Lech's conjecture on deformations of singularities and second Harrison cohomology

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Abstract

Let B_0 be a local singularity of dimension d. Then we consider the problem of Lech, whether for every deformation $(A, m) \longrightarrow (B, n)$ of B_0 the inequality $H_A^{d+1} \leq H_B^1$ between the Hilbert functions is true, and give a positive answer in the case, that the formal versal deformation of B_0 is a base change of an algebraic family $(R, M) \longrightarrow (S, N)$, where R is regular and $\dim S = \dim R + d$.

So one should lift versal deformations in that way. There are obstructions against this in certain second Harrison cohomology groups.

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Introduction

In 1959 C. Lech [Le 59] stated the problem whether the multiplicities of local rings (A, m) and (B, n) being base, respectively total space, of a deformation $(A, m) \longrightarrow (B, n)$ of a local ring $B_0 = B/mB$ satisfy the inequality

$$e_0(A) \leq e_0(B). \tag{1}$$

Note that the only condition on such a homomorphism to be a deformation is its flatness.

A generalization of this is the analogous inequality

$$H_A^{d+i} \leq H_B^i \tag{2}$$

between sum transforms of the Hilbert functions (, where d denotes the dimension of the fiber B_0). Here sum transforms are defined inductively by

$$H_A^j(l) := \sum_{k=0}^{l} H_A^{j-1}(k)\,,$$

where H_A^0 is the usual Hilbert function

$$H_A^0(l) := \dim_{A/m} m^l/m^{l+1}\,.$$

The inequality between two functions $H, H' : \mathbf{N} \longrightarrow \mathbf{N}$ is always to be understood in its total sense, i.e. $H(l) \leq H'(l)$ for all l.

In 1970 H. Hironaka [Hi 70] asked whether inequality (2) is always true with $i = 1$, since that would simplify his proof of the existence of a resolution of singularities in characteristic zero [Hi 64].

These inequalities are established in very few cases, only. The most interesting result in that direction is due to Lech himself. It says that

$$H_A^1 \leq H_B^1$$

in the case, that the special fiber B_0 is a zero dimensional complete intersection [Le 64]. B. Herzog generalized this to the situation that B_0 corresponds to a regular point $[B_0]$ of the Hilbert scheme [He 90]. This includes all complete intersections and all singularities with embedding dimension less than 3 [Fo].

On the other hand, Larfeldt and Lech [LL] showed that the general problem (2) of Hironaka is equivalent to the following statement:

For every local ring A and every coheight one prime P in A the inequality

$$H_{A_P}^1 \leq H_A^0$$

is true.

This one and its immediate corollaries are usually referred as Bennett's inequality. Note that this problem can not be easy, since it generalizes Serre's result [Se], that the localization of a regular local ring by a prime ideal is again regular. It is solved in the case A is excellent ([Be], [Si]).

We note, that there is also a completely different approach to the Lech-Hironaka problem. One can consider singularities with tangentially flat deformations only as in [He 91]. A generalization of that may be found in the doctoral thesis of the author [J].

In this paper we give a weak generalization of the main Theorem 6 of [He 90]. There the Lech-Hironaka inequality is proved for base changes of deformations, i.e. flat local homomorphisms, with regular base. We will show, that the flatness assumption can be replaced by the weaker condition to have a fiber of the minimal possible dimension. So our goal is to lift the formal versal deformation of some singularity in such a way, that the base becomes regular. Of course, such a lift will be no more flat, but it is required to have a fiber of minimal dimension.

Then we restrict to the case of zero dimensional singularities. We show that, in this situation, the minimality condition on the dimension of the fiber is more or less equivalent to injectivity. In the last section we show what the "lift"-problem could have to do with second Harrison cohomology. When one tries to lift step by step there are obstructions against that in those cohomologies. Unfortunately, the author does not understand the behaviour of the obstructions, when such a step was done.

We shall use the conventions and notations of commutative algebra as in [Ma]. Further all local rings are assumed to be Noetherian. An A-algebra is a homomorphism of the ring A into some ring, a homomorphism of A-algebras is a commutative triangle. k will always denote a fixed ground field. Note that we use "local k-algebra" for algebras $k \longrightarrow (A, m)$, where (A, m) is local and $k \longrightarrow A/m$ is an isomorphism. In particular, "complete local k-algebras" form just the category $\hat{C}$ of [Schl]. By a deformation of a local k-algebra B_0 we mean a flat local homomorphism of local k-algebras with special fiber B_0.

At some point we will use the language of Schlessinger's paper [Schl]. Note, that we call the "pro- representable hull" of the deformation functor, he constructs, "formal versal embedded deformation". It is well known, and can easily be derived from the universal property of the Hilbert scheme, that the completed local ring of the Hilbert scheme at $[B_0]$ is nothing but the base of that formal versal embedded deformation of the singularity B_0.

At the end of the introduction the following principal remark: We consider only deformations $f : (A, m) \longrightarrow (B, n)$ of a local k-algebra, which are itself homomorphisms of local k-algebras, i.e. where A and B are equicharacteristic and f is residually rational. Using Cohen's structure theory one could really generalize that, at least one can replace "residually rational" by "residually separable". We will omit the proof for that, since it does not seem to make sense to consider the abstract situation, when almost nothing is known in the "geometric case".

1 Base changes of algebraic families with regular base

1.1 Remark. B. Herzog [He 90] proved, that Hironaka's inequality is true for base changes of flat homomorphisms, i.e. deformations, with regular base. In this section we will verify, that the flatness assumption can be replaced by the weaker condition, that the special fiber has the minimal possible dimension.

1.2 Proposition. *Let the commutative diagram*

$$\begin{array}{ccc} (R, M) & \longrightarrow & (S, N) \\ \downarrow & \square & \downarrow \\ (A, m) & \longrightarrow & (B, n) \end{array}$$

of local rings and local homomorphisms be cartesian, i.e. $B \cong A \otimes_R S$, and assume the following conditions.

1. The special fiber of $R \longrightarrow S$ has the minimal dimension, i.e.

$$\dim S = \dim R + \dim S/MS.$$

2. $R \longrightarrow A$ is residually rational, i.e. induces an isomorphism of the residue fields.

3. R is regular and contains a field.

Then

$$H_A^{d+1} \leq H_B^1 \ ,$$

where $d := \dim B/mB$ $(= \dim S/MS)$.

Proof. We start with several straightforward steps.

First step. *We may assume A to be an Artin local ring.* Note that this implies by Cohen's structure theory ([Ma], Theorem 28.3 or [EGA IV$_0$], §19), that A is even a finite algebra over its residue field k.

For proving

$$H_A^{d+1}(n) \leq H_B^1(n)$$

for arbitrary given n the local rings A and B can be replaced by A/m^{n+1} and $B/m^{n+1}B$, respectively.

Second step. *We may assume, that B, R and S are complete local rings.*

Replace the local rings of the diagram above by their completions. Since $A \otimes N$ is an n-primary ideal in $B = A \otimes_R S$, the canonical topology on B is that as a finite S-module. Therefore

$$B^\wedge = (A \otimes_R S)^\wedge = A \otimes_{R^\wedge} S^\wedge.$$

Third step. *In the cartesian diagram above we may replace R and S in such a way that $R \longrightarrow A$ becomes surjective.*

Adjoin indeterminates T_i to R and S, which are mapped to some system of generators of the maximal ideal m in A and consider the resulting commutative diagram

$$\begin{array}{ccc} R[[T]] & \longrightarrow & S[[T]] \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \quad ,$$

where T denotes $\{T_1, \ldots, T_t\}$.

Since A is Artin, $R[[T]] \longrightarrow A$ factors through $R_l := R[[T]]/(T)^l$ for some l. Therefore we see, using that $S[[T]]/(T)^l$ is a free S-module,

$$B \cong A \otimes_R S \cong A \otimes_{R_l} R_l \otimes_R S \cong A \otimes_{R_l} S[[T]]/(T)^l \cong$$

$$\cong A \otimes_{R_l} R_l \otimes_{R[[T]]} S[[T]] \cong A \otimes_{R[[T]]} S[[T]],$$

meaning that the new commutative square is cartesian, too.

We replace R and S by $R[[T]]$ and $S[[T]]$, respectively. Then a system of generators of m may be lifted to R. Since all rings are complete and $R \longrightarrow A$ is residually rational, this implies it is surjective.

Fourth step. *In the cartesian diagram above we may replace S and B such that S/MS becomes an Artin local ring.*

Choose some prime ideal P in S satisfying $MS \subseteq P$ and

$$\dim S/P = \dim S/MS \quad (= d).$$

Then the special fiber of the induced homomorphism $R \longrightarrow S_P$ becomes zero dimensional and the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S_P \\ \downarrow & & \downarrow \\ A & \longrightarrow & B_P \end{array}$$

is again cartesian: $B_P \cong B \otimes_S S_P \cong A \otimes_R S \otimes_S S_P \cong A \otimes_R S_P$. Further, B_P is a local ring as a factor of the local ring S_P and all the homomorphisms in the diagram above are local. That is trivial, except for $A \longrightarrow B_P$, and there it follows from the simple reason that m consists of nilpotent elements only, which cannot be mapped to units.

Since B is complete, one has Bennett's inequality (Lemma 1.3)

$$H^{d+1}_{B_P} \leq H^1_B.$$

So we have to prove $H^{d+1}_A \leq H^{d+1}_{B_P}$, for which $H^1_A \leq H^1_{B_P}$ would be sufficient, obviously.

Further it turns out that the dimension of the fiber of $R \longrightarrow S_P$ is minimal. Here that means simply $\dim S_P = \dim R$. But this is clear by $\dim S = \dim R + d$ and $\dim S/P = d$, when one notes that S is complete and, therefore, catenary.

We replace S and B by S_P and B_P, respectively.

Fifth step. *This is the key step. We will prove that $B = A \otimes_R S$ is a factor of $B' := A \otimes_k S$ in a very specific way.*

Note that B' is Noetherian as a finitely generated S-algebra. Let n' be a maximal ideal in B'. Then $(m) := m \otimes S \subseteq n'$, since m is nilpotent, and $n'/(m)$ is maximal in $B'/(m) = A \otimes_k S/m \otimes S \cong S$. Therefore,

$$n' = m \otimes S + A \otimes N,$$

which shows B' to be local.

Now let $\{x_1, \ldots, x_s\}$ be a regular system of parameters of R. We denote by a_i the image of $x_i \in R$ in A and put $d_i := 1 \otimes x_i - a_i \otimes 1 \in B'$. Then, using the fact that A is finite over k, one obtains

$$\begin{aligned} B &\cong A \otimes_R S \\ &\cong (A[[X]]/(X_1 - a_1, \ldots, X_s - a_s)) \otimes_R S \\ &\cong (A \otimes_k R/(1 \otimes x_1 - a_1 \otimes 1, \ldots, 1 \otimes x_s - a_s \otimes 1)) \otimes_R S. \end{aligned}$$

By construction, the two R-module structures on the left factor coincide, which implies

$$\begin{aligned} B &\cong A \otimes_k S/(1 \otimes x_1 - a_1 \otimes 1, \ldots, 1 \otimes x_s - a_s \otimes 1) \\ &= B'/(d_1, \ldots, d_s). \end{aligned}$$

Here we remark, that

$$\dim B'/(d_1, \ldots, d_s) = \dim B = 0$$

by our reduction steps before. On the other hand $\dim S = \dim R$, since the dimension of the fiber of $R \longrightarrow S$ is the minimal one and we reduced that fiber to be Artin, and, furthermore,

$$\dim B' = \dim A \otimes_k S = \dim A/m \otimes_k S = \dim S = \dim R,$$

when we use m is nilpotent. Altogether that means, that $\{d_1, \ldots, d_s\}$ is a system of parameters for B'.

Sixth step. *Now we are in the position to complete the proof in the same way as B. Herzog did in the flat case. Note that we give a more elementary proof here, which avoids the technical concept of tangential flatness.*

Let

$$C_i := B'/(d_1, \ldots, d_i).$$

By the fact that $\{d_1, \ldots, d_s\}$ is a system of parameters for B' we know

$$\dim C_i = s - i.$$

Therefore we may take a chain of prime ideals

$$P_0 \subseteq \ldots \subseteq P_s$$

in B' such that
i) $\dim B'/P_i = \dim C_i = s - i$,
ii) $P_{i+1} \supseteq (P_i, d_{i+1})$.
Then by Lemma 1.3

$$H^1_{(C_i)_{P_i}} \leq H^0_{(C_i)_{P_{i+1}}} \leq H^1_{(C_{i+1})_{P_{i+1}}},$$

hence

$$H^1_{B'_{P_0}} \leq H^1_{(B'/(d_1,\ldots,d_s))_{P_s}} \leq H^1_{(B'/(d_1,\ldots,d_s))} = H^1_B$$

by the previous step. So it would be sufficient to show

$$H^1_A \leq H^1_{B'_{P_0}}. \tag{11}$$

For that we identify S with its canonical image in $B' := A \otimes_k S$ and put $P := P_0 \cap S$. Then the canonical homomorphism

$$S_P \longrightarrow B'_{P_0}$$

is well defined, local and factors through $A \otimes_k S_P$, a ring being local with maximal ideal

$$M_P := m \otimes_k S_P + A \otimes PS_P$$

(use that m is nilpotent). The induced homomorphism

$$A \otimes_k S_P \longrightarrow (A \otimes_k S)_{P_0}$$

turns out to be local, when we note once more that m is nilpotent. But, on the other hand, the ring on the right is obtained from the ring on the left by a further localization. So this homomorphism is even an isomorphism. In particular,

$$H^1_{B'_{P_0}} = H^1_{A \otimes_k S_P}. \tag{12}$$

But now we see directly

$$\begin{aligned} H^1_{A\otimes_k S_P}(l) &= \ell(A \otimes_k S_P / M_P^{l+1}) \\ &= \ell(A \otimes_k S_P / (m \otimes S_P + A \otimes PS_P)^{l+1}) \\ &\geq \ell(A \otimes_k S_P / m^{l+1} \otimes S_P + A \otimes PS_P) \\ &= \ell(A/m^{l+1}) \\ &= H^1_A(l). \end{aligned}$$

Putting that together with (12) we obtain

$$H^1_A \le H^1_{B'_{P_0}},$$

being just the required inequality (11).

$\square$

1.3 Lemma. *Let (A,m) be a local ring and $x \in m$ be an element. Then*

$$H^0_A \le H^1_{A/xA}.$$

If, moreover, A is excellent (e.g. complete), then for any prime ideal $P \in \mathrm{Spec}(A)$

$$H^d_{A_P} \le H^0_A$$

($d := \dim A/P$).

Proof. The first statement is easily proved by the reader. Alternatively, see [Si], Theorem 1. The second part of the Lemma is just Bennett's inequality ([Be], Theorem (2)) in the improved version due to Singh (see [Si], p.202). For a comment on Singh's proof see [He 90], Proof of Lemma 2.

$\square$

1.4 Theorem. *Let (B_0, n_0) be a local k-algebra of dimension d. Suppose, that the formal versal deformation $(R', M') \longrightarrow (S', N')$ of its completion $B_0^\wedge$ is a base change of some local homomorphism $(R, M) \longrightarrow (S, N)$ of local k-algebras*

$$\begin{array}{ccc} (R,M) & \longrightarrow & (S,N) \\ \downarrow & \square & \downarrow \\ (R',M') & \longrightarrow & (S',N') \end{array} ,$$

where

1. *The special fiber of $R \longrightarrow S$ has minimal dimension, i.e.*

$$\dim S = \dim R + d$$

and

2. *R is regular.*

Then for every deformation $(A,m) \longrightarrow (B,n)$ of the local k-algebra B_0 the inequality

$$H^{d+1}_A \le H^1_B$$

is true.

Proof. First step. *We may assume the local k-algebras B_0, A and B to be complete.*

Replace A and B by their completions. Then the induced homomorphism $A^\wedge \longrightarrow B^\wedge$ is again flat and its fiber is $B^\wedge/m^\wedge B^\wedge = B_0^\wedge$. Of course, there is no effect on the Hilbert series.

Second step. *$A \longrightarrow B$ is a base change of the formal versal deformation of B_0.*

Schlessinger [Schl] calls the morphism $h_{R'} \longrightarrow D_{B_0/k}$, induced by $R' \longrightarrow S'$ the "pro-representable hull" of the deformation functor

$$D_{B_0/k} : \{\text{Artin local } k\text{-algebras (with residue field } k)\} \longrightarrow \{\text{Sets}\}$$

of B_0. By [Schl], Remark (2.4), the induced morphism

$$h^{\wedge}_{R'} = \mathrm{Hom}_{\mathrm{local}\, k-\mathrm{alg}}(R', \,.\,) \longrightarrow D^{\wedge}_{B_0/k}$$

between the canonical prolongations to

$$\{\text{complete (Noetherian) local } k\text{-algebras (with residue field } k)\}$$

is objectwise surjective.

Down the earth this means nothing but the existence of a cartesian diagram

$$\begin{array}{ccc} R' & \longrightarrow & S' \\ \downarrow & \square & \downarrow \\ A & \longrightarrow & B \end{array} \quad .$$

So our assumption gives another cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & \square & \downarrow \\ A & \longrightarrow & B \end{array}$$

and the clain comes from the Proposition above.

$\square$

2 Injectivity

2.1 Remark. By Theorem 1.4 it is our goal to lift the formal versal deformation of some singularity in such a way, that the base becomes regular. Of course, such a lift will no longer be flat, but it is required to have a fiber of minimal dimension.

This "lift"-problem does not seem to be easy and we will not be able to solve it in this paper. In order to come relatively close to it, we note, that a versal deformation $R/I \longrightarrow S'$ (, where R is regular), can, of course, always be lifted, for instance in a trivial way to $R \longrightarrow S'$. That's why in this section we will try to understand what it means for lifts to have a fiber of minimal dimension.

For that we will restrict to deformations of zero dimensional singularities, i.e. to the case $d = 0$. Note that this is not an essential restriction, since the general Lech-Hironaka problem (2) can be reduced to that situation using Bennett's inequality for complete local rings (Lemma 1.3, cf. fourth step of the proof of Proposition 1.2).

2.2 Fact. *Let (R, M) be a local ring and*

$$(R, M) \longrightarrow (S, N)$$

be a residually finite injective *local homomorphism of local rings, the special fiber of which is Artin. Then*

$$\dim S = \dim R.$$

2.3 Warning. May be, one would try to generalize that fact.

a) Without the assumption on the fiber to be Artin, the inequality

$$\dim S \geq \dim R$$

seems to be natural, and this one implies (for an integral domain R) even

$$\dim S \geq \dim R + \dim S \otimes_R Q(R)\,,$$

where $Q(R)$ denotes the quotient field of R, i.e. $S \otimes_R Q(R)$ is the generic fiber of $R \longrightarrow S$.

But these inequalities are wrong!

Proof. Put $R := k[Y,Z]_{(Y,Z)}$. Then the generic fiber of

$$R \longrightarrow R[[X]]$$

is *two*dimensional! (For $R = k[Y,Z]$ this is shown in [Ma], §15.2, Remark 2 and the same proof works in our situation.)

So we see there is a chain of prime ideals

$$P_0 \subsetneq P_1 \subsetneq P_2,$$

lying over the zero ideal in R. Then

$$R \hookrightarrow R[[X]]/P =: S$$

is still injective, but

$$\dim R = 2 \quad \text{and} \quad \dim S = 1.$$

Note that $R \hookrightarrow S$ is residually rational and R is regular. Further one can complete R and S without any effect on the injectivity or on the dimensions.

□

b) The assumption on the homomorphism to be residually finite is necessary, too. Consider the local homomorphism

$$i: \quad R := \mathbf{Q}[[X,Y]] \longrightarrow S := \mathbf{R}[[X]],$$

where $X \mapsto X$ and $Y \mapsto \pi X$. Then i is obviously injective and its special fiber is Artin. (It is even the field $\mathbf{R}$.) But

$$\dim R = 2 \quad \text{and} \quad \dim S = 1.$$

2.4 Proposition. *Let the commutative diagram*

$$\begin{array}{ccc} (R,M) & \longrightarrow & (S,N) \\ \downarrow & \square & \downarrow \\ (A,m) & \longrightarrow & (B,n) \end{array}$$

of local rings and local homomorphisms be cartesian and assume the following statements to be true.

1. $\dim B/mB \quad (= \dim S/MS) = 0.$
2. $R \hookrightarrow S$ *is injective and residually finite.*
3. $R \longrightarrow A$ *is residually rational.*
4. R *is regular and contains a field.*

Then

$$H_A^1 \le H_B^1 .$$

2.5 Theorem. *Let* (B_0, n_0) *be a local* k*-algebra of dimension* 0*. Suppose, that the formal versal deformation* $(R', M') \longrightarrow (S', N')$ *of its completion* $B_0^\wedge$ *is a base change of some local homomorphism* $(R, M) \longrightarrow (S, N)$ *of local* k*-algebras*

$$\begin{array}{ccc} (R, M) & \longrightarrow & (S, N) \\ \downarrow & \square & \downarrow \\ (R', M') & \longrightarrow & (S', N') \end{array} \quad ,$$

where

1. $R \hookrightarrow S$ *is injective*

and

2. R *is regular.*

Then for every deformation $(A, m) \longrightarrow (B, n)$ *of the local* k*-algebra* B_0 *the inequality*

$$H_A^1 \le H_B^1$$

is true.

2.6 Proof of the Theorem. By Theorem 1.4 we only have to show

$$\dim S = \dim R.$$

But this comes directly from the Fact.

$\square$

2.7 Proof of the Proposition. This is a direct corollary of Proposition 1.2 and the Fact.

$\square$

2.8 Proof of the Fact. First step. *We may assume* R *to be complete.*
Replace R and S by their completions. Then

$$R^\wedge \longrightarrow S^\wedge$$

is, of course, still injective and residually finite. Its fiber

$$S^\wedge / M^\wedge S^\wedge = (S/MS)^\wedge = S/MS$$

is Artin. There is no effect on the dimensions.

Second step. *S is a finite R-module.*
In the sense of M-adic topology

a) R is complete and separated

and

b) S is separated (, since it is even separated in the N-adic topology).

Further

c) S/MS is a finite module over R/M (, since it is an Artin ring and the homomorphism was residually finite).

Now the claim is implied by the Lemma below.

Third step. Since S is finite over R, it is an integral extension by injectivity. But

$$\dim S = \dim R$$

for integral extensions is standard [Ma].

$\square$

2.9 Lemma. *Let A be a ring, I be an ideal in A and M be an A-module. Assume, that, in the I-adic topology,*

a) A is complete

and

b) M is separated.

Assume further the

c) A/I-module M/IM to be finite.

Then M is a finite A-module.

Proof. Choose elements $m_1, \ldots, m_r \in M$, the residue classes of which generate M/IM. These elements generate a submodule $M' \subseteq M$ with

$$M' + IM = M.$$

Iterating we obtain for every k

$$M' + I^k M = M. \tag{21}$$

Therefore M' is dense in M, in the sense of the I-adic topology on M. Our goal is to show $M' = M$, which would give the claim.

For that we observe

$$I^k M' = I^k M \cap M'. \tag{22}$$

"$\subseteq$" trivial

"$\supseteq$" Let $x \in I^k M \cap M'$. By (21) we get $x \in I^k M' + I^{2k} M \cap M'$, i.e. there is $x_1 \in I^k M'$ such that

$$x \in x_1 + I^{2k} M \cap M'.$$

Inductively we obtain a sequence $\{x_i\}_{i \in \mathbf{N}}$ with $x_i \in I^{ik} M'$ and

$$x \in x_1 + x_2 + \ldots + x_l + I^{(l+1)k} M \cap M'.$$

Now M' is I-adically complete as a finite module over A. Therefore the infinite series $\sum_{i=0}^{\infty} x_i$, giving rise to a Cauchy sequence, converges in M'. Let

$$x' := x_1 + x_2 + \ldots \in I^k M'.$$

be its sum. Then

$$x - x' \in \bigcap_{l=1}^{\infty} I^{kl} M = 0$$

and therefore $x \in I^k M'$. (22) is proved.

So M' is complete in the topology induced by that on M. Combining this with density we see $M = M'$.

$\square$

3 Second Harrison cohomology

3.1 Generalities

3.1.1 Remark. In this section we go on considering the case $d = 0$ only. So we have a homomorphism

$$R/I \longrightarrow S\,,$$

where R is regular, and want to lift it to an injection.

$$\begin{array}{ccc} R & \hookrightarrow & S' \\ \downarrow & \square & \downarrow \\ R/I & \longrightarrow & S \end{array}$$

For that we have to understand the extensions of S.

Our idea is to lift step by step. Therefore we are interested in the most simple, so called singular extensions of rings.

3.1.2 Let S be a local k-algebra. A *singular extension* of S by the S-module M is a k-algebra S' being isomorphic to $S \oplus M$ as a k-vector space, where

1. $S' \longrightarrow S$ is a homomorphism of rings,
2. $(a, 0) \cdot (0, y) = (0, ay)$ for all $a \in S$ and $y \in M$

and

3. $M^2 = 0$.

A *morphism* of singular extensions of S by M is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & S' & \longrightarrow & S & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & S'' & \longrightarrow & S & \longrightarrow & 0 \end{array}\,,$$

where $S' \longrightarrow S''$ is a homomorphism of k-algebras. Such a morphism is automatically an isomorphism.

3.1.3 Remark. A singular extension S' of a local k-algebra S will be local again, since M is nilpotent and, therefore, contained in any prime ideal.

3.1.4 Remark. The multiplication in a singular extension of S by an S-module M is of the type

$$(a,x)\cdot(b,y) = (ab, ay + bx + f(a,b))\,, \tag{31}$$

where $f : S \times S \longrightarrow M$ is a k-bilinear form.

Of course, not every k-bilinear form f induces a ring structure that way. But one can easily write down conditions, necessary and sufficient for that.

First one observes we have an abelian group structure for addition and (31) guarantees distributivity.

Associativity. Explicit computations give

$$\begin{aligned}
[(a,x)\cdot(b,y)]\cdot(c,z) &= (abc\,,\ abz + acy + bcx + cf(a,b) + f(ab,c)) \\
(a,x)\cdot[(b,y)\cdot(c,z)] &= (abc\,,\ abz + acy + bcx + af(b,c) + f(a,bc))\,.
\end{aligned}$$

So the condition on f, necessary and sufficient for associativity, is

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0\,.$$

Such k-bilinear forms are usually called *Hochschild 2-cocycles.*

Commutativity. Obviously, the symmetry of f

$$f(a,b) = f(b,a)$$

is necessary and sufficient for commutativity of the ring defined by (31).

3.1.5 Remark. Different 2-cocycles f_1, f_2 can induce isomorphic singular extensions.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & S \oplus_{f_1} M & \longrightarrow & S & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & S \oplus_{f_2} M & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

The vertical homomorphism sends

$$(a,x) \mapsto (a, h(a) + x)\,,$$

where $h : S \longrightarrow M$ is a k-linear map. For this is a homomorphism of rings we get the following diagram

$$\begin{array}{ccccc}
(a,x) & \cdot & (b,y) & = & (ab, ay + bx + f_1(a,b)) \\
\big\downarrow\!\!\!\!{\scriptstyle\top} & & \big\downarrow\!\!\!\!{\scriptstyle\top} & & \big\downarrow\!\!\!\!{\scriptstyle\top} \\
 & & & & (ab, ay + bx + h(ab) + f_1(a,b)) \\
 & & & & \|! \\
(a, h(a) + x) & \cdot & (b, h(b) + x) & = & (ab, ay + bx + ah(b) + h(a)b + f_2(a,b))
\end{array}\ .$$

This gives

$$f_1(a,b) = f_2(a,b) + ah(b) - h(ab) + h(a)b\,,$$

i.e. the 2-cocycle f can just be changed by *Hochschild 2-coboundaries* in order to obtain isomorphic ring structures.

Note that Hochschild 2-coboundaries are automatically symmetric.

3.1.6 The one. Up to now we did not care about a neutral element for multiplication in the extensions defined by (31).

In order to make $(1,0)$ into the one we would need

$$(f(1,\,.\,)) = f(\,.\,,1) = 0\,.$$

But the cocycle condition, applied for $b = c = 1$ gives already

$$f(a,1) = a \cdot f(1,1)$$

and f can be changed by a coboundary, defined by h with $h(1) = -f(1,1)$.

Therefore extensions defined by (31) automatically have a one. Note that the k-linear map

$$k \longrightarrow S \oplus_f M\,, \qquad 1 \mapsto 1$$

is a homomorphism of rings, giving $S \oplus_f M$ the structure of a k-algebra.

So we have summarized the result, that singular extensions of the k-algebra S by the S-module M are *classified* by

$$\text{"}\textit{symmetric Hochschild } 2-\textit{cocyles}\text{"} \,/\, \text{"}\textit{Hochschild } 2-\textit{coboundaries}\text{"}\ ,$$

i.e. by the second Harrison cohomology $\mathrm{Har}^2_k(S,M)$.

3.1.7 Remark. All that about singular extensions of algebras and second Harrison cohomology can be found, for example, in [EGA IV_0, §18] in a very explicit way. Note that they use the name *second symmetric Hochschild cohomology*, denoted by $\mathrm{H}^2_k(S,M)^s$, for what we call second Harrison cohomology.

There are a number of other interesting tractises on that. See [CE] for a very solid introduction to various homology theories, [Ge] for Hochschild and Harrison cohomology in general and connections to deformation theory of algebras and [An] for a more abstract point of view on cohomology of commutative algebras.

3.2 The special case $M = k$

3.2.1 Remark. Here we deal with the easiest case of singular extensions. We consider "small" extensions by $M = S/N\,(= k)$. In that situation one can describe the cohomology group $\mathrm{Har}^2_k(S,k)$ explicitly and, therefore, give a necessary and sufficient criterion for injective (flat) local homomorphisms to be liftable to another injection by such a "small step".

3.2.2 Fact. *Let $R = k[[X_1, \ldots, X_r]]$ be a complete regular local k-algebra. Then there are no nontrivial singular extensions of R by the R-module k.*

$$\mathrm{Har}^2_k(R,k) = 0$$

Proof. Let $R' \longrightarrow R$ be some singular extension of R by k. Then R' is automatically complete as the following 5-lemma situation shows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow & & \| & & \\
0 & \longrightarrow & kR'^{\wedge} & \longrightarrow & R'^{\wedge} & \longrightarrow & R^{\wedge} & \longrightarrow & 0
\end{array}$$

Now choose preimages $X'_1, \dots, X'_r \in R'$ of $X_1, \dots, X_r$ and define a local homomorphism of local k-algebras

$$R = k[[X_1, \dots, X_r]] \longrightarrow R' \quad , X_i \mapsto X'_i .$$

Obviously this is a section and the extension must have been trivial.

$\square$

3.2.3 Remark. *Let $(R, M) = k[[X_1, \dots, X_r]]$ be a complete regular local k-algebra and $I \subseteq M$ some ideal. Then there is a canonical k-linear map*

$$(I/MI)^* \longrightarrow \mathrm{Har}_k^2(R/I, k) .$$

Construction. Consider $h \in (I/MI)^*$ as a k-linear map $R \longrightarrow k$ vanishing on MI and outside I. Then the k-bilinear form $\widetilde{f_h}$ defined by

$$\widetilde{f_h}(x, y) := -\overline{x}h(y) + h(xy) - h(x)\overline{y} ,$$

where $\overline{x}$ denotes the residue class of x in R/M, is a 2-coboundary on R. By $\mathrm{Har}_k^2(R, k) = 0$ it is even a symmetric 2-cocycle.

We claim, that $\widetilde{f_h}$ vanishes on $I \times R$ (and $R \times I$.)
Indeed, let $x \in I$ and $y \in M$ first. Then we have $\overline{x} = 0$, $\overline{y} = 0$ and $xy \in MI$, implying

$$\widetilde{f_h}(x, y) = -\overline{x}h(y) + h(xy) - h(x)\overline{y} = 0 .$$

In the second place, let $x \in I$ and $y \in k \subseteq R$. Then $\overline{x} = 0$ and $h(xy) = h(x)\overline{y}$ give

$$\widetilde{f_h}(x, y) = -\overline{x}h(y) + h(xy) - h(x)\overline{y} = 0 ,$$

too.

So $\widetilde{f_h}$ induces a 2-cocycle f_h on $R/I \times R/I$, i.e. a cohomology class $[f_h]$ in $\mathrm{Har}_k^2(R/I, k)$.

3.2.4 Proposition. *Let $(R, M) = k[[X_1, \dots, X_r]]$ be a complete regular local k-algebra and $I \subseteq M^2$ some ideal. Then the k-linear map, constructed above, is even an isomorphism.*

$$\mathrm{Har}_k^2(R/I, k) \xleftarrow{\cong} (I/MI)^*$$

Proof. Injectivity. We have

$$f_h(x, y) = -\overline{x}h(y) + h(xy) - h(x)\overline{y}$$

with $h \in (I/MI)^*$. Assume f_h is a 2-coboundary, i.e.

$$f_h(x, y) = -\overline{x}h'(y) + h'(xy) - h'(x)\overline{y} ,$$

where $h' \in (R/I)^*$. For $x, y \in M$ only the middle summands can be nonzero, i.e. $h = h'$ on M^2, in particular on I. But h' vanishes on I. Hence h must vanish on I, too, and $h = 0$.

Surjectivity. Let $[f] \in \mathrm{Har}_k^2(R/I, k)$ be a cohomology class and $f : R/I \times R/I \longrightarrow k$ be a corresponding 2-cocycle

$$f \in Z^2_{sym}(R/I, k)\,.$$

This one induces a 2-cocycle $\widetilde{f} : R \times R \longrightarrow k$ on R. By $\mathrm{Har}_k^2(R, k) = 0$ that must be even a 2-coboundary;

$$\widetilde{f}(x, y) = -\overline{x}\widetilde{h}(y) + \widetilde{h}(xy) - \widetilde{h}(x)\overline{y}$$

for some $\widetilde{h} \in R^*$.

We have necessarily $\widetilde{h}|_{MI} = 0$, since $\widetilde{f}$ vanishes for $x \in M$ and $y \in I$. So $\widetilde{h} \in (R/MI)^*$.

Further $h' \in (R/I)^*$ generates the zero class in $\mathrm{Har}_k^2(R/I, k)$. So one can change the k-linear map $\widetilde{h} : R/MI \longrightarrow k$ in such a way, that it vanishes outside I/MI.

$$h \in (I/MI)^*$$

□

3.2.5 Remark. This result is not surprising when one has Fact 3.2.2. It means, there is a trivial extension of R/I, corresponding to the zero class, being of type

$$R/I[[T]]/(T, M\mathrm{mod} I \cdot T) \longrightarrow R/I\,,$$

and the other extensions are restrictions of I "by length one" to some ideal J with $MI \subseteq J \subseteq I$.

3.2.6 Remark. Let $R = k[[X_1, \ldots, X_r]]$ be a complete regular local k-algebra and

$$g : R \xrightarrow{p} R/I \overset{g'}{\hookrightarrow} S$$

be some homomorphism. Then we are interested in lifts

$$(g, h) : R \longrightarrow S \oplus_f k\,,$$

where, of course, $h(I)$ is supposed to be nonzero in order to make the kernel properly smaller than I.

Note that

$$R/\ker(g, h) \longrightarrow S \oplus_f k$$

is really a lift of $R/I \xrightarrow{g'} S$ in the sense of sections 1 and 2, since $h(I) \neq 0$ implies that the extension by k is killed by base change to R/I.

3.2.7 Proposition. *There exists a lift, as it was described in the Remark above, if and only if the canonical map*

$$\mathrm{Har}_k^2(S, k) \longrightarrow \mathrm{Har}_k^2(R/I, k)$$

is nonzero.

Proof. Investigating the conditions, necessary and sufficient for (g,h) is a homomorphism of rings, we obtain the following diagram

$$\begin{array}{ccccc} x & \cdot & y & = & xy \\ \downarrow & & \downarrow & & \downarrow \\ & & & & (g(xy),h(xy)) \\ & & & & \overset{!}{\|} \\ (g(x),h(x)) & \cdot & (g(y),h(y)) & = & (g(xy),\overline{x}h(y)+h(x)\overline{y}+f(g(x),g(y))) \end{array},$$

using we are in residually rational situation. So the condition for (g,h) respects the multiplicative structures is

$$f(g(x),g(y)) = -\overline{x}h(y) + h(xy) - h(x)\overline{y}\,. \tag{32}$$

From 3.1.6 we know that we can restrict to 2-cocycles, where

$$f(1,\,.\,) = 0\,.$$

Then $(1,0)$ is the neutral element for multiplication in $S \oplus_f M$. So the condition on (g,h), to respect the one, is simply

$$h(1) = 0\,.$$

But (32), applied to $x = y = 1$ gives just that relation.

So (32) is the only condition on (g,h) to be a homomorphism of local k-algebras. It means that there has to be a cohomology class $[f] \in \mathrm{Har}_k^2(S,k)$, such that $g'^*(f) \in \mathrm{Har}_k^2(R/I,k)$ is induced by some $h \in R^*$ (, where $h(I) \neq 0$). But by Proposition 3.2.4 every cohomology class in $\mathrm{Har}_k^2(R/I,k)$ comes from such an $h \in (I/MI)^*$. So we simply need

$$0 \neq h \in (I/MI)^* \cong \mathrm{Har}_k^2(R/I,k)$$

belonging to the image of g'^*.

That is just the assertion.

$\square$

3.2.8 Remark. Let R and T be complete *regular* local k-algebras and

$$\tilde{g} : R \longrightarrow T$$

be some local homomorphism. Further consider some ideal $J \subseteq T$, let $I := J \cap R$ be its restriction to R and let

$$g : R/I \hookrightarrow T/J$$

be the canonical injection. Then, in order to lift g, we need

$$(I/MI \longrightarrow J/NJ) \neq 0\,,$$

i.e.

$$(I =)\, J \cap R \neq NJ \cap R\,.$$

Then one has to try to lift again and hopes to end "at infinity" at the base R. Unfortunately, the author does not understand the behaviour of that condition under lifts. He would suggest other people to tackle this problem. May be one can prove substancially new results on the Lech-Hironaka conjecture.

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