K3 surfaces of Picard rank one which are double covers of the projective plane

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Abstract. We construct explicit examples of K3 surfaces over $\mathbb{Q}$ which are of degree 2 and the geometric Picard rank of which is equal to 1. We construct, particularly, examples in the form $w^2 = \det M$ where M is a symmetric (3×3)-matrix of ternary quadratic forms or a symmetric (6×6)-matrix of ternary linear forms. Our method is based on reduction modulo p for $p = 3$ and 5.

Keywords. K3 surface, Picard group, plane sextic, tritangent, conic tangent in six points, Frobenius, Tate conjecture

Introduction

In the projective plane, let a smooth curve B of degree 6 be given by $f_6(x,y,z) = 0$. Then, $w^2 = f_6(x,y,z)$ defines an algebraic surface S in a weighted projective space. We have a double cover $\pi\colon S \to \mathbf{P}^2$ ramified at $\pi^{-1}(B)$.

According to the Enriques classification of algebraic surfaces, S is an example of a K3 surface (of degree two). In general, a K3 surface is a simply connected, projective algebraic surface the canonical class of which is trivial.

Examples 1.1 K3 surfaces embedded into $\mathbf{P}^n$ are automatically of even degree.

K3 surfaces of small degree may be described, explicitly. A K3 surface of degree two is a double cover of $\mathbf{P}^2$, ramified in a smooth sextic. K3 surfaces of degree four are smooth quartics in $\mathbf{P}^3$. A K3 surface of degree six is a smooth complete intersection of a quadric and a cubic in $\mathbf{P}^4$. And, finally, K3 surfaces of degree eight are smooth complete intersections of three quadrics in $\mathbf{P}^5$.

The Picard group of a K3 surface is known to be isomorphic to $\mathbb{Z}^n$ where n may range from 1 to 20. It is generally known that a generic K3 surface over $\mathbb{C}$ is of Picard rank one.

Nevertheless, it seems that the first explicit examples of K3 surfaces of geometric Picard rank one have been constructed as late as in 2005 [5]. All these examples are of degree four.

The goal of the work described in this article is to provide explicit examples of K3 surfaces defined over $\mathbb{Q}$ which are of degree two and geometric Picard rank one.

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Let $\mathscr{S}$ be a K3 surface over a finite field $\mathbb{F}_q$. Then, we have the first Chern class

$$c_1\colon \operatorname{Pic}(\mathscr{S}) \longrightarrow H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_l(1))$$

into l-adic cohomology at our disposal. There is a natural operation of the Frobenius on $H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_l(1))$. All eigenvalues are of absolute value 1. The Frobenius operation on the Picard group is compatible with the operation on cohomology.

Every divisor is defined over a finite extension of the ground field. Consequently, on the subspace $\operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_l \hookrightarrow H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_l(1))$, all eigenvalues are roots of unity. Those correspond to eigenvalues of the Frobenius operation on $H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_l)$ which are of the form $q\zeta$ for ζ a root of unity.

We may therefore estimate the rank of the Picard group $\operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_q})$ from above by counting how many eigenvalues are of this particular form.

Estimates from below may be obtained by explicitly constructing divisors. Under certain circumstances, it is possible, that way, to calculate $\operatorname{rk}\operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_q})$, exactly.

Our general strategy is to use reduction modulo p. If S is a K3 surface over $\mathbb{Q}$ then there is the inequality

$$\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \leq \operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_p})$$

which is true for every prime p of good reduction.

Remark 1.2 Consider a complex K3 surface S. Since $H^1(S, \mathscr{O}_S) = 0$, the Picard group of S is discrete and the first Chern class

$$c_1\colon \operatorname{Pic}(S) \to H^2(S, \mathbb{Z}) \subset H^2(S, \mathbb{C})$$

is an injection. For divisors, numerical and homological equivalence are known to coincide [4, Corollary 1]. This shows, $\operatorname{Pic}(S)$ equals the group of divisors modulo numerical equivalence.

2. Explicit divisors - Geometric constructions over $\overline{\mathbb{F}}_p$

In order to estimate the rank of the Picard group from below, one needs to explicitly construct divisors. Calculating discriminants, it is possible to show that the corresponding divisor classes are linearly independent.

Assumption 2.1 For the algebro-geometric considerations described in this section, we assume that we work over a ground field which is algebraically closed of characteristic $\neq 2$.

Construction 2.2 i) One possible construction is to start with a branch curve "$f_6 = 0$" which allows a tritangent line G. The pull-back of G to the K3 surface $\mathscr{S}$ is a divisor splitting into two irreducible components. The corresponding divisor classes are linearly independent.

ii) A second possibility is to use a conic which is tangent to the branch sextic in six points.

Both constructions yield a lower bound of 2 for the rank of the Picard group.

Tritangent. Assume, the line G is a tritangent to "$f_6 = 0$". This means, the restriction of f_6 to $G \cong \mathbf{P}^1$ is a section of $\mathscr{O}(6)$, the divisor of which is divisible by 2 in $\mathrm{Div}(G)$. As G is of genus 0, this implies $f_6|_G$ is the square of a section $f \in \Gamma(G, \mathscr{O}(3))$. The form f_6 may, therefore, be written as $f_6 = \widetilde{f}^2 + lq_5$ for l a linear form defining G, $\widetilde{f}$ a cubic form lifting f, and a quintic form q_5.

Consequently, the restriction of π to $\pi^{-1}(G)$ is given by an equation of the form $w^2 = f^2(s,t)$. We, therefore, have $\pi^*(G) = D_1 + D_2$ where D_1 and D_2 are the two irreducible divisors given by $w = \pm f(s,t)$. Both curves are isomorphic to G. In particular, they are projective lines.

The adjunction formula shows $-2 = D_1(D_1 + K) = D_1^2$. Analogously, $D_2^2 = -2$. Finally, we have $G^2 = 1$. It follows that $(D_1 + D_2)^2 = 2$ which yields $D_1D_2 = 3$. For the discriminant, we find

$$\begin{vmatrix} -2 & 3 \\ 3 & -2 \end{vmatrix} = -5 \neq 0$$

guaranteeing $\operatorname{rk}\operatorname{Pic}(\mathscr{S}) \geq 2$.

Remark 2.3 We note explicitly that this argument works without modification if two or all three points of tangency coincide.

Conic tangent in six points. If C is a conic tangent to the branch curve "$f_6 = 0$" in six points then, for the same reasons as above, we have $\pi^*(C) = C_1 + C_2$ where C_1 and C_2 are irreducible divisors. Again, C_1 and C_2 are isomorphic to C and, therefore, of genus 0. This shows $C_1^2 = C_2^2 = -2$. Further, $C^2 = 4$ which implies $(C_1 + C_2)^2 = 8$ and $C_1C_2 = 6$. Here, for the discriminant, we obtain

$$\begin{vmatrix} -2 & 6 \\ 6 & -2 \end{vmatrix} = -32 \neq 0.$$

Thus, $\operatorname{rk}\operatorname{Pic}(\mathscr{S}) \geq 2$ in this case, too.

Remark 2.4 Further tritangents or further conics which are tangent in six points lead to even larger Picard groups.

3. Explicit divisors – Practical tests over $\mathbb{F}_q$

A test for tritangents. The property of a line of being a tritangent may easily be written down as an algebraic condition. Therefore, tritangents may be searched for, in practice, by investigating a Gröbner basis.

More precisely, a general line in $\mathbf{P}^2$ can be described by a parametrization

$$g_{a,b}\colon t \mapsto [1 : t : (a+bt)]\,.$$

$g_{a,b}$ is a (possibly degenerate) tritangent of the sextic "$f_6 = 0$" if and only if $f_6 \circ g_{a,b}$ is a perfect square in $\overline{\mathbb{F}}_q[t]$. This means,

$$f_6(g_{a,b}(t)) = (c_0 + c_1 t + c_2 t^2 + c_3 t^3)^2$$

is an equation which encodes the tritangent property of $g_{a,b}$. Comparing coefficients, this yields a system of seven equations in c_0, c_1, c_2, and c_3 which is solvable if and only if $g_{a,b}$ is a tritangent. The latter may be understood as well as a system of equations in a, b, c_0, c_1, c_2, and c_3 encoding the existence of a tritangent of the form above.

Using `Magma`, we compute the length of $\mathbb{F}_q[a,b,c_0,c_1,c_2,c_3]$ modulo the corresponding ideal I. This is twice the number of the tritangents detected.

The remaining one dimensional family of lines may be tested analogously using the parametrizations $g_a\colon t \mapsto [1 : a : t]$ and $g\colon t \mapsto [0 : 1 : t]$.

Remarks 3.1 a) To compute the length of $\mathbb{F}_q[a,b,c_0,c_1,c_2,c_3]/I$, a Gröbner basis of I is needed. The time required to compute such a basis over a finite field is usually a few seconds. From the Gröbner basis, the tritangents may be read off, explicitly.

b) Since the existence of a tritangent is a codimension 1 condition, one occasionally finds tritangents on randomly chosen examples.

A test for conics tangent in six points. A non-degenerate conic in $\mathbf{P}^2$ allows a parametrization of the form

$$c\colon t \mapsto [(c_0 + c_1 t + c_2 t^2) : (d_0 + d_1 t + d_2 t^2) : (e_0 + e_1 t + e_2 t^2)]\,.$$

With the sextic "$f_6 = 0$", all intersection multiplicities are even if and only if $f_6 \circ c$ is a perfect square in $\overline{\mathbb{F}}_q[t]$. This may easily be checked by factoring $f_6 \circ c$.

For small q, that allows, at least, to search for conics which are defined over $\mathbb{F}_q$ and tangent in six points. To achieve this, we listed all $q^2(q^3-1)$ non-degenerate conics over $\mathbb{F}_q$ for $q = 3$ and 5.

Remark 3.2 A general approach, analogous to the one described above, which would be able to find conics defined over $\overline{\mathbb{F}}_q$ does not succeed. The Gröbner basis required becomes too large.

4. Upper bounds – The Frobenius operation on l-adic cohomology

The Lefschetz trace formula. A method to understand the Frobenius operation on $H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l)$ works as follows.

Count the points on $\mathscr{S}$ over $\mathbb{F}_{p^d}$ and apply the Lefschetz trace formula [6] to compute the trace of the Frobenius $\phi_{\mathbb{F}_{p^d}} = \phi^d$. In our situation, it yields

$$\mathrm{Tr}(\phi^d) = \#\mathscr{S}(\mathbb{F}_{p^d}) - p^{2d} - 1\,.$$

We have $\mathrm{Tr}(\phi^d) = \lambda_1^d + \cdots + \lambda_{22}^d =: \sigma_d(\lambda_1, \ldots, \lambda_{22})$ when we denote the eigenvalues of ϕ by $\lambda_1, \ldots, \lambda_{22}$. Newton's identity [8]

$$s_k(\lambda_1, \ldots, \lambda_{22}) = \frac{1}{k} \sum_{r=0}^{k-1} (-1)^{k+r+1} \sigma_{k-r}(\lambda_1, \ldots, \lambda_{22}) s_r(\lambda_1, \ldots, \lambda_{22})$$

shows that, doing this for $d = 1, \ldots, k$, one obtains enough information to determine the coefficient $(-1)^k s_k$ of t^{22-k} of the characteristic polynomial f_p of ϕ.

Observe that we also have the functional equation

$$(*) \qquad p^{22} f_p(t) = \pm t^{22} f_p(p^2/t)$$

at our disposal. It may be used to convert the coefficient of t^i into the one of t^{22-i}.

Methods for counting points. The number $\#\mathscr{S}(\mathbb{F}_q)$ of the points may be determined as the sum

$$\sum_{[x:y:z] \in \mathbf{P}^2(\mathbb{F}_q)} \left[1 + \chi\big(f_6(x,y,z)\big)\right].$$

Here, χ is the quadratic character. The sum is well-defined since $f_6(x,y,z)$ is uniquely determined up to a sixth-power residue. To count the points naively, one would need $q^2 + q + 1$ evaluations of f_6 and χ.

Here, a number of possibilities arise for optimization. We use two of them which we describe below.

i) Symmetry: If f_6 is defined over $\mathbb{F}_p$ then the summands for $[x : y : z]$ and $\phi([x : y : z])$ are equal. This means, over $\mathbb{F}_{p^d}$, we may save a factor of d if, on the affine chart "$x = 1$", we put in for y only values from a fundamental domain of the Frobenius.

ii) Decoupling: Suppose, f_6 contains only monomials of the form $x^i y^{6-i}$ or $x^i z^{6-i}$. Then, on the affine chart "$x = 1$", the form f_6 may be written as a sum of a function in y and a function in z.

   In $O(q \log q)$ steps, for each of the two functions, we build up a table stating how many times it adopts each of its values. Again, we may restrict one of the tables to a fundamental domain of the Frobenius. We tabulate the quadratic character, too. After these preparations, less than q^2 additions suffice to determine the number of points.

   The advantage of a decoupled situation is, therefore, that an evaluation of a polynomial in $\mathbb{F}_{p^d}$ gets replaced by an addition.

Remark 4.1 Having implemented the point counting in `C`, these optimizations allow to determine the number of $\mathbb{F}_{3^{10}}$-rational points on a K3 surface $\mathscr{S}$ within half an hour (without decoupling) on an AMD Opteron processor.

In a decoupled situation, the number of $\mathbb{F}_{5^9}$-rational points may be counted within two hours. In a few cases, we determined the numbers of points over $\mathbb{F}_{5^{10}}$. This took around two days. Without decoupling, the same counts would have taken around one day or 25 days, respectively.

This shows, using the methods above, we may effectively compute the traces of $\phi_{\mathbb{F}_{p^d}} = \phi^d$ for $d = 1, \ldots, 9, (10)$.

An upper bound for rk Pic($\mathscr{S}_{\overline{\mathbb{F}}_p}$) having counted till $d = 10$.
We know that f_p, the characteristic polynomial of the Frobenius, has a zero at p since the pull-back of a line in $\mathbf{P}^2$ is a divisor defined over $\mathbb{F}_p$. Suppose, we determined $\mathrm{Tr}(\phi^d)$ for $d = 1, \ldots, 10$. We may achieve an upper bound for $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p})$ as follows.

i) First, assume the minus sign in the functional equation $(*)$. Then, f_p automatically has coefficient 0 at t^{11}. Therefore, the numbers of points counted suffice in this case to determine f_p, completely.

ii) Then, assume that, on the other hand, the plus sign is present in $(*)$. In this case, the data collected immediately allow to compute all coefficients of f_p except that at t^{11}. Use the known zero at p to determine that final coefficient.

iii) Use the numerical test, described below, to decide which sign is actually present.

iv) Factor $f_p(pt)$ into irreducible polynomials. Check which of the factors are cyclotomic polynomials and add their degrees. That sum is an upper bound for $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p})$. If step iii) had failed then one has to work with both candidates for f_p and deal with the maximum.

Verifying rk Pic($\mathscr{S}_{\overline{\mathbb{F}}_p}$) $= 2$ having counted till $d = 9$, only.
Assume, $\mathscr{S}$ is a K3 surface over $\mathbb{F}_p$ given by Construction 2.2.i) or ii). We, therefore, know that the rank of the Picard group is at least equal to 2. We suppose that the divisor constructed by pull-back splits already over $\mathbb{F}_p$. This ensures p is a double zero of f_p. There is the following method to verify $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p}) = 2$.

i) First, assume the minus sign in the functional equation $(*)$. This forces another zero of f_p at $(-p)$. The data collected are then sufficient to determine f_p, completely. The numerical test, described below, may indicate a contradiction.

Otherwise, the verification fails. (In that case, we could still find an upper bound for $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p})$ which is, however, at least equal to 4.)

ii) As we have the plus sign in $(*)$, the data immediately suffice to compute all coefficients of f_p with the exception of those at t^{10}, t^{11}, and t^{12}. The functional equation yields a linear relation for the three remaining coefficients of f_p. From the known double zero at p, one computes another linear condition.

iii) Let n run through all natural numbers such that $\varphi(n) \leq 20$. (The largest such n is 66.) Assume, in addition, that there is another zero of the form $p\zeta_n$. This yields further linear relations. Inspecting this system of linear equations, one either achieves a contradiction or determines all three remaining coefficients. In the latter case, the numerical test may indicate a contradiction.

If each value of n turned out to be contradictory then we found that $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p}) = 2$.

Consequently, the equality $\operatorname{rk}\operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p}) = 2$ may be effectively provable having determined $\operatorname{Tr}(\phi^d)$ for $d = 1,\ldots,9,(10)$.

A numerical test. Given a polynomial f of degree 22, we calculate all its zeroes as floating point numbers. If at least one of them is clearly not of absolute value p then f can not be the characteristic polynomial of the Frobenius for any K3 surface over $\mathbb{F}_p$.

Remarks 4.2 i) This approach will always yield an even number for the upper bound of the Picard rank. Indeed, the bound is

$$\begin{aligned}\operatorname{rk}\operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p}) \;\leq\; & \dim(H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l)) \\ & \qquad -\#\{\,\text{zeroes of } f_p \text{ which are not of the form } \zeta_n p\}\,.\end{aligned}$$

The relevant zeroes come in pairs of complex conjugate numbers. Hence, for a K3 surface the bound is always even.

ii) There is a famous conjecture due to John Tate [7] which implies that the canonical injection $c_1\colon \operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p}) \to H^2_{\text{ét}}(\mathscr{S}_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_l(1))$ maps actually onto the sum of all eigenspaces for the eigenvalues which are roots of unity. Together with the conjecture of J.-P. Serre claiming that the Frobenius operation on étale cohomology is always semisimple, this would imply that the bound above is actually sharp.

It is a somewhat surprising consequence of the Tate conjecture that the Picard rank of a K3 surface over $\overline{\mathbb{F}}_p$ is always even. For us, this is bad news. The obvious strategy to prove $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$ for a K3 surface S over $\mathbb{Q}$ would be to verify $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_p}) = 1$ for a suitable place p of good reduction. The Tate conjecture indicates that there is no hope for such an approach.

5. How to prove $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$

Using the methods described above, on one hand, we can construct even upper bounds for the Picard rank. On the other hand, we can generate lower bounds by explicitly stating divisors. In an optimal situation, this may establish an equality $\operatorname{rk}\operatorname{Pic}(\mathscr{S}_{\overline{\mathbb{F}}_p}) = 2$. How is it possible that way to reach Picard rank 1 for a surface defined over $\mathbb{Q}$?

For this, a trick due to R. van Luijk [5, Remark 2] is helpful.

Fact 5.1 (van Luijk) *Assume, we are given a K3 surface $\mathscr{S}^{(3)}$ over $\mathbb{F}_3$ and a K3 surface $\mathscr{S}^{(5)}$ over $\mathbb{F}_5$ which are both of geometric Picard rank 2. Suppose further that the discriminants of the intersection forms on $\operatorname{Pic}(\mathscr{S}^{(3)}_{\overline{\mathbb{F}}_3})$ and $\operatorname{Pic}(\mathscr{S}^{(5)}_{\overline{\mathbb{F}}_5})$ are essentially different, i.e. their quotient is not a perfect square in $\mathbb{Q}$.*

Then, every K3 surface S such that its reduction at 3 is isomorphic to $\mathscr{S}^{(3)}$ and its reduction at 5 is isomorphic to $\mathscr{S}^{(5)}$ is of geometric Picard rank one.

Proof. The reduction maps $\iota_p\colon \operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \to \operatorname{Pic}(S_{\overline{\mathbb{F}}_p}) = \operatorname{Pic}(\mathscr{S}^{(p)}_{\overline{\mathbb{F}}_p})$ are injective [3, Example 20.3.6]. Observe here, $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})$ is equal to the group of divisors on $S_{\overline{\mathbb{Q}}}$ modulo numerical equivalence.

This immediately leads to the bound $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \leq 2$. Assume, by contradiction, that equality holds. Then, the reductions of $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})$ are sublattices of maximal rank in both, $\operatorname{Pic}(S_{\overline{\mathbb{F}}_3}) = \operatorname{Pic}(\mathscr{S}^{(3)}_{\overline{\mathbb{F}}_3})$ and $\operatorname{Pic}(S_{\overline{\mathbb{F}}_5}) = \operatorname{Pic}(\mathscr{S}^{(5)}_{\overline{\mathbb{F}}_5})$.

The intersection product is compatible with reduction. Therefore, the quotients $\operatorname{Disc}\operatorname{Pic}(S_{\overline{\mathbb{Q}}})/\operatorname{Disc}\operatorname{Pic}(\mathscr{S}^{(3)}_{\overline{\mathbb{F}}_3})$ and $\operatorname{Disc}\operatorname{Pic}(S_{\overline{\mathbb{Q}}})/\operatorname{Disc}\operatorname{Pic}(\mathscr{S}^{(5)}_{\overline{\mathbb{F}}_5})$ are perfect squares. This is a contradiction to the assumption. $\square$

Remark 5.2 Suppose that $\mathscr{S}^{(3)}$ and $\mathscr{S}^{(5)}$ are K3 surfaces of degree two given by explicit branch sextics in $\mathbf{P}^2$. Then, using the Chinese Remainder Theorem, they can easily be combined to a K3 surface S over $\mathbb{Q}$.

If one of them allows a conic tangent in six points and the other a tritangent then the discriminants of the intersection forms on $\operatorname{Pic}(\mathscr{S}^{(3)}_{\overline{\mathbb{F}}_3})$ and $\operatorname{Pic}(\mathscr{S}^{(5)}_{\overline{\mathbb{F}}_5})$ are essentially different as shown in section 2.

Remark 5.3 Suppose S is a K3 surface over $\mathbb{Q}$ constructed that way. Then, S cannot be isomorphic, not even over $\overline{\mathbb{Q}}$, to a K3 surface $S' \subset \mathbf{P}^3$ of degree 4. In particular, the explicit examples, which we will describe in the next sections, are different from those of R. van Luijk [5].

Indeed, $\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = \mathbb{Z}\cdot\langle\mathscr{L}\rangle$ and $\deg S = 2$ mean that the intersection form on $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})$ is given by $\langle\mathscr{L}^{\otimes n}, \mathscr{L}^{\otimes m}\rangle = 2nm$. All self-intersection numbers of invertible sheaves on $S_{\overline{\mathbb{Q}}}$ are of the form $2n^2$ which is always different from 4.

6. An explicit K3 surface of degree two

Examples 6.1 We consider two particular K3 surfaces over finite fields.

i) By $\mathscr{X}^0$, we denote the surface over $\mathbb{F}_3$ given by the equation

$$\begin{aligned} w^2 &= (y^3 - x^2y)^2 + (x^2 + y^2 + z^2)(2x^3y + x^3z + 2x^2yz + x^2z^2 + 2xy^3 + 2y^4 + z^4) \\ &= 2x^5y + x^5z + x^4y^2 + 2x^4yz + x^4z^2 + x^3y^3 + x^3y^2z + 2x^3yz^2 + x^3z^3 \\ &\qquad + 2x^2y^3z + x^2y^2z^2 + 2x^2yz^3 + 2x^2z^4 + 2xy^5 + 2xy^3z^2 + 2y^4z^2 + y^2z^4 + z^6 . \end{aligned}$$

ii) Further, let $\mathscr{Y}^0$ be the K3 surface over $\mathbb{F}_5$ given by

$$w^2 = x^5y + x^4y^2 + 2x^3y^3 + x^2y^4 + xy^5 + 4y^6 + 2x^5z + 2x^4z^2 + 4x^3z^3 + 2xz^5 + 4z^6 .$$

Theorem 6.2 *Let S be any K3 surface over $\mathbb{Q}$ such that its reduction modulo 3 is isomorphic to $\mathscr{X}^0$ and its reduction modulo 5 is isomorphic to $\mathscr{Y}^0$. Then,* $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

Proof. We follow the strategy described in Remark 5.2. For the branch locus of $\mathscr{X}^0$, the conic given by $x^2 + y^2 + z^2 = 0$ is tangent in six points. The branch locus of $\mathscr{Y}_0$ has a tritangent given by $z - 2y = 0$. It meets the branch locus at $[1:0:0]$, $[1:3:1]$, and $[0:1:2]$.

It remains necessary to show that $\operatorname{rk}\operatorname{Pic}(\mathscr{X}^0_{\overline{\mathbb{F}}_3}) \leq 2$ and $\operatorname{rk}\operatorname{Pic}(\mathscr{Y}^0_{\overline{\mathbb{F}}_5}) \leq 2$. To verify these assertions, we used the methods described in section 4. We counted points over $\mathbb{F}_{3^d}$ and $\mathbb{F}_{5^d}$, respectively, for $d \leq 10$. For $\mathscr{Y}^0$, we could use the faster method since the sextic form on the right hand side is decoupled. $\square$

Corollary 6.3 *Let S be the K3 surface over $\mathbb{Q}$ given by*

$$\begin{aligned} w^2 \;=\; & -4x^5y+7x^5z+x^4y^2+5x^4yz+7x^4z^2+7x^3y^3-5x^3y^2z+5x^3yz^2+4x^3z^3 \\ & +6x^2y^4+5x^2y^3z-5x^2y^2z^2+5x^2yz^3+5x^2z^4-4xy^5+5xy^3z^2-3xz^5 \\ & -6y^6+5y^4z^2-5y^2z^4+4z^6\,. \end{aligned}$$

i) *Then,* $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

ii) *Further,* $S(\mathbb{Q}) \neq \emptyset$. *For example,* $[2\,;\,0:0:1] \in S(\mathbb{Q})$.

Remarks 6.4 i) For the K3 surface $\mathscr{X}^0$, our calculations show the following.

The numbers of the points defined over $\mathbb{F}_{3^d}$ for $d = 1,\dots,10$ are, in this order, 14, 92, 758, 6 752, 59 834, 532 820, 4 796 120, 43 068 728, 387 421 463, and 3 487 077 812. The traces of the Frobenius $\phi_{\mathbb{F}_{p^d}} = \phi^d$ on $H^2_{\text{ét}}(\mathscr{X}^0_{\overline{\mathbb{F}}_3}, \overline{\mathbb{Q}}_l)$ are equal to 4, 10, 28, 190, 784, 1 378, 13 150, 22 006, 973, and 293 410.

The sign in the functional equation is positive. For the decomposition of the characteristic polynomial f_p of the Frobenius, we find (after scaling to zeroes of absolute value 1)

$$\begin{aligned} (t-1)^2\,(3t^{20}+2t^{19}+2t^{18}+2t^{17}+t^{16}-2t^{13}-2t^{12}-t^{11}-2t^{10} \\ -t^9-2t^8-2t^7+t^4+2t^3+2t^2+2t+3)/3 \end{aligned}$$

with an irreducible polynomial of degree 20. The assumption of the negative sign leads to zeroes the absolute values of which range (without scaling) from 2.598 to 3.464.

ii) For the K3 surface $\mathscr{Y}^0$, our calculations yield the following results.

The numbers of points over $\mathbb{F}_{5^d}$ are, in this order, 41, 751, 15 626, 392 251, 9 759 376, 244 134 376, 6 103 312 501, 152 589 156 251, 3 814 704 296 876, and 95 367 474 609 376. The traces of the Frobenius on $H^2_{\text{ét}}(\mathscr{Y}^0_{\overline{\mathbb{F}}_5}, \overline{\mathbb{Q}}_l)$ are 15, 125, 0, 1 625, $-6\,250$, $-6\,250$, $-203\,125$, 1 265 625, 7 031 250, and 42 968 750.

The sign in the functional equation is positive. For the decomposition of the scaled characteristic polynomial of the Frobenius, we find

$$\begin{aligned} (t-1)^2\,(5t^{20}-5t^{19}-5t^{18}+10t^{17}-2t^{16}-3t^{15}+4t^{14}-2t^{13}-2t^{12}+t^{11} \\ +3t^{10}+t^9-2t^8-2t^7+4t^6-3t^5-2t^4+10t^3-5t^2-5t+5)/5\,. \end{aligned}$$

The assumption of the negative sign leads to zeroes the absolute values of which range (without scaling) from 3.908 to 6.398.

7. An explicit K3 surface of degree two given by a symmetric (3×3)-determinant

Examples 7.1 Consider the following two K3 surfaces over finite fields.

i) By $\mathscr{X}$, we denote the surface over $\mathbb{F}_3$ given by the equation $w^2 = f_6(x,y,z)$ for

$$
\begin{aligned}
f_6(x,y,z) &= \det \begin{pmatrix} 2xy+2y^2+yz & 2x^2+xy+xz+yz+2z^2 & 2x^2+xz+yz+z^2 \\ 2x^2+xy+xz+yz+2z^2 & 2x^2+xy & xy+y^2+yz+2z^2 \\ 2x^2+xz+yz+z^2 & xy+y^2+yz+2z^2 & 2x^2+2xy+2y^2+2yz \end{pmatrix} \\
&= 2x^6+2x^5y+2x^5z+2x^4y^2+x^4yz+x^3y^3+x^3yz^2+x^3z^3+2x^2y^4 \\
&\qquad +x^2y^3z+2x^2y^2z^2+xy^5+xy^2z^3+y^6+y^5z+y^2z^4+yz^5+2z^6 \,.
\end{aligned}
$$

ii) Further, let $\mathscr{Y}$ be the K3 surface over $\mathbb{F}_5$ given by $w^2 = f_6(x,y,z)$ for

$$
\begin{aligned}
f_6(x,y,z) &= \det \begin{pmatrix} 4x^2+4xz+y^2 & 2x^2+3z^2 & 4x^2+2xy+2xz+4y^2+3yz+2z^2 \\ 2x^2+3z^2 & 2x^2+4xy+4y^2+yz+3z^2 & 4xy+2xz+y^2+4yz+4z^2 \\ 4x^2+2xy+2xz+4y^2+3yz+2z^2 & 4xy+2xz+y^2+4yz+4z^2 & 4x^2+xz+3z^2 \end{pmatrix} \\
&= 4x^6+2x^5y+x^5z+x^4y^2+x^4z^2+x^3y^3+4x^3z^3 \\
&\qquad +2x^2y^4+2x^2z^4+4xy^5+xz^5+4z^6 \,.
\end{aligned}
$$

Theorem 7.2 *Let S be any K3 surface over $\mathbb{Q}$ such that its reduction modulo 3 is isomorphic to $\mathscr{X}$ and its reduction modulo 5 is isomorphic to $\mathscr{Y}$. Then,* $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

Proof. Consider the branch locus of $\mathscr{X}$. For the conic C, given by $x^2+xy+2xz+z^2 = 0$, there is the parametrization

$$q\colon u \mapsto [u^2 : 2 : (2u^2+2u)].$$

We find

$$f_6(q(u)) = (u+1)^2(u^5+u^4+u^3+u+1)^2,$$

i.e. C admits the property of being tangent in six points and the corresponding divisor on $\mathscr{X}$ splits already over $\mathbb{F}_3$. The branch sextic of $\mathscr{Y}$ has a degenerate tritangent given by $x = 0$.

To verify that $\operatorname{rk}\operatorname{Pic}(\mathscr{X}_{\overline{\mathbb{F}}_3}) \le 2$ and $\operatorname{rk}\operatorname{Pic}(\mathscr{Y}_{\overline{\mathbb{F}}_5}) \le 2$, again, we used the methods described in section 4. We counted points over $\mathbb{F}_{3^d}$, respectively $\mathbb{F}_{5^d}$, for $d \le 10$. Observe that, for $\mathscr{Y}$, we could use the faster method since the sextic form on the right hand side is decoupled. $\square$

Corollary 7.3 *Let S be the K3 surface over $\mathbb{Q}$ given by $w^2 = f_6(x,y,z)$ for*

$$
\begin{aligned}
f_6(x,y,z) &= \det \begin{pmatrix} -6x^2+5xy-6xz-4339y^2-5yz & 2x^2-5xy-5xz-150y^2-5yz-7z^2 & -x^2-3xy+7xz-6y^2-2yz+7z^2 \\ 2x^2-5xy-5xz-150y^2-5yz-7z^2 & 2x^2+4xy-6y^2+6yz+3z^2 & 4xy-3xz+y^2+4yz-z^2 \\ -x^2-3xy+7xz-6y^2-2yz+7z^2 & 4xy-3xz+y^2+4yz-z^2 & -x^2+5xy+6xz+5y^2+5yz+3z^2 \end{pmatrix} \\
&= 14x^6-118x^5y-64x^5z+8\,021x^4y^2+220x^4yz-114x^4z^2 \\
&\quad -20\,249x^3y^3-47\,700x^3y^2z-635x^3yz^2+4x^3z^3 \\
&\quad -64\,753x^2y^4-247\,925x^2y^3z+26\,045x^2y^2z^2-2\,745x^2yz^3-153x^2z^4 \\
&\quad -33\,821xy^5-107\,100xy^4z-463\,245xy^3z^2-62\,450xy^2z^3 \\
&\qquad -3\,075xyz^4-384xz^5 \\
&\quad +24\,025y^6-77\,345y^5z-143\,880y^4z^2-201\,885y^3z^3-39\,455y^2z^4 \\
&\qquad -1\,055yz^5-196z^6 \,.
\end{aligned}
$$

i) *Then,* $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

ii) *Further,* $S(\mathbb{Q}) \neq \emptyset$*. For example,* $[155\ ;\ 0:1:0] \in S(\mathbb{Q})$.

Remarks 7.4 i) For the K3 surface $\mathscr{X}$, our calculations show the following.

The numbers of the points defined over $\mathbb{F}_{3^d}$ for $d = 1, \ldots, 10$ are, in this order, 14, 88, 800, 6664, 59114, 531136, 4782344, 43029952, 387550223 and 3486755578. The traces of the Frobenius $\phi_{\mathbb{F}_{3^d}} = \phi^d$ on $H^2_{\text{ét}}(\mathscr{X}_{\overline{\mathbb{F}}_3}, \overline{\mathbb{Q}}_l)$ are 5, 7, 71, 103, 65, −305, −625, −16769, 129734, and −28823.

The decomposition of the scaled characteristic polynomial is

$$(t-1)^2(3t^{20}+t^{19}+2t^{18}+t^{16}+t^{15}+2t^{14}+2t^{13}+3t^{12} \\ +2t^{10}+3t^8+2t^7+2t^6+t^5+t^4+2t^2+t+3)/3\,.$$

By consequence, the geometric Picard rank is equal to 2.

ii) For the K3 surface $\mathscr{Y}$, our calculations yield the following results.

The numbers of points over $\mathbb{F}_{5^d}$ are, in this order, 33, 669, 15522, 391861, 9768668, 244132734, 6103019942, 152588860821, 3814709624898, and 95367420137974. The traces of the Frobenius on $H^2_{\text{ét}}(\mathscr{Y}_{\overline{\mathbb{F}}_5}, \overline{\mathbb{Q}}_l)$ are 8, 44, −103, 1236, 3043, −7891, −495683, 970196, 12359273, and −11502651.

The decomposition of the scaled characteristic polynomial is

$$(t-1)^2(5t^{20}+2t^{19}+t^{18}+5t^{17}+2t^{16}+2t^{15}+5t^{14}+8t^{13}+4t^{12}+2t^{11} \\ +8t^{10}+2t^9+4t^8+8t^7+5t^6+2t^5+2t^4+5t^3+t^2+2t+5)/5\,.$$

Consequently, the geometric Picard rank is equal to 2.

A probabilistic method to construct symmetric (3×3)-matrices with decoupled determinant.
A general ternary sextic has 28 coefficients. It is decoupled if 15 of these vanish. Thus, a randomly chosen sextic form in $\mathbb{F}_q[x,y,z]$ is decoupled with a probability of q^{-15}. This is too low for our purposes.

On the other hand, we can think of decoupling as solving an non-linear system of 15 equations in 36 variables. One could try to attack this system by a Gröbner base calculation. We use a mixture of both methods. More precisely, we do the following.

Method 7.5 We construct the matrix M in the particular form

$$M := \begin{pmatrix} a(x,y,z) & b(x,z) & c_1(x,y,z) \\ b(x,z) & c_2(x,y,z) & c_3(x,y,z) \\ c_1(x,y,z) & c_3(x,y,z) & d(x,z) \end{pmatrix}.$$

i) We choose the quadratic forms c_1, c_2, c_3, and d, randomly.

ii) In a second step, we have to fix the nine coefficients of the quadratic forms a and b. The coefficients of $\det M$ at $x^{6-i-j}y^iz^j$ for $i,j > 0$ are linear functions of the coefficients of a and b. Observe that the summand $-b^2d$ does not contribute to these critical coefficients.

Thus, we have to solve a system of 15 linear equations in nine variables. Naively, such a system is solvable with a probability of q^{-6}.

If it is not solvable then we go back to the first step.

Remarks 7.6 i) We randomly generated a sample of 30 surfaces over $\mathbb{F}_3$. For each of them, the branch locus was smooth and had passed the two tests described in section 3, to exclude the existence of a tritangent and to ensure there was exactly one conic over $\mathbb{F}_3$ tangent in six points.

We could establish the equality $\operatorname{rk}\operatorname{Pic}(\mathscr{X}_{\overline{\mathbb{F}}_3}) = 2$ in three of the examples. Example 7.1.i) reproduces one of them.

ii) Using the probabilistic method described above, we generated a sample of 50 surfaces over $\mathbb{F}_5$. We made sure that, for each of them, the branch sextic was smooth, had exactly one tritangent, and no conic over $\mathbb{F}_5$ tangent in six points. Further, it was decoupled by construction. It took `Magma` approximately one hour to generate that sample.

Having counted points over $\mathbb{F}_{5^d}$ for $d \leq 9$, we could establish the equality $\operatorname{rk}\operatorname{Pic}(\mathscr{Y}_{\overline{\mathbb{F}}_5}) = 2$ in two of the examples. For those, we determined, in addition, the numbers of points over $\mathbb{F}_{5^{10}}$. Example 7.1.ii) reproduces one of the two.

8. An explicit K3 surface of degree two given by a symmetric (6×6)-determinant

Examples 8.1 Consider the following K3 surfaces over finite fields.

i) By $\mathscr{X}'$, we denote the surface over $\mathbb{F}_3$ given by the equation $w^2 = f_6(x,y,z)$ for

$$f_6(x,y,z) = \det\left[x\begin{pmatrix} 0&0&0&0&2&2\\ 0&1&0&1&0&1\\ 0&0&2&1&1&2\\ 0&1&1&1&1&1\\ 2&0&1&1&2&2\\ 2&1&2&1&2&0 \end{pmatrix} + y\begin{pmatrix} 0&2&1&1&2&1\\ 2&2&2&2&2&2\\ 1&2&0&1&1&0\\ 1&2&1&1&1&1\\ 2&2&1&1&2&2\\ 1&2&0&1&2&0 \end{pmatrix} + z\begin{pmatrix} 2&1&1&1&2&0\\ 1&1&0&1&2&1\\ 1&0&2&1&1&2\\ 1&1&1&0&0&1\\ 2&2&1&0&2&2\\ 0&1&2&1&2&0 \end{pmatrix}\right]$$

$$\begin{aligned} &= x^6 + x^5y + 2x^5z + 2x^4y^2 + 2x^4yz + 2x^2y^3z + x^2z^4 \\ &\qquad + 2xy^5 + 2xy^4z + 2y^6 + 2y^5z + y^2z^4 + 2yz^5\,. \end{aligned}$$

ii) Further, let $\mathscr{Y}'$ be the K3 surface over $\mathbb{F}_5$ given by $w^2 = f_6(x,y,z)$ for

$$f_6(x,y,z) = \det\left[x\begin{pmatrix} 3&4&3&4&4&1\\ 4&3&0&2&1&0\\ 3&0&4&0&3&0\\ 4&2&0&2&1&3\\ 4&1&3&1&0&2\\ 1&0&0&3&2&1 \end{pmatrix} + y\begin{pmatrix} 0&0&3&3&0&1\\ 0&2&0&1&3&1\\ 3&0&5&0&3&5\\ 3&1&0&3&5&5\\ 0&3&3&5&0&1\\ 1&1&5&5&1&3 \end{pmatrix} + z\begin{pmatrix} 2&1&1&1&0&5\\ 1&0&4&4&4&4\\ 1&4&2&3&0&2\\ 1&4&3&1&2&2\\ 0&4&0&2&3&1\\ 5&4&2&2&1&0 \end{pmatrix}\right]$$

$$= 2x^6 + x^5y + 2x^4y^2 + 3x^4z^2 + x^3y^3 + 2x^3z^3 + x^2y^4 + 3x^2z^4 + 2xy^5 + 4z^6\,.$$

Theorem 8.2 *Let S be any K3 surface over $\mathbb{Q}$ such that its reduction modulo 3 is isomorphic to $\mathscr{X}'$ and its reduction modulo 5 is isomorphic to $\mathscr{Y}'$. Then,* $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

Proof. Consider the branch locus of $\mathscr{X}'$. For the conic C, given by $xz+y^2+2yz+2z^2=0$, there is the parametrization

$$q\colon u \mapsto [(2u^2+u+1) : u : 1].$$

We find

$$f_6(q(u)) = (u^2+u+2)^2(u^4+u+2)^2,$$

i.e. C admits the property of being tangent in six points and the corresponding divisor on $\mathscr{X}'$ splits already over $\mathbb{F}_3$. The branch sextic of $\mathscr{Y}'$ has a degenerate tritangent given by $x=0$.

To verify that $\operatorname{rk}\operatorname{Pic}(\mathscr{X}'_{\overline{\mathbb{F}}_3}) \leq 2$ and $\operatorname{rk}\operatorname{Pic}(\mathscr{Y}'_{\overline{\mathbb{F}}_5}) \leq 2$, again, we used the methods described in section 4. We counted points over $\mathbb{F}_{3^d}$ and $\mathbb{F}_{5^d}$, respectively, for $d \leq 10$. Observe, for $\mathscr{Y}'$, we could use the faster method since the sextic form on the right hand side is decoupled. $\square$

Corollary 8.3 *Let S be the K3 surface over $\mathbb{Q}$ given by $w^2 = f_6(x,y,z)$ for*

$$f_6(x,y,z) = \det\left[x\begin{pmatrix} -2382 & -21 & 3 & -6 & -1 & -4 \\ -21 & 28 & 0 & 7 & 6 & -5 \\ 3 & 0 & -1 & -5 & -2 & 5 \\ -6 & 7 & -5 & 7 & 1 & -2 \\ -1 & 6 & -2 & 1 & 5 & 2 \\ -4 & -5 & 5 & -2 & 2 & 6 \end{pmatrix} + y\begin{pmatrix} 0 & 5 & -2 & -2 & 5 & 1 \\ 5 & 2 & 5 & -4 & -7 & -4 \\ -2 & 5 & 0 & -5 & -2 & 0 \\ -2 & -4 & -5 & -2 & -5 & -5 \\ 5 & -7 & -2 & -5 & 5 & -4 \\ 1 & -4 & 0 & -5 & -4 & 3 \end{pmatrix} + z\begin{pmatrix} 2 & 1 & 1 & 1 & 5 & 0 \\ 1 & -5 & -6 & 4 & -1 & 4 \\ 1 & -6 & 2 & -2 & -5 & 2 \\ 1 & 4 & -2 & 6 & -3 & 7 \\ 5 & -1 & -5 & -3 & -7 & -4 \\ 0 & 4 & 2 & 7 & -4 & 0 \end{pmatrix}\right]$$

$$\begin{aligned} &= 76\,139\,167x^6 + 231\,184\,081x^5y + 210\,075\,725x^5z \\ &\quad + 25\,609\,337x^4y^2 + 487\,337\,315x^4yz - 314\,154\,987x^4z^2 \\ &\quad - 141\,937\,719x^3y^3 + 283\,035\,180x^3y^2z - 434\,149\,815x^3yz^2 - 5\,367\,468x^3z^3 \\ &\quad - 175\,763\,034x^2y^4 + 168\,686\,090x^2y^3z - 421\,490\,010x^2y^2z^2 \\ &\qquad\qquad + 160\,009\,155x^2yz^3 - 153\,566\,957x^2z^4 \\ &\quad - 90\,295\,273xy^5 + 175\,779\,575xy^4z - 285\,747\,180xy^3z^2 \\ &\qquad\qquad + 327\,585\,255xy^2z^3 - 215\,766\,345xyz^4 + 94\,479\,045xz^5 \\ &\quad + 133\,220y^6 + 31\,145y^5z + 380\,715y^4z^2 - 324\,195y^3z^3 - 476\,810y^2z^4 \\ &\qquad\qquad + 402845yz^5 - 174\,261z^6 . \end{aligned}$$

i) *Then,* $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) = 1$.

ii) *Further,* $S(\mathbb{Q}) \neq \emptyset$. *For example,* $[1286\,;\,1:1:1] \in S(\mathbb{Q})$.

Remarks 8.4 i) For the K3 surface $\mathscr{X}'$, our calculations show the following.

The numbers of the points defined over $\mathbb{F}_{3^d}$ for $d = 1,\ldots,10$ are, in this order, 12, 90, 783, 6534, 59697, 535329, 4793661, 43079526, 387521091, and 3487248045. The traces of the Frobenius $\phi_{\mathbb{F}_{p^d}} = \phi^d$ on $H^2_{\text{ét}}(\mathscr{X}'_{\overline{\mathbb{F}}_3}, \overline{\mathbb{Q}}_l)$ are 3, 9, 54, –27, 648, 3888, 10692, 32805, 100602, and 463644.

The decomposition of the scaled characteristic polynomial is

$$(t-1)^2(3t^{20}+3t^{19}+3t^{18}+2t^{17}+3t^{16}+2t^{15}-2t^{13}-3t^{12}-4t^{11}\\ -6t^{10}-4t^9-3t^8-2t^7+2t^5+3t^4+2t^3+3t^2+3t+3)/3\,.$$

Consequently, the geometric Picard rank is equal to 2.

ii) For the K3 surface $\mathscr{Y}'$, our calculations yield the following results.

The numbers of points over $\mathbb{F}_{5^d}$ are, in this order, 36, 666, 15 711, 391 706, 9 763 601, 244 152 021, 6 103 934 341, 152 589 189 186, 3 814 705 355 181, and 95 367 412 593 451. The traces of the Frobenius on $H^2_{\text{ét}}(\mathscr{Y}'_{\overline{\mathbb{F}}_5},\overline{\mathbb{Q}}_l)$ are 11, 41, 86, 1 081, $-2\,024$, 11 396, 418 716, 1 298 561, 8 089 556, and $-19\,047\,174$.

The decomposition of the scaled characteristic polynomial is

$$(t-1)^2(5t^{20}-t^{19}+t^{18}+2t^{17}+3t^{15}+t^{14}-2t^{13}+t^{12}-t^{11}\\ +2t^{10}-t^9+t^8-2t^7+t^6+3t^5+2t^3+t^2-t+5)/5\,.$$

By consequence, the geometric Picard rank is equal to 2.

A probabilistic method to construct symmetric (6×6)-matrices with decoupled determinant.

Method 8.5 a) We construct a symmetric (6×6)-matrix M_0 the entries of which are linear forms only in y and z. The goal is that its determinant is decoupled, i.e.

$$\det M_0 = ay^6+bz^6$$

for certain $a,b\in\mathbb{F}_q$, not both vanishing.

This leads to five conditions for the coefficients.

i) We choose all entries in M_0 randomly except for $(M_0)_{11}$.

ii) The determinant is linear in the coefficients of $(M_0)_{11}$. Therefore, we have a system of five linear equations in two variables. Such a system is solvable with a probability of q^{-3} which is enough for our purposes.

If there is no solution then we return to step i).

b) We construct M in the form

$$M := M_0+xA$$

for A a symmetric matrix with entries in $\mathbb{F}_q$.

i) First, look at the monomials xy^iz^{5-i} for $i=1,\dots,4$, only. To make their coefficients vanish leads to a system of four linear equations. In general, its solutions form a 17-dimensional vector space.

ii) For decoupling, there are six further coefficients which are required to be zero. This means, we are left with 17 parameters and six non-linear equations.

We choose the parameters randomly and iterate this procedure until a solution has been found. Naively, the probability to hit a solution is q^{-6}, each time.

Remarks 8.6 i) We randomly generated a sample of 50 surfaces over $\mathbb{F}_3$. For each of them, the branch sextic was smooth and had passed the two tests described in section 3, to exclude the existence of a tritangent and to ensure there was exactly one conic over $\mathbb{F}_3$ tangent in six points.

We established $\operatorname{rk}\operatorname{Pic}(\mathscr{X}'_{\overline{\mathbb{F}}_3}) = 2$ in eleven of the examples. Example 8.1.i) is one of them.

ii) Using the probabilistic method described above, we generated a sample of 120 surfaces over $\mathbb{F}_5$. For each of them, the branch sextic was decoupled, by construction. We made sure, in addition, that it was smooth, had exactly one tritangent, and no conic, defined over $\mathbb{F}_5$, which was tangent in six points. It should be remarked that it took `Magma` half a day to generate that sample.

Having counted points over $\mathbb{F}_{5^d}$ for $d \leq 9$, we could establish the equality $\operatorname{rk}\operatorname{Pic}(\mathscr{Y}'_{\overline{\mathbb{F}}_5}) = 2$ in three of the examples. For those, we determined, in addition, the numbers of points over $\mathbb{F}_{5^{10}}$. Example 8.1.ii) reproduces one of the three.

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