extension and $\sqrt{D} \in K_\mathfrak{l}(\sqrt{D})$ is a unit. But then t_4 is a unit, in contradiction to equation (2.2).

Second, we claim that $t_0 + At_1$ or $t_0 + Bt_1$ is a unit. Indeed, since $A \not\equiv B \pmod{\mathfrak{l}}$, the assumption $\mathfrak{l}|t_0 + At_1, t_0 + Bt_1$ implies $\mathfrak{l}|t_0, t_1$.

We have thus shown that one of the four expressions $(t_0+At_1)/t_0$, $(t_0+At_1)/t_1$, $(t_0+Bt_1)/t_0$, and $(t_0 + Bt_1)/t_1$ is a unit. Write q for that quotient. As the local extension $K_\mathfrak{l}(\sqrt{D})/K_\mathfrak{l}$ is unramified of degree two, we see that $(q, D)_\mathfrak{l} = 1$. Proposition 3.2.b.i) implies the assertion. $\square$

If $\mathfrak{l}$ is a split prime then an even stronger statement is true.

Lemma 4.2 (Split primes). — *Let K be a number field, $A, B, D \in \mathscr{O}_K$, and $\mathfrak{l} \subset \mathscr{O}_K$ a prime ideal that is split under $K(\sqrt{D})/K$. Consider the surface $S := S^{(D;A,B)}$.*

a) *Then $S(K_\mathfrak{l}) \neq \emptyset$.*

b) *Furthermore, if S is non-singular and $\alpha \in \mathrm{Br}(S)$ is the Brauer class, described in Proposition 3.2.a), then the local evaluation map $\mathrm{ev}_{\alpha,\mathfrak{l}}\colon S(K_\mathfrak{l}) \to \mathbb{Q}/\mathbb{Z}$ is constantly zero.*

Proof. a) The assumption that $\mathfrak{l}$ is split under the field extension $K(\sqrt{D})/K$ is equivalent to $\sqrt{D} \in K_\mathfrak{l}$. Therefore, the point $(1\!:\!0\!:\!1\!:\!\frac{1}{\sqrt{D}}\!:\!0)$ is defined over $K_\mathfrak{l}$. In particular, $S(K_\mathfrak{l}) \neq \emptyset$.

b) This is the assertion of Proposition 3.2.b.ii). $\square$

Remark 4.3. — If $\mathfrak{l}$ is inert, $0 \not\equiv A \equiv B \pmod{\mathfrak{l}}$, and $(A/D \bmod \mathfrak{l}) \in \mathscr{O}_K/\mathfrak{l}$ is a non-square then the assertion of Proposition 4.2.b) is true, too.

Indeed, t_0 or t_1 must be a unit by the same argument as before. The assumption $\mathfrak{l}|t_0+At_1, t_0+Bt_1$ does not lead to an immediate contradiction, but to $\mathfrak{l}|t_2, t_4$ and $t_0/t_1 \equiv -A \pmod{\mathfrak{l}}$. In particular, both t_0 and t_1 must be units. But then equation (2.1) implies the congruence $-At_1^2 \equiv -Dt_3^2 \pmod{\mathfrak{l}}$.

Remark 4.4 (Inert primes–the case of residue characteristic 2). — We note that a statement analogous to Proposition 4.2.a) is true for any inert prime $\mathfrak{l}$ under some more restrictive conditions on the coefficients A and B.

For this suppose that A, B, $D \in \mathscr{O}_K$ and that $\mathfrak{l} \subset \mathscr{O}_K$ is a prime ideal that is inert under $K(\sqrt{D})/K$. Let e be a positive integer such that $x \equiv 1 \pmod{\mathfrak{l}^e}$ is enough to imply that $x \in K_\mathfrak{l}$ is a square. Assume that $\nu_\mathfrak{l}(B-1) = f \geqslant 1$ and that $\nu_\mathfrak{l}(A)$ is an odd number such that $\nu_\mathfrak{l}(A) \geqslant 2f+e$. Then $S(K_\mathfrak{l}) \neq \emptyset$.

Indeed, let us show that there exists a point $(t_0 : t_1 : t_2 : t_3 : t_4) \in S(K_\mathfrak{l})$ such that $t_3 = t_4$ and $t_1 \neq 0$. This leads to the equation $(T_0 + AT_1)(T_0 + BT_1) = T_0T_1$, or

$$T_0^2 + (A + B - 1)T_0T_1 + ABT_1^2 = 0\,.$$

The discriminant of this binary quadric is $(A + B - 1)^2 - 4AB = (B-1)^2 + A(A - 2B-2)$, which is a square in $K_\mathfrak{l}$ by virtue of our assumptions. Thus, there are two solutions in $K_\mathfrak{l}$ for T_0/T_1 and their product is AB, which is of odd valuation. We may therefore choose a solution t_0/t_1 such that $\nu_\mathfrak{l}(t_0/t_1)$ is even. This is enough to imply that $(t_0 + At_1)(t_0 + Bt_1) = t_0t_1$ is a norm from $K_\mathfrak{l}(\sqrt{D})$.

Remark 4.5 (Archimedean primes). — i) Let $\sigma\colon K \hookrightarrow \mathbb{R}$ be a real prime. Then, for A, $B \in K$ arbitrary and $D \in K$ non-zero, one has $S_\sigma(\mathbb{R}) \neq \emptyset$.

Indeed, we can put $t_1 := 1$ and choose $t_0 \in \mathbb{R}$ such that t_0, $t_0 + \sigma(A)$, and $t_0 + \sigma(B)$ are positive. Then $C := t_0 > 0$ and $C' := (t_0 + \sigma(A))(t_0 + \sigma(B)) > 0$ and we have to show that the system of equations

$$\begin{aligned} T_2^2 - \sigma(D)T_3^2 &= C \\ T_2^2 - \sigma(D)T_4^2 &= C' \end{aligned}$$

is solvable in $\mathbb{R}$. For this one may choose t_2 such that $t_2^2 \geqslant \max(C, C')$ if $\sigma(D) > 0$ and such that $t_2^2 \leqslant \min(C, C')$, otherwise. In both cases it is clear that there exist real numbers t_3 and t_4 such that the resulting point is contained in $S_\sigma(\mathbb{R})$.

Moreover if $\sigma(D) > 0$ then the local evaluation map $\mathrm{ev}_{\alpha,\sigma}\colon S(K_\sigma) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is constantly zero. Indeed, then one has $(q, D)_\sigma = 1$ for every $q \in K_\sigma \cong \mathbb{R}$, different from zero.

ii) For $\tau\colon K \hookrightarrow \mathbb{C}$ a complex prime and A, B, and $D \in K$ arbitrary, we clearly have that $S(K_\tau) \neq \emptyset$. Furthermore, $(q, D)_\tau = 1$ for every non-zero $q \in K_\tau \cong \mathbb{C}$.

5. Ramification–Reduction to the union of four planes

The goal of this section is to study the evaluation of the Brauer class at ramified primes $\mathfrak{l}$. Under certain congruence conditions on the parameters A and B we deduce that the evaluation map is constant on the $K_\mathfrak{l}$-rational points on S, and we determine its value depending on A and B.

PROPOSITION 5.1 (Ramified primes in residue characteristic $\neq 2$). —
Let K be a number field, $A, B, D \in \mathscr{O}_K$, and $\mathfrak{l} \subset \mathscr{O}_K$ a prime ideal such that $\#\mathscr{O}_K/\mathfrak{l}$ is not a power of 2 and that is ramified under the field extension $K(\sqrt{D})/K$. Suppose that $\overline{A} := (A \bmod \mathfrak{l})$ $\in \mathscr{O}_K/\mathfrak{l}$ is a square, different from 0 and (-1), that $\overline{A}^2 + \overline{A} + 1 \neq 0$, and that

$$B \equiv -\tfrac{A}{A+1} \pmod{\mathfrak{l}}\,.$$

Consider the surface $S := S^{(D;A,B)}$.

a) *Then $S(K_\mathfrak{l}) \neq \emptyset$.*

b) *Assume that S is non-singular and let $\alpha \in \mathrm{Br}(S)$ be the Brauer class, described in Proposition 3.2.a).*

i) *If $\overline{A}{+}1 \in \mathscr{O}_K/\mathfrak{l}$ is a square then the local evaluation map $\mathrm{ev}_{\alpha,\mathfrak{l}}\colon S(K_\mathfrak{l}) \to \mathbb{Q}/\mathbb{Z}$ is constantly zero.*

ii) *If $\overline{A}{+}1 \in \mathscr{O}_K/\mathfrak{l}$ is a non-square then the local evaluation map $\mathrm{ev}_{\alpha,\mathfrak{l}}\colon S(K_\mathfrak{l}) \to \mathbb{Q}/\mathbb{Z}$ is constant of value $\frac{1}{2}$.*

Proof. First of all, we note that $\nu_\mathfrak{l}(D)$ is odd. Indeed, assume the contrary. We may then normalise D to be a unit and write $K_\mathfrak{l}^\mathrm{n}$ for the unramified quadratic extension of $K_\mathfrak{l}$. Then $(D \bmod \mathfrak{l}\mathscr{O}_{K_\mathfrak{l}^\mathrm{n}})$ is a square and, since $\mathscr{O}_{K_\mathfrak{l}^\mathrm{n}}/\mathfrak{l}\mathscr{O}_{K_\mathfrak{l}^\mathrm{n}}$ is a field of characteristic different from 2, Hensel's Lemma ensures that D is a square in $K_\mathfrak{l}^\mathrm{n}$. I.e., $K_\mathfrak{l}(\sqrt{D}) \subseteq K_\mathfrak{l}^\mathrm{n}$, a contradiction.

Let us normalise D such that $\nu_\mathfrak{l}(D) = 1$. Then the reduction $S_\mathfrak{l}$ of S is given by the equations

$$T_0T_1 = T_2^2\,, \tag{5.1}$$

$$(T_0 + \overline{A}T_1)(T_0 - \tfrac{\overline{A}}{\overline{A}+1}T_1) = T_2^2\,, \tag{5.2}$$

which geometrically define a cone over four points in $\mathbf{P}^2$.

a) We write $\ell := \#\mathscr{O}_K/\mathfrak{l}$. It suffices to verify the existence of a regular $\mathbb{F}_\ell$-rational point on $S_\mathfrak{l}$. For this, it is clearly enough to show that one of the four points in $\mathbf{P}^2$, defined by the equations (5.1) and (5.2), is simple and defined over $\mathbb{F}_\ell$.

Equating the two terms on the left hand side, one finds the equation

$$T_0^2 + \tfrac{\overline{A}^2-\overline{A}-1}{\overline{A}+1}T_0T_1 - \tfrac{\overline{A}^2}{\overline{A}+1}T_1^2 = 0\,,$$

which obviously has the two solutions $T_0/T_1 = 1$ and $T_0/T_1 = -\frac{\overline{A}^2}{\overline{A}+1}$. By virtue of our assumptions, both are $\mathbb{F}_\ell$-rational points in $\mathbf{P}^1$, different from 0 and ∞. They are different from each other, since $\overline{A}^2 + \overline{A} + 1 \neq 0$.

Consequently, the four points defined by the equations (5.1) and (5.2) are all simple. The two points corresponding to $(t_0\!:\!t_1) = 1$ are defined over $\mathbb{F}_\ell$. The two others are defined over $\mathbb{F}_\ell$ if and only if $(-\overline{A} - 1) \in \mathbb{F}_\ell$ is a square.

b) Let $(t_0 : t_1 : t_2 : t_3 : t_4) \in S(K_\mathfrak{l})$ be any point. We normalise the coordinates such that $t_0, \ldots, t_4 \in \mathscr{O}_{K_\mathfrak{l}}$ and at least one of them is a unit. Then $\mathfrak{l}$ cannot divide both t_0 and t_1. Indeed, this would imply $\mathfrak{l}^2 | t_2^2 - Dt_3^2$ and $\mathfrak{l}^2 | t_2^2 - Dt_4^2$ and, as $\nu_\mathfrak{l}(D) = 1$, this is possible only for $\mathfrak{l} | t_2, t_3, t_4$.

Therefore, $((t_0{+}At_1)/t_1 \bmod \mathfrak{l}) = \overline{A}{+}(t_0/t_1 \bmod \mathfrak{l})$ is either equal to $(\overline{A}{+}1)$ or to $\overline{A} - \frac{\overline{A}^2}{\overline{A}+1} = \frac{\overline{A}}{\overline{A}+1}$. Both terms are squares in $\mathbb{F}_\ell$ under the assumptions of b.i), while, under the assumptions of b.ii), both are non-squares.

As a unit in $\mathscr{O}_{K_\mathfrak{l}}$ is a norm from the ramified extension $K_\mathfrak{l}(\sqrt{D})$ if and only if its residue modulo $\mathfrak{l}$ is a square, for $q := (t_0 + At_1)/t_1$, we find that $(q, D)_\mathfrak{l} = 1$ in case i) and $(q, D)_\mathfrak{l} = -1$ in case ii). Proposition 3.2.b.ii) implies the assertion. $\square$

6. Zariski density in the coarse moduli scheme

We are now in the position to formulate sufficient conditions on A, B, D, under which the corresponding surface $S^{(D;A,B)}$ violates the Hasse principle.

Theorem 6.1. — *Let $D \in K$ be non-zero and $(D) = (\mathfrak{q}_1^{k_1} \cdot \ldots \cdot \mathfrak{q}_l^{k_l})^2\, \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_k$ its decomposition into prime ideals with $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ being distinct. Suppose that*

i) $k \geqslant 1$,

ii) *the quadratic extension* $K(\sqrt{D})/K$ *is unramified at all primes of* K *lying over the rational prime* 2,

iii) *for every real prime* $\sigma\colon K \hookrightarrow \mathbb{R}$, *one has* $\sigma(D) > 0$.

iv) *For every prime* $\mathfrak{l}$ *of* K *that lies over the rational prime* 2 *and is inert under* $K(\sqrt{D})/K$, *assume*

$$\text{that } \nu_{\mathfrak{l}}(B-1) = f_{\mathfrak{l}} \geqslant 1\,, \quad \text{that } \nu_{\mathfrak{l}}(A) \text{ is odd}\,, \quad \text{and that } \nu_{\mathfrak{l}}(A) \geqslant 2f_{\mathfrak{l}} + e_{\mathfrak{l}}\,,$$

for $e_{\mathfrak{l}}$ *a positive integer such that* $x \equiv 1 \pmod{\mathfrak{l}^{e_{\mathfrak{l}}}}$ *is enough to ensure that* $x \in K_{\mathfrak{l}}$ *is a square.*

v) *For every* $i = 1, \ldots, k$, *suppose that*

• $(A \bmod \mathfrak{p}_i) \in \mathscr{O}_K/\mathfrak{p}_i$ *is a square, different from* 0, (-1), *and the primitive third roots of unity. If* $\#\mathscr{O}_K/\mathfrak{p}_i$ *is a power of* 3 *then assume* $(A \bmod \mathfrak{p}_i) \neq 1$, *too.*

• $B \equiv -\frac{A}{A+1} \pmod{\mathfrak{p}_i}$.

• $1 + (A \bmod \mathfrak{p}_i) \in \mathscr{O}_K/\mathfrak{p}_i$ *is a non-square for* $i = 1, \ldots, b$, *for an odd integer* b, *and a square for* $i = b+1, \ldots, k$.

vi) *Finally, assume that* $(A - B)$ *is a product of only split primes.*

Then $S^{(D;A,B)}(\mathbb{A}_K) \neq \emptyset$. *If* $S^{(D;A,B)}$ *is non-singular then* $S^{(D;A,B)}(K) = \emptyset$.

Remark 6.1. — Without any change, one may assume $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ distinct, too. Note, on the other hand, that we do not suppose $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ and $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_l\}$ to be disjoint.

Proof of Theorem 6.1. By i), D is not a square in K, hence $K(\sqrt{D})/K$ is a proper quadratic field extension. It is clearly ramified at $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. According to ii), these are the only ramified primes. In view of assumption iv), $S(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ follows from Proposition 4.2.a) and Proposition 5.1.a), together with Lemma 4.2.a), Remark 4.4, and Remark 4.5.

On the other hand, let $\alpha \in \mathrm{Br}(S)$ be the Brauer class, described in Proposition 3.2.a). Then, in view of assumptions vi), v) and iii), Proposition 4.2.b) and Proposition 5.1.b), together with Lemma 4.2.b) and Remark 4.5, show that the local evaluation map $\mathrm{ev}_{\alpha,\mathfrak{l}}$ is constant of value $\frac{1}{2}$ for $\mathfrak{l} = \mathfrak{p}_1, \ldots, \mathfrak{p}_b$ and constantly zero for all others. Proposition 3.1 proves that S is a counterexample to the Hasse principle. □

Example 6.2. — Let S be the surface in $\mathbf{P}^4_{\mathbb{Q}}$, given by

$$\begin{aligned} T_0T_1 &= T_2^2 - 17T_3^2\,, \\ (T_0 + 9T_1)(T_0 + 11T_1) &= T_2^2 - 17T_4^2\,. \end{aligned}$$

Then $S(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ but $S(\mathbb{Q}) = \emptyset$.

Proof. We have $K = \mathbb{Q}$ and $D = 17$. Furthermore, $A = 9$ and $B = 11$ such that Proposition 2.1 ensures that $S = S^{(D;A,B)}$ is non-singular.

The extension $L := \mathbb{Q}(\sqrt{17})/\mathbb{Q}$ is real-quadratic, i.e. $D > 0$, and ramified only at 17. Under $\mathbb{Q}(\sqrt{17})/\mathbb{Q}$, the prime 2 is split, which completes the verification of i)–iii) and shows that iv) is fulfilled trivially.

For v), note that $17 \not\equiv 1 \pmod 3$, such that there are no nontrivial third roots of unity in $\mathbb{F}_{17}$. Furthermore, $9 \neq 0, (-1)$ is a square modulo 17, but 10 is not, and $11 \equiv -\frac{9}{10} \pmod{17}$. Finally, for vi), note that $(A - B) = (-2) = (2)$ is a prime that is split in $\mathbb{Q}(\sqrt{17})$. □

Remark 6.3. — The assumption on S to be non-singular may be removed from Theorem 6.1. Indeed, the elementary argument described at the very end of section 3 works in the singular case, too.

The goal of the next lemma is to construct discriminants $D \in K$, for which we will later be able to construct counterexamples to the Hasse principle, via the previous theorem.

LEMMA 6.4. — *Let K be an arbitrary number field and $\mathfrak{p}, \mathfrak{r}_1, \ldots, \mathfrak{r}_n$ be distinct prime ideals such that $\mathscr{O}_K/\mathfrak{p}$ and $\mathscr{O}_K/\mathfrak{r}_i$ are of characteristics different from 2. Then there exists some $D \in K$ such that*

i) *the prime $\mathfrak{p}$ is ramified in $K(\sqrt{D})$,*

ii) *all primes lying over the rational prime 2 are split in $K(\sqrt{D})$.*

iii) *For every real prime $\sigma\colon K \hookrightarrow \mathbb{R}$, one has $\sigma(D) > 0$.*

iv) *The primes $\mathfrak{r}_i$ are unramified in $K(\sqrt{D})$.*

In particular, assumptions i)–iv) of Theorem 6.1 are fulfilled.

Proof. Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_m$ be the primes of K that lie over the rational prime 2. We impose the congruence conditions $D \equiv 1 \pmod{\mathfrak{l}_1^{e_1}}, \ldots, D \equiv 1 \pmod{\mathfrak{l}_m^{e_m}}$, for $e_1, \ldots, e_m$ large enough that this implies that D is a square in $K_{\mathfrak{l}_1}, \ldots, K_{\mathfrak{l}_m}$.

Furthermore, the assumptions imply that $\mathfrak{p}, \mathfrak{r}_1, \ldots, \mathfrak{r}_n$ are different from $\mathfrak{l}_1, \ldots, \mathfrak{l}_m$. We impose, in addition, the conditions $D \in \mathfrak{p} \backslash \mathfrak{p}^2$ and $D \notin \mathfrak{r}_1, \ldots, \mathfrak{r}_n$.

According to the Chinese remainder theorem, these conditions have a simultaneous solution D'. Put $D := D' + k \cdot \#(\mathscr{O}_K/\mathfrak{l}_1^{e_1} \ldots \mathfrak{l}_m^{e_m} \mathfrak{p}^2 \mathfrak{r}_1 \ldots \mathfrak{r}_n)$, for k an integer that is sufficiently large to ensure $\sigma(D) > 0$ for every real prime $\sigma\colon K \hookrightarrow \mathbb{R}$. Then assertion iii) is true. Furthermore, the congruences $D \equiv 1 \pmod{\mathfrak{l}_i^{e_i}}$ imply ii), while $D \in \mathfrak{p} \backslash \mathfrak{p}^2$ yields assertion i) and $D \notin \mathfrak{r}_1, \ldots, \mathfrak{r}_n$ ensures that iv) is true. $\square$

Before we come to the next main theorem of this section, we need to formulate two technical lemmata.

LEMMA 6.5. — *Let K be a number field, $I \subset \mathscr{O}_K$ an ideal, and $x \in \mathscr{O}_K \setminus I$ any element. Then there exists an infinite sequence of pairwise non-associated elements $y_i \in \mathscr{O}_K$ such that, for each $i \in \mathbb{N}$, one has that (y_i) is a prime ideal and $y_i \equiv x \pmod{I}$.*

Proof. The invertible ideals in K relatively prime to I modulo the principal ideals generated by elements from the residue class $(1 \bmod I)$ form an abelian group that is canonically isomorphic to the ray class group $Cl_K^I \cong C_K/C_K^I$ of K [17, Chapter VI, Proposition 1.9]. Thus, the Chebotarev density theorem applied to the ray class field K^I/K, which has the Galois group $\mathrm{Gal}(K^I/K) \cong Cl_K^I$, shows that there exist infinitely many prime ideals $\mathfrak{r}_i \subset \mathscr{O}_K$ with the property below.

There exist some $u_i, v_i \in \mathscr{O}_K$, $u_i \equiv v_i \equiv 1 \pmod{I}$ such that

$$\mathfrak{r}_i \cdot (u_i) = (x) \cdot (v_i)\,.$$

Take one of these prime ideals. Then $\mathfrak{r}_i \cdot (u_i) = (xv_i)$. As $\mathfrak{r}_i \subset \mathscr{O}_K$, this shows that xv_i is divisible by u_i. Put $y_i := xv_i/u_i$. Then $(y_i) = \mathfrak{r}_i$. Furthermore, $y_i \equiv x \pmod{I}$. $\square$

LEMMA 6.6. — *Let $\mathbb{F}_q$ be a finite field of characteristic $\neq 2$ having > 25 elements. Then there exist elements a_{00}, a_{01}, a_{10}, and $a_{11} \in \mathbb{F}_q$, different from $0, (-1), (-2)$ and such that $a_{ij}^2 + a_{ij} + 1 \neq 0$, that fulfil the conditions below.*

i) *a_{00}, $(a_{00}+1)$, and $(a_{00}+2)$ are squares in $\mathbb{F}_q$.*

ii) *a_{01} and $(a_{01}+1)$ are squares in $\mathbb{F}_q$, but $(a_{01}+2)$ is not.*

iii) *a_{10} and $(a_{10}+2)$ are squares in $\mathbb{F}_q$, but $(a_{10}+1)$ is not.*

iv) *a_{11} is a square in $\mathbb{F}_q$, but $(a_{11}+1)$ and $(a_{11}+2)$ are not.*

Proof. Let $C_1 \in \mathbb{F}_q^*$ be a square in the cases i) and ii), and a non-square, otherwise. Similarly, let $C_2 \in \mathbb{F}_q^*$ be a square in the cases i) and iii), and a non-square, otherwise. The problem then translates into finding an $\mathbb{F}_q$-rational point on the curve E, given in $\mathbf{P}^3$ by

$$\begin{aligned} U_1^2 + \ U_0^2 &= C_1 U_2^2\,, \\ U_1^2 + 2U_0^2 &= C_2 U_3^2\,, \end{aligned}$$

such that $U_i \neq 0$ for $i = 0, \ldots, 3$ and $(U_1/U_0)^4 + (U_1/U_0)^2 + 1 \neq 0$. Note that the conditions $U_2 \neq 0$ and $U_3 \neq 0$ imply that $\left(\frac{U_1}{U_0}\right)^2 \neq -1, -2$.

Since the characteristic of the base field is different from two, a direct calculation shows that E is non-singular, i.e. a smooth curve of genus 1. The extra conditions define an open subscheme $\widetilde{E} \subset E$ that excludes not more than 32 points. Thus, Hasse's bound yields $\#\widetilde{E}(\mathbb{F}_q) \geqslant q - 2\sqrt{q} - 31$, which is positive for $q > 44$.

An experiment shows that the four affine curves have points, too, over $\mathbb{F}_{27}$, $\mathbb{F}_{29}$, $\mathbb{F}_{31}$, $\mathbb{F}_{37}$, $\mathbb{F}_{41}$, and $\mathbb{F}_{43}$. $\square$

The following theorem provides us with Hasse counterexamples in the family $S^{(D;A,B)}$ for suitable discriminants D. For us, the important feature is that one may choose the parameters A and B to lie in (almost) arbitrary congruence classes modulo some prime ideal $\mathfrak{l} \subset \mathscr{O}_K$, unramified in $K(\sqrt{D})$, provided only that $A \not\equiv B \pmod{\mathfrak{l}}$.

Theorem 6.2. — *Let K be an arbitrary number field and $D \in K$ a non-zero element. Write $(D) = (\mathfrak{q}_1^{k_1} \cdot \ldots \cdot \mathfrak{q}_l^{k_l})^2 \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_k$ for its decomposition into prime ideals, the $\mathfrak{p}_i$ being distinct. Assume that*

i) $k \geqslant 1$,

ii) *all primes lying over the rational prime 2 are split in $K(\sqrt{D})$,*

iii) *for every real prime $\sigma\colon K \hookrightarrow \mathbb{R}$, one has $\sigma(D) > 0$,*

iv) *all primes with residue field $\mathbb{F}_3$ are unramified in $K(\sqrt{D})$.*

Suppose further that among the primes $\mathfrak{p}$ of K that are ramified in $K(\sqrt{D})$, there is one such that $\#\mathscr{O}_K/\mathfrak{p} > 25$.

Then, for every prime $\mathfrak{l} \subset \mathscr{O}_K$, unramified in $K(\sqrt{D})$, and all $a, b \in \mathscr{O}_K/\mathfrak{l}$ such that $a \neq b$, there exist $A, B \in \mathscr{O}_K$ such that $(A \bmod \mathfrak{l}) = a$, $(B \bmod \mathfrak{l}) = b$, and $S^{(D;A,B)}(\mathbb{A}_K) \neq \emptyset$, but $S^{(D;A,B)}(K) = \emptyset$.

Proof. *First step.* Construction of A and B.
Let $M \in \{1, \ldots, k\}$ be such that $\#\mathscr{O}_K/\mathfrak{p}_M > 25$. Besides

$$(A \bmod \mathfrak{l}) = a \qquad \text{and} \qquad (B \bmod \mathfrak{l}) = b\,, \tag{6.1}$$

we will impose further congruence conditions on A and B. For each $i \neq M$, we choose a square $a_i \in \mathscr{O}_K/\mathfrak{p}_i$ such that $a_i \neq 0, (-1), (-2)$ and $a_i^2 + a_i + 1 \neq 0$. This is possible since $\mathscr{O}_K/\mathfrak{p}_i$ is of characteristic $\neq 2$ and $\#\mathscr{O}_K/\mathfrak{p}_i > 3$. For instance, $a_i := 1$ may be taken except when $\mathfrak{p}_i$ is of residue characteristic 3.

We require

$$(A \bmod \mathfrak{p}_i) = a_i \qquad \text{and} \qquad (B \bmod \mathfrak{p}_i) = -\frac{a_i}{a_i + 1}\,. \tag{6.2}$$

Finally, we choose a square $a_M \in \mathscr{O}_K/\mathfrak{p}_M$ such that $a_M \neq 0, (-1), (-2)$ and $a_M^2 + a_M + 1 \neq 0$, satisfying the additional conditions below.

• If, among the elements $a_1+1,\dots,a_{M-1}+1,a_{M+1}+1,\dots,a_k+1$, there are an odd number of non-squares then a_M+1 is a square. Otherwise, a_M+1 is a non-square.

• If, among the elements $a_1+2,\dots,a_{M-1}+2,a_{M+1}+2,\dots,a_k+2$, there are an odd number of non-squares then a_M+2 is a square. Otherwise, a_M+2 is a non-square.

Lemma 6.6 guarantees that such an element $a_M \in \mathscr{O}_K/\mathfrak{p}_M$ exists. We impose the final congruence condition

$$(A \bmod \mathfrak{p}_M) = a_M \qquad \text{and} \qquad (B \bmod \mathfrak{p}_M) = -\frac{a_M}{a_M+1}\,. \tag{6.3}$$

According to the Chinese remainder theorem, one may choose an algebraic integer $B \in \mathscr{O}_K$ such that the conditions on the right hand sides of (6.1), (6.2), and (6.3) are fulfilled. Then, by Lemma 6.5, there exist infinitely many non-associated elements $y_i \in \mathscr{O}_K$ such that (y_i) is a prime ideal and (y_i+B,B) a simultaneous solution of the system of congruences (6.1, 6.2, 6.3).

We choose some $i \in \mathbb{N}$ such that $\mathfrak{r} := (y_i)$ is of residue characteristic different from 2, that $\mathfrak{r} \neq \mathfrak{p}_1,\dots,\mathfrak{p}_k,\mathfrak{q}_1,\dots,\mathfrak{q}_l$, and such that $A^2-2AB+B^2-2A-2B+1 \neq 0$ for $A := y_i+B$. Note that $\mathfrak{r} \neq \mathfrak{p}_1,\dots,\mathfrak{p}_k,\mathfrak{q}_1,\dots,\mathfrak{q}_l$ is equivalent to $\mathfrak{r} \not\ni D$.

Second step. The surface $S := S^{(D;A,B)}$ is a counterexample to the Hasse principle.
To show this, let us use Theorem 6.1. Our assumptions on D imply that assumptions i)–iv) of Theorem 6.1 are fulfilled. Assumption v) is satisfied, too, by consequence of the construction of the elements a_i. Observe, in particular, that among the elements $a_1+1,\dots,a_k+1$, there are an odd number of non-squares. Furthermore, S is non-singular.

It therefore remains to check assumption vi). The only prime $\mathfrak{p} \subset \mathscr{O}_K$, for which $A \equiv B \pmod{\mathfrak{p}}$, is $\mathfrak{p} = \mathfrak{r}$ $(=(A-B))$. We have to show that $\mathfrak{r}$ is split under $K(\sqrt{D})/K$.

For this, we observe that, for $i=1,\dots,k$,

$$A-B \equiv A+\tfrac{A}{A+1} = A\tfrac{A+2}{A+1} \pmod{\mathfrak{p}_i}\,.$$

As A is a square modulo $\mathfrak{p}_i$, this shows

$$\prod_{i=1}^k (A-B,D)_{\mathfrak{p}_i} = \prod_{i=1}^k (A+2,D)_{\mathfrak{p}_i} \Big/ \prod_{i=1}^k (A+1,D)_{\mathfrak{p}_i}\,.$$

Here, by our construction, both $1+(A \bmod \mathfrak{p}_i)$ and $2+(A \bmod \mathfrak{p}_i)$ are non-squares, an odd number of times. Consequently,

$$\prod_{i=1}^k (A-B,D)_{\mathfrak{p}_i} = 1\,.$$

On the other hand, D is a square in $K_{\mathfrak{l}_i}$ for l_i the primes of residue characteristic 2 and for every real prime, by assumption iii). Thus, $(A-B,D)_{\mathfrak{l}} = 1$ unless $\mathfrak{l}$ divides either $(A-B)$ or D. I.e. for $\mathfrak{l} \neq \mathfrak{r},\mathfrak{p}_1,\dots,\mathfrak{p}_k,\mathfrak{q}_1,\dots,\mathfrak{q}_l$. Moreover, $(A-B,D)_{\mathfrak{q}} = 1$ for $\mathfrak{q} \in \{\mathfrak{q}_1,\dots,\mathfrak{q}_l\}\backslash\{\mathfrak{p}_1,\dots,\mathfrak{p}_k\}$ since both arguments of the Hilbert symbol are of even $\mathfrak{q}$-adic valuation. The Hilbert reciprocity law [17, Chapter VI, Theorem 8.1] therefore reveals the fact that

$$(A-B,D)_{\mathfrak{r}} \cdot \prod_{i=1}^k (A-B,D)_{\mathfrak{p}_i} = 1\,.$$

Altogether, this implies $(A-B,D)_{\mathfrak{r}} = 1$. Consequently, the prime ideal $\mathfrak{r}$ splits in $K(\sqrt{D})$. $\square$

SUBLEMMA 6.7. — *The rational map* $\kappa\colon \mathbf{A}^2/S_2 \dashrightarrow \mathscr{U}/(S_5\times \mathrm{PGL}_2)$, *given on points by*

$$\overline{(a_1,a_2)} \mapsto \overline{(a_1,a_2,0,-1,\infty)}\,,$$

is dominant.

Proof. It suffices to prove that the rational map $\widetilde{\kappa}\colon \mathbf{A}^2 \dashrightarrow \mathscr{U}/(S_5\times \mathrm{PGL}_2)$, given by $(a_1,a_2)\mapsto \overline{(a_1,a_2,0,-1,\infty)}$ is dominant. For this, recall that dominance may be tested after base extension to the algebraic closure. Moreover, it is well-known that three distinct points on $\mathbf{P}^1_{\overline{K}}$ may be sent to 0, (-1), and ∞ under the operation of $\mathrm{PGL}_2(\overline{K})$. □

LEMMA 6.8. — *Let K be a field of characteristic $\neq 2$ and $0\neq D\in K$. Let $\pi\colon \mathscr{S}\to U$ be the family of degree four del Pezzo surfaces over an open subscheme $U\subset \mathbf{A}^2_K$, given by*

$$\begin{aligned} T_0T_1 &= T_2^2 - DT_3^2\,,\\ (T_0+a_1T_1)(T_0+a_2T_1) &= T_2^2 - DT_4^2\,. \end{aligned}$$

I.e., the fibre of π over (a_1,a_2) is exactly the surface $S^{(D;a_1,a_2)}$. Then the invariant map

$$I_\pi\colon U \longrightarrow \mathbf{P}(1,2,3)$$

associated with π is dominant.

Proof. As dominance may be tested after base extension to the algebraic closure, let us assume that the base field K is algebraically closed. Write

$$\begin{aligned} Q_1(a_1,a_2;T_0,\ldots,T_4) &:= T_0T_1-(T_2^2-DT_3^2) \qquad\text{and}\\ Q_2(a_1,a_2;T_0,\ldots,T_4) &:= (T_0+a_1T_1)(T_0+a_2T_1)-(T_2^2-DT_4^2), \end{aligned}$$

and consider the family $(uQ_1+vQ_2)_{(u:v)\in\mathbf{P}^1}$ of pencils of quadrics that is parametrised by $(a_1,a_2)\in\mathbf{A}^2(K)$.

We see that, independently of the values of the parameters, degenerate quadrics occur for $(u:v)=0$, ∞, and (-1). The two other degenerate quadrics appear for $(u:v)$ the zeroes of the determinant

$$\begin{vmatrix} 1 & (a_1+a_2+t)/2\\ (a_1+a_2+t)/2 & a_1a_2 \end{vmatrix} = -\tfrac{1}{4}[t^2+2(a_1+a_2)t+(a_1-a_2)^2]\,.$$

Thus, I_π is the composition of the rational map $\rho\colon \mathbf{A}^2\supset U \dashrightarrow \mathbf{A}^2/S_2$, sending (a_1,a_2) to the pair of roots of $t^2+2(a_1+a_2)t+(a_1-a_2)^2$, followed by the rational map $\kappa\colon \mathbf{A}^2/S_2 \dashrightarrow \mathscr{U}/(S_5\times\mathrm{PGL}_2)$, studied in Sublemma 6.7, and the open embedding $\iota\colon \mathscr{U}/(S_5\times\mathrm{PGL}_2)\hookrightarrow \mathbf{P}(1,2,3)$, defined by the fundamental invariants. It remains to prove that $\rho\colon U\dashrightarrow \mathbf{A}^2/S_2$ is dominant.

For this, as coordinates on $\mathbf{A}^2/S_2$ one may choose the sum and the product of the coordinates on $\mathbf{A}^2$. Indeed, these generate the field of S_2-invariant functions on $\mathbf{A}^2$. Thus, we actually claim that the map $\mathbf{A}^2\to\mathbf{A}^2$, given by $(a_1,a_2)\mapsto(-2(a_1+a_2),(a_1-a_2)^2)$ is dominant, which is obvious. □

We are now, finally, in the position to prove that the set of counterexamples to the Hasse principle is Zariski dense in the moduli scheme of del Pezzo surfaces of degree four. For this, we will consider the family $S^{(D;A,B)}$ for some fixed discriminant D and use Theorem 6.2.

THEOREM 6.3. — *Let K be a number field, $U_\mathrm{reg}\subset \mathrm{Gr}(2,15)_K$ the open subset of the Graßmann scheme that parametrises degree four del Pezzo surfaces, and $\mathscr{HC}_K\subset U_\mathrm{reg}(K)$ be the set of all degree four del Pezzo surfaces over K that are counterexamples to the Hasse principle.*

Then the image of $\mathscr{HC}_K$ under the invariant map

$$I\colon U_{\mathrm{reg}} \longrightarrow \mathbf{P}(1,2,3)_K$$

is Zariski dense.

Proof. According to Lemma 6.4, there exists an algebraic integer $D \in \mathscr{O}_K$ fulfilling the assumptions of Theorem 6.2. Suppose that the image of I were not Zariski dense. By Lemma 6.8, this implies that there exists a (possibly reducible) curve $C \subset \mathbf{A}^2$ of certain degree d such that, for all surfaces of the form

$$\begin{aligned} T_0T_1 &= T_2^2 - DT_3^2\,, \\ (T_0 + AT_1)(T_0 + BT_1) &= T_2^2 - DT_4^2 \end{aligned}$$

that violate the Hasse principle, one has $(A, B) \in C(K)$.

On the other hand, let $\mathfrak{l} \subset \mathscr{O}_K$ be an unramified prime and put $\ell := \#\mathscr{O}_K/\mathfrak{l}$. Then, by Theorem 6.2, we know counterexamples to the Hasse principle having $\ell(\ell-1)$ distinct reductions modulo $\mathfrak{l}$. But an affine plane curve of degree d has $\leqslant \ell d$ points over $\mathbb{F}_\ell$ [4, the lemma in Chapter 1, Paragraph 5.2]. For a prime ideal $\mathfrak{l}$ such that $\ell \geqslant d + 2$, this yields a contradiction. □

7. Zariski density in the Hilbert scheme

This section is devoted to Zariski density of the counterexamples to the Hasse principle in the Hilbert scheme. Our result is, in fact, an application of the Zariski density in the moduli scheme established above.

THEOREM 7.1. — *Let K be a number field, $U_{\mathrm{reg}} \subset \mathrm{Gr}(2,15)_K$ the open subset of the Graßmann scheme that parametrises degree four del Pezzo surfaces, and $\mathscr{HC}_K \subset U_{\mathrm{reg}}(K)$ be the set of all degree four del Pezzo surfaces over K that are counterexamples to the Hasse principle.*

Then $\mathscr{HC}_K$ is Zariski dense in $\mathrm{Gr}(2,15)_K$.

Proof. Let us fix an algebraic closure $\overline{K}$ and an embedding of K into $\overline{K}$. Assume for the sake of assertion that $\mathscr{HC}_K \subset U_{\mathrm{reg}} \subset \mathrm{Gr}(2,15)_K$ is not Zariski dense. It is well-known that the Graßmann scheme $\mathrm{Gr}(2,15)_K$ is irreducible and projective of dimension $(15-2)\cdot 2 = 26$. The Zariski closure $\overline{\mathscr{HC}_K} \subset \mathrm{Gr}(2,15)_K$ is therefore of dimension at most 25.

By Theorem 6.3, the invariant map $\overline{\mathscr{HC}_K} \to \mathbf{P}(1,2,3)$ is dominant. Its generic fibre thus must be of dimension at most 23. In particular, outside of a finite union of curves $C \subset \mathbf{P}(1,2,3)$, the special fibres are of dimension $\leqslant 23$, as well.

Now, let us choose a K-rational point $s \in [\mathbf{P}(1,2,3) \backslash C](K)$ that is the image of a degree four del Pezzo surface $S \in \mathscr{HC}_K$ under the invariant map. The geometric fibre $I^{-1}(s)_{\overline{K}}$ over s of the full invariant map $I\colon U_{\mathrm{reg}} \to \mathbf{P}(1,2,3)$ parametrises all reembeddings of S into $\mathbf{P}^4_{\overline{K}}$ and is therefore a torsor under $\mathrm{PGL}_5(\overline{K})/\operatorname{Aut}(S_{\overline{K}})$. In particular, $I^{-1}(s)_{\overline{K}}$ is of dimension 24.

This implies that $I^{-1}(s) \not\subseteq \overline{\mathscr{HC}_K}$. But the orbit of s under $\mathrm{PGL}_5(K)$ parametrises counterexamples to the Hasse principle, and is therefore contained in $\mathscr{HC}_K$. As $\mathrm{PGL}_5(K)$ is Zariski dense in $\mathrm{PGL}_5(\overline{K})$, this is a contradiction. □

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