On the quasi group of a cubic surface over a finite field

Andreas-Stephan Elsenhans[a], Jörg Jahnel[b]

[a] *Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany*
[b] *Département Mathematik, Universität Siegen, Walter-Flex-Straße 3, D-57068 Siegen, Germany*

Abstract

We study the Mordell-Weil group $\mathrm{MW}(V)$ for cubic surfaces V over finite fields that are not necessarily irreducible and smooth. We construct a surjective map from $\mathrm{MW}(V)$ to a group that can by computed explicitly. For $\#\mathrm{MW}(V)$, this yields a lower bound, which is (often but) not always trivial. To distinguish cases, we follow the classification of cubic surfaces, originally due to Schläfli and Cayley.

On the other hand, we describe an algorithm that a priori gives an upper bound for $\mathrm{MW}(V)$. We report on our experiments for "randomly" chosen surfaces of the various types, showing that in all but one case lower and upper bounds agree.

Finally, we give two applications to the number field case. First, we prove that the number of generators of $\mathrm{MW}(V)$ is unbounded. A second application explains why, for many reduction types, the Brauer-Manin obstruction may not distinguish points reducing to the smooth part.

Key words: Cubic surface, Schläfli-Cayley classification, Quasi group, Mordell-Weil group, Finite field, Suslin's homology
2000 MSC: 11G25, 11D25, 11G35, 14J26

1. Introduction

The quasi group of a cubic surface.

1.1. — According to Yu. I. Manin [Ma1], a cubic surface V carries a structure of a *quasi group*. For us, this shall simply mean the collinearity relation.

Definition. Let V be a cubic surface over a finite field K. Then by the *quasi group* associated to V, we mean the following ternary relation on $V^{\mathrm{reg}}(K)$.

$$[P_1, P_2, P_3] :\Longleftrightarrow P_1, P_2, P_3 \in V^{\mathrm{reg}}(K), \text{ intersection points of } V \text{ with a line } l, l \text{ not contained in } V\,.$$

Here, as usual, intersection points are taken with multiplicities.

1.2. Definition. — Let $(\Gamma, [\,])$ be a quasi group and $(G, +)$ be an abelian group. By a homomorphism $g\colon \Gamma \to G$, we mean a mapping satisfying the following condition.

There exists an element $h \in G$ such that for all triples $(P_1, P_2, P_3) \in \Gamma^3$ fulfilling $[P_1, P_2, P_3]$,

$$g(P_1) + g(P_2) + g(P_3) = h\,.$$

Email addresses: `Stephan.Elsenhans@uni-bayreuth.de` (Andreas-Stephan Elsenhans), `~jjahnel@mathematik.uni-siegen.de` (Jörg Jahnel)

URL: `http://www.staff.uni-bayreuth.de/~btm216` (Andreas-Stephan Elsenhans), `http://www.uni-math.gwdg.de/jahnel` (Jörg Jahnel)

1.3. Lemma-Definition. —— *The category of all homomorphisms from a quasi group* Γ *to abelian groups has an initial object. The corresponding abelian group is* $\underline{\Gamma} := \mathbb{Z}\Gamma/N$*, for* N *the subgroup generated by* $1\cdot P_1 + 1\cdot P_2 + 1\cdot P_3 - 1\cdot P'_1 - 1\cdot P'_2 - 1\cdot P'_3$ *for all* $[P_1, P_2, P_3]$ *and* $[P'_1, P'_2, P'_3]$.

The group $\underline{\Gamma}$ *carries a surjective augmentation homomorphism* $s\colon \underline{\Gamma} \to \mathbb{Z}$*. We will call* $\ker s$ *the abelian* group associated *with* Γ. □

1.4. Definitions. —— Let V be a cubic surface over a field K.

i) We will call the abelian group associated with the quasi group $V^{\mathrm{reg}}(K)$ the *Mordell-Weil group* of V. It will be denoted by $\mathrm{MW}(V)$.

ii) Two points $P_1, P_2 \in V^{\mathrm{reg}}(K)$ will be said to be *equivalent* if $[P_1] - [P_2] = 0 \in \mathrm{MW}(V)$.

1.5. Remark. —— This definition coincides with the one used by Yu. I. Manin in [Ma2], but not with the one suggested in [Ma1]. The difference is whether three points on a line lying entirely on the surface cause a relation or not. If there is no K-rational line contained in V then the two definitions agree. In general, the Mordell-Weil group considered in this article canonically surjects to the one from [Ma1].

1.6. Example. —— Let V be the *Cayley cubic* given by the equation

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$$

in $\mathbf{P}^3$ over a field K. Then for a non-singular point $P = (x_0 : x_1 : x_2 : x_3) \in V(K)$, either no coordinate vanishes or exactly two of them. Accordingly, define $c\colon V(K) \to K^*$ by

$$c(P) := \begin{cases} x_0x_1x_2x_3 & \text{if } x_i \neq 0, i = 1, \ldots, 4\,, \\ -\prod\limits_{i\colon x_i\neq 0} x_i & \text{otherwise}\,. \end{cases}$$

Further, let l be a line in $\mathbf{P}^3$ not contained in V and denote by P_1, P_2, and P_3 the intersection points with V, taken with multiplicity. Suppose P_1, P_2, and P_3 to be non-singular and K-rational.

Then $c(P_1)c(P_2)c(P_3)$ is a square in K.

Proof. This observation is easily checked by calculations in `maple`, treating the possible cases separately. Let us present the generic case of a line through three points with all coordinates different from zero. We parametrize the line by $t \mapsto (a_0 + tb_0 : a_1 + tb_1 : a_2 + tb_2 : a_3 + tb_3)$ for $a_i, b_i \in K$. The product $c(P_1)c(P_2)c(P_3)$ then evaluates to

$$\frac{1}{(b_0b_1b_2 + b_0b_1b_3 + b_0b_2b_3 + b_1b_2b_3)^4} \prod_{0\leq i<j\leq 3} (a_ib_j - a_jb_i)^2\,.$$ □

The map c therefore induces a homomorphism of groups

$$\overline{c}\colon \mathrm{MW}(V) \longrightarrow K^*/(K^*)^2\,.$$

Further, this homomorphism is a surjection. Indeed, to get the square class of $x \in K^*$, take the point $(1 : (-x) : 0 : 0) \in V(K)$. If the base field is not of characteristic two and not $\mathbb{F}_3$ then the points

$$((-2x) : (-2x) : (x + 1) : x(x + 1)) \in V(K)$$

for $x \neq 0, -1$ have non-zero coordinates and yield every square class.

For instance, when V is the Cayley cubic over $\mathbb{Q}$, the group $\mathrm{MW}(V)$ has a surjection to $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. In particular, is not finitely generated.

On the other hand, for the Cayley cubic over a finite field $\mathbb{F}_q$ of odd characteristic, we have a surjection $\mathrm{MW}(V) \twoheadrightarrow \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 = \mathbb{Z}/2\mathbb{Z}$. There are (at least) two different kinds of smooth points on V. Two points $P_1, P_2 \in V^{\mathrm{reg}}(\mathbb{F}_q)$ may be equivalent only if $c(P_1)c(P_2)$ is a square.

The purpose of this article is to investigate this phenomenon more systematically. It seems natural to ask whether $\overline{c}$ is a bijection and whether it may be somehow generalized to an arbitrary cubic surface over $\mathbb{F}_q$.

The results.

1.7. —— In this article, for V a cubic surface over a finite field $\mathbb{F}_q$, we will compare $\mathrm{MW}(V)$ with a group more tractable from the theoretical point of view. For V geometrically irreducible and not a cone, this will be $\mathrm{A}_0(V^{\mathrm{reg}})$, the degree-0 part of Suslin's homology group $h_0(V^{\mathrm{reg}})$. We will establish a canonical homomorphism

$$\pi_V\colon\ \mathrm{MW}(V) \longrightarrow \mathrm{A}_0(V^{\mathrm{reg}}),$$

which we will prove to be surjective for $q > 23$. Further, we will show that $\mathrm{A}_0(V^{\mathrm{reg}}) = \mathbb{Z}/2\mathbb{Z}$ when V is of types $4A_1$, $A_3 + 2A_1$, or $A_5 + A_1$, $\mathrm{A}_0(V^{\mathrm{reg}}) = \mathbb{Z}/3\mathbb{Z}$ when V is of type $3A_2$ and some extra condition is fulfilled, and $\mathrm{A}_0(V^{\mathrm{reg}}) = 0$, otherwise. In the case of a reducible cubic surface or a cone, we will describe a surjective map from $\mathrm{MW}(V)$ in an elementary manner.

Finally, we will report experimental evidence for $\mathrm{MW}(V)$ and $\mathrm{A}_0(V^{\mathrm{reg}})$ being actually isomorphic as long as the base field is not too small.

1.8. Plan of the article. —— We will start section 2 by recalling the classification of cubic surfaces. After this, we will consider two degenerate cases at first, the situation of a cone and the reducible case. In the cone case, there is the surjection to the Mordell-Weil group of the underlying curve. In the reducible case, there is a surjection $\mathrm{MW}(V) \twoheadrightarrow \mathbb{Z}^{k-1}$ that simply distinguishes the k components. Furthermore, in Example 2.3.2, we will present one reducible type where not all points on one component are equivalent. The argument is similar to that for the standard Cayley cubic, cf. Example 1.6.

Then, in section 3, we will construct the homomorphism $\pi_V\colon\ \mathrm{MW}(V) \to \mathrm{A}_0(V^{\mathrm{reg}})$. After this, we will establish the isomorphisms

$$\mathrm{A}_0(V^{\mathrm{reg}}) \cong \pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}})^{\mathrm{ab}} \cong [(\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_{\text{prime to } p} \otimes_{\mathbb{Z}} \mu_\infty^\vee)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}]^\vee .$$

The remaining cases of the classification of cubic surfaces are then dealt with as follows. For cubic ruled surfaces, we observe that $\pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}}) = 0$. Finally, for each of the remaining cases of the classification of cubic surfaces, we will compute the torsion of the geometric Picard group.

At the end of the main body of the article, we will describe our experiments comparing $\mathrm{MW}(V)$ and $\mathrm{A}_0(V^{\mathrm{reg}})$ in a large sample of examples. An appendix is devoted to efficient algorithms for the computation of $\mathrm{MW}(V)$ for concrete cubic surfaces. Although our algorithms a priori yield only an upper bound for $\mathrm{MW}(V)$, it turned out in practice that the bounds are sharp.

2. Elementary cases

2.1. The classification of cubic surfaces

2.1.1. Proposition (L. Schläfli and A. Cayley)**.** —— *Over an algebraically closed field of characteristic $\neq 2$, let V be an integral cubic surface. Then exactly one of the following is true.*

I) *V is a normal cubic surface. Then it is either*

i) *in one of the 21 classes of surfaces with finitely many double points, listed in [Do, Table 9.1]. This includes the case of a smooth cubic surface.*

ii) *Or the cone over a smooth cubic curve C.*

II) *V is a non-normal, irreducible cubic surface. Then it is either*

i) *a cubic ruled surface. There are two types of those, ordinary and Cayley's cubic ruled surfaces.*

ii) *Or the cone over a singular cubic curve. This might be a cubic with a self-intersection or a cusp.*

Proof. This classification was given by L. Schläfli in [Schl], back in 1858. An independent presentation was given by A. Cayley [Ca]. There is a modern proof due to I. Dolgachev [Do, Section 9.2]. For normal cubic surfaces, Dolgachev's proof actually works in arbitrary characteristic. Further, the list after [Do, Lemma 9.2.5] shows that each type may be realized over the prime field. The reader should also consult [BW] where, for many of the types, the dimension of the moduli stack is discussed. $\square$

2.1.2. — In the situation of a finite base field, the classification of geometrically integral cubic surfaces is actually a little finer.

I.i) Among these types, $2A_1$, $3A_1$, $2A_2$, $A_2 + 2A_1$, $4A_1$, $2A_2 + A_1$, $A_3 + 2A_1$, and $3A_2$ have symmetries. Hence, the Frobenius may permute the singular points. For example, the type $4A_1$ breaks into five subtypes according to the possible operations. All in all, when allowing singularities that are defined over extensions of the ground field, the number of types rises from 21 to 34.

II.i) An ordinary cubic ruled surface may have its normal form $xz^2 + yw^2 = 0$ only over a quadratic extension. This causes a third type of cubic ruled surfaces over a finite field.

II.ii) In the case of the cone over a cubic curve with self-intersection, there are two variants as to whether the two tangent directions at the point of intersection are defined over the ground field or not.

2.1.3. Remark (Reducible cubic surfaces)**.** — We will restrict ourselves to reduced cubic surfaces. In other words, the following types of reducible surfaces are allowed.

i) A reducible cubic surface might consist of a quadric and a plane. There are four cases where the quadric is nondegenerate. In fact, the quadric may split over the ground field or not and the plane may be tangent or not. There are four more cases when the quadric is a cone. The intersection with the plane might be a conic, two lines, a double line, or a point.

ii) Finally, the surface might be reducible into three planes. There are two cases as to whether their intersection is a point or a line. Observe that it is possible that the decomposition into three planes is defined only after a finite field extension.

2.2. Cones

2.2.1. — Let $C \subset \mathbf{P}^2$ be a reduced cubic curve over a field K. Then in a manner analogous to the surface case, there are a quasi group structure on C and the *Mordell-Weil group* $\mathrm{MW}(C)$. This group is known in every case.

i) It may happen that C is reducible and all K-rational smooth points are contained in one component being a line. Then the quasi group structure is empty and $\mathrm{MW}(C) = \ker(\mathrm{sum}\colon \mathbb{Z}[C(K)] \to \mathbb{Z})$. This degenerate case automatically occurs whenever C contains a line defined over a proper extension of K. Otherwise,

ii) $\mathrm{MW}(C) = J(C)(K)$ for C smooth. Here, $J(C)$ denotes the Jacobian of C.

iii) $\mathrm{MW}(C) = K^+$ if C is a cubic curve with a cusp.

iv) If C is a cubic curve with a node then $\mathrm{MW}(C) = K^*$ in the case that the two tangent directions at the node are defined over K. If the tangent directions are defined over the quadratic extension F/K then $\mathrm{MW}(C) = \ker(\mathrm{N}\colon F^* \to K^*)$.

v) When C is reducible into a line and a conic then $\mathrm{MW}(C) = \ker(\mathrm{N}\colon F^* \to K^*) \oplus \mathbb{Z}$, $\mathrm{MW}(C) = K^+ \oplus \mathbb{Z}$ or $\mathrm{MW}(C) = K^* \oplus \mathbb{Z}$ depending on whether, over K, there are no, one, or two points of intersection.

vi) When C is reducible into three components then $\mathrm{MW}(C) = K^+ \oplus \mathbb{Z}^2$ or $\mathrm{MW}(C) = K^* \oplus \mathbb{Z}^2$ depending on whether the three points of intersection coincide or not.

Proof. Parts ii), iii), and iv) are standard, cf. [Sil, Propositions III.2.2 and III.2.5]. The other assertions are elementary, but have to be checked, one by one.

For example, let us explain the subcase of vi), where there are three points of intersection. The summand $\mathbb{Z}^2 \cong \ker(\mathrm{sum}\colon \mathbb{Z}^3 \to \mathbb{Z})$ collects the three augmentation maps. The summand K^* appears as the collinearity of three points on a triangle is characterized multiplicatively, according to Menelaus' Theorem [VY, Theorem 22]. $\square$

2.2.2. Lemma (Cones)**. —–** *For V a cone over a cubic curve C, we have a canonical surjection* $\mathrm{MW}(V) \to \mathrm{MW}(C)$.

Proof. Denote by $\pi\colon V^{\mathrm{reg}} \to C^{\mathrm{reg}}$ the canonical projection. If $P_1, P_2, P_3 \in V^{\mathrm{reg}}$ are collinear then $\pi(P_1), \pi(P_2), \pi(P_3) \in C^{\mathrm{reg}}$ are collinear, too. Thus, π induces a homomorphism $\pi_*\colon \mathrm{MW}(V) \to \mathrm{MW}(C)$, which is clearly surjective. $\square$

2.3. Reducible cubic surfaces

2.3.1. —– Let V be a reducible cubic surface over a field K. Then there are two essentially different cases.

i) There are two irreducible components, a plane E and a quadric, but the quadric consists of two planes defined over a quadratic extension. Then only the plane E contains K-rational smooth points. We have an empty quasi group structure and $\mathrm{MW}(V) = \ker(\mathrm{sum}\colon \mathbb{Z}[E(K)] \to \mathbb{Z})$.

ii) Otherwise, when V decomposes into $k = 2, 3$ components, there is a canonical surjection $\mathrm{MW}(V) \twoheadrightarrow \ker(\mathrm{sum}\colon \mathbb{Z}^k \to \mathbb{Z}) \cong \mathbb{Z}^{k-1}$.

There is one very interesting situation where this surjection is systematically non-bijective. Consider the following example.

2.3.2. Example. —– Over a finite field $\mathbb{F}_q$ of characteristic $\neq 2$, let V be a reducible cubic surface consisting of a nondegenerate quadratic cone Q and a plane E. Suppose that E does not meet the vertex of Q. Then there is a canonical surjection

$$\mathrm{MW}(V) \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\,.$$

Proof. The homomorphism to $\mathbb{Z}$ is that from 2.3.1.ii). It remains to construct the homomorphism to $\mathbb{Z}/2\mathbb{Z}$.

For this, we fix coordinates such that the cusp is in $(1:0:0:0)$ and the plane E is given by $x = 0$. Further, we assume without restriction that the plane "$y = 0$" is tangent to the cone Q. Then the cone is given by, say, $yz + Lw^2 = 0$ for $L \neq 0$. The whole cubic surface has the equation

$$x(yz + Lw^2) = 0\,.$$

On the plane E, we define the homomorphism $\pi\colon \mathrm{MW}(V) \to \mathbb{Z}/2\mathbb{Z}$ simply as $\chi_2(L(yz + Lw^2))$ for χ_2 the quadratic character on $\mathbb{F}_q^*$. On the cone "$yz + Lw^2 = 0$", we take $\chi_2(xy)$, respectively $\chi_2(-Lxz)$ when $y = 0$.

We have to show that this definition is indeed compatible with the quasi group structure. For this, let $P_1 = (x:y:z:w) \in Q(\mathbb{F}_q)$, $P_2 = (x':y':z':w') \in Q(\mathbb{F}_q)$, and $P_3 = (0:y'':z'':w'') \in E(\mathbb{F}_q)$ be three collinear points. Then we clearly have $(0 : y'' : z'' : w'') = (0 : (x'y - xy') : (x'z - xz') : (x'w - xw'))$. Furthermore,

$$\begin{aligned}(xy)(x'y')L(y''z'' + Lw''^2) &= Lxx'yy'[(x'y - xy')(x'z - xz') + L(x'w - xw')^2] \\ &= -Lxx'yy'[xx'(yz' + y'z + 2Lww')] \\ &= -Lx^2x'^2(-Ly^2w'^2 - Ly'^2w^2 + 2Lyy'ww') \\ &= L^2x^2x'^2(yw' - y'w)^2\end{aligned}$$

is a square. Hence, when $y, y' \neq 0$, we see that $\pi(P_1) + \pi(P_2) + \pi(P_3) = 0$. The remaining cases are treated analogously. □

3. Irreducible cubic surfaces not being cones

When a cubic surface V is irreducible, but geometrically reducible, then it consists of three planes acted upon transitively by the Galois group. In this case, $V^{\mathrm{reg}}(K) = \emptyset$ and, therefore, $\mathrm{MW}(V) = 0$. Thus, we may restrict ourselves to the geometrically irreducible case.

3.1. Suslin's singular homology group h_0

3.1.1. — For a scheme of finite type over a field K, the singular homology groups $h_*(S)$ were introduced by A. Suslin [SV]. We will only need $h_0(S)$, for which there is the following elementary description.

3.1.2. Proposition. — *Let S be an integral scheme of finite type over a field K. Then*

$$h_0(S) = \mathrm{Z}_0(S)/\,\mathrm{Rat}'_0(S)\,.$$

Here, $\mathrm{Z}_0(S)$ is the group of 0-cycles, i.e., the free abelian group over all closed points of S. $\mathrm{Rat}'_0(S)$ is generated by all 0-cycles of the following kind.

Let $C \subset S$ be an irreducible curve, C' its normalization, and $\overline{C}$ the corresponding smooth, proper model. Then take all the cycles $\mathrm{div}(f)$ where $f \not\equiv 0$ is a rational function on C which, after pull-back to $\overline{C}$, is constantly 1 on $\overline{C}\backslash C'$.

Proof. See [Schm, Theorem 5.1]. □

3.1.3. Notation. — i) $h_0(S)$ is equipped with a natural map $\deg\colon h_0(S) \to \mathbb{Z}$. We will denote its kernel by $\mathrm{A}_0(S)$.

ii) Let $i\colon S_1 \to S_2$ be an arbitrary morphism of quasi-projective varieties over K. Then there is the induced homomorphism $i_*\colon h_0(S_1) \longrightarrow h_0(S_2), [P] \mapsto [i(P)]$. This immediately yields a map $i_*\colon \mathrm{A}_0(S_1) \longrightarrow \mathrm{A}_0(S_2)$.

3.1.4. Remark. — When S is proper, $h_0(S)$ coincides with the Chow group of zero cycles on S. Then $\mathrm{A}_0(S)$ is nothing but the subgroup of zero cycles of degree zero.

3.1.5. Lemma. — *Let V be a geometrically irreducible cubic surface over $\mathbb{F}_q$. Then there is a canonical homomorphism*

$$\pi_V\colon \mathrm{MW}(V) \longrightarrow \mathrm{A}_0(V^{\mathrm{reg}})\,.$$

Proof. To each combination $a_1[P_1] + \ldots + a_k[P_k]$ for $P_1, \ldots, P_k \in V^{\mathrm{reg}}(K)$ and $a_1 + \ldots + a_k = 0$, the homomorphism i_* assigns the corresponding cycle. We take this as a definition for π_V. To show that π_V is well-defined, we have to verify the following.

Assume that X_1, X_2, X_3 are collinear and X'_1, X'_2, X'_3 are collinear, too. Suppose that the connecting lines are not contained in V. Then

$$[X_1] + [X_2] + [X_3] - [X'_1] - [X'_2] - [X'_3] = 0 \in \mathrm{A}_0(V^{\mathrm{reg}})\,.$$

For this, consider the pencil of planes through X_1, X_2, X_3. Generically, the intersection with V is a curve, smooth at X_1, X_2 and X_3. The only possible exceptions are the tangent planes. We claim that the generic intersection curve is irreducible, too. Indeed, the contrary would mean that all intersection curves contained a line. Suppose, this is a line through X_1. Then V contains a pencil of lines through X_1, which implies V contains a plane through X_1. Hence, V is reducible, a contradiction.

Thus, take a plane through X_1, X_2, X_3, generating an irreducible intersection curve C that is smooth in X_1, X_2 and X_3. Further, take a plane through X'_1, X'_2, X'_3 generating an irreducible intersection curve C' that is smooth in X'_1, X'_2 and X'_3 and meets C only in smooth points X''_1, X''_2, X''_3. The sublemma below, applied to C and C', immediately yields the assertion. □

3.1.6. Sublemma. — *Let C be an irreducible cubic curve. Assume that $P_1, P_2, P_3 \in C^{\mathrm{reg}}$ as well as $Q_1, Q_2, Q_3 \in C^{\mathrm{reg}}$ are triples of collinear points such that $\{P_1, P_2, P_3\} \cap \{Q_1, Q_2, Q_3\} = \emptyset$.*
Then there is a rational function f on C having simple zeroes at P_1, P_2, P_3, simple poles at Q_1, Q_2, Q_3, no other zeroes or poles, and the value 1 at the possible singular point.

Proof. According to J. Plücker, an irreducible cubic curve may have at most one singular point. We may therefore put $f := K \cdot l_1/l_2$ for forms l_1 and l_2 defining the lines. By assumption, these do not meet the singular point. If necessary, we choose the constant K such that the value at the singularity is normalized to 1. □

Surjectivity.

3.1.7. Corollary. — *Let V be a geometrically irreducible cubic surface over $\mathbb{F}_q$, not being a cone. If*

$$\pi_V\colon\ \mathrm{MW}(V) \longrightarrow \mathrm{A}_0(V^{\mathrm{reg}})$$

is not surjective then V^{reg} has a nontrivial finite covering which is trivial over every $\mathbb{F}_q$-rational point.

Proof. Under the assumption, the image of the canonical map $V^{\mathrm{reg}}(\mathbb{F}_q) \to h_0(V^{\mathrm{reg}})$ generates a subgroup which is not dense. Hence, there are $l > 1$ and a surjective, continuous homomorphism $\alpha\colon h_0(V^{\mathrm{reg}}) \to \mathbb{Z}/l\mathbb{Z}$ sending the whole image of $V^{\mathrm{reg}}(\mathbb{F}_q)$ to zero.

The same is true for the composition $\alpha \circ \iota'_{V^{\mathrm{reg}}}\colon \pi_1^t(V^{\mathrm{reg}}) \to \mathbb{Z}/l\mathbb{Z}$. But this simply means that the l-sheeted covering of V^{reg} defined by $\alpha \circ \iota'_{V^{\mathrm{reg}}}$ has exactly l $\mathbb{F}_q$-rational points above every $x \in V^{\mathrm{reg}}(\mathbb{F}_q)$. □

Suppose, π_V were not surjective. Then according to the lemma, we have a nontrivial covering W such that $\#W(\mathbb{F}_q) = l \cdot \#V^{\mathrm{reg}}(\mathbb{F}_q)$. The Weil conjectures, proven by P. Deligne, assure that this may be possible only for very small values of q.

3.1.8. Proposition. — *Let V be a geometrically irreducible cubic surface over the finite field $\mathbb{F}_q$, not being a cone. Then the canonical homomorphism*

$$\pi_V\colon\ \mathrm{MW}(V) \longrightarrow \mathrm{A}_0(V^{\mathrm{reg}})$$

is surjective for $q > 23$.

Proof. Assume the contrary. Then according to Corollary 3.1.7, we have a twofold covering $\pi\colon V' \to V$ ramified at the $1 \leq s \leq 4$ singularities such that, over every smooth $\mathbb{F}_q$-rational point of V, there are two of V'. Being a geometrically irreducible cubic surface, V has at least $q^2 - 5q + 1$ points. Hence, $\#V^{\text{reg}}(\mathbb{F}_q) \geq q^2 - 5q - 3$.

On the other hand, $\chi_{\text{top}}(V) = 9 - 2s$, as V^{reg} is $\mathbf{P}^2$, blown up in six points, with s A-, D-, or E-configurations of lines deleted. Therefore, $\chi_{\text{top}}(V') \leq 18 - 3s \leq 15$. Indeed, V' consists of the two sheets above V^{reg} and $\leq s$ points of ramification.

We claim $\#V'(\mathbb{F}_q) \leq q^2 + 13q + 1$. For this, first observe that V' is simply connected as, otherwise, V^{reg} had more coverings than the twofold one. Let k be the number of blow-ups necessary in order to desingularize V'. Then $\dim H^2_{\text{ét}}(\overline{V}, \mathbb{Q}_l) \leq k + 13$ and one has the naive estimate $\#\overline{V}(\mathbb{F}_q) \leq q^2 + (k+13)q + 1$. The claim follows.

Consequently, $2(q^2 - 5q - 3) \leq 2 \cdot \#V^{\text{reg}}(\mathbb{F}_q) \leq \#(V')^{\text{reg}}(\mathbb{F}_q) \leq q^2 + 13q + 1$, which implies $q \leq 23$, immediately. $\square$

3.2. Computing h_0. The tame fundamental group

3.2.1. — Let S be a smooth surface over the finite field $\mathbb{F}_q$ for $q = p^r$ and let $\overline{S} \supseteq S$ be a smooth compactification. Then the *tame fundamental group* $\pi_1^t(S)$ of S classifies all finite coverings of S which are tamely ramified at $\overline{S} \setminus S$.

The group $\pi_1^t(S)$ is independent of the choice of the compactification $\overline{S}$. $\pi_1^t(S)$ is a quotient of $\pi_1^{\text{ét}}(S)$. By the purity of the branch locus [SGA1, Exp. X, Théorème 3.1], one has

$$\pi_1^t(S)^{\text{ab}}_{\text{tors}} \cong (\pi_1^{\text{ét}}(S)^{\text{ab}})_{\text{prime to } p} \oplus (\pi_1^{\text{ét}}(\overline{S})^{\text{ab}})_{p\text{-power}} .$$

Again, this decomposition is independent of the choice of $\overline{S}$.

The structural morphism $S \to \operatorname{Spec} \mathbb{F}_q$ induces a surjection $\pi_1^t(S) \to \pi_1(\operatorname{Spec} \mathbb{F}_q)$ the kernel of which we will denote by $\pi_1^{t,\text{geo}}(S)$. Note that $\pi_1^{t,\text{geo}}(S)$ differs from $\pi_1^t(S_{\overline{\mathbb{F}}_q})$. The point is that the analogue of the natural short exact sequence [SGA1, Exp. IX, Théorème 6.1] is only right exact for the tame fundamental group.

3.2.2. Theorem (Schmidt, Spieß). — *Let S be a surface over a finite field $\mathbb{F}_q$ which is smooth and geometrically irreducible, but not necessarily proper.*

i) *Then $\mathrm{A}_0(S)$ is a finite abelian group.*

ii) *There is a canonical isomorphism $\iota_S\colon \mathrm{A}_0(S) \longrightarrow \pi_1^{t,\text{geo}}(S)^{\text{ab}}$.*

Proof. See [SchS, Theorem 0.1]. $\square$

3.2.3. Remarks. — a) Concretely, ι_S is given as follows.

i) For a point $x\colon \operatorname{Spec} \mathbb{F}_{q'} \to S$, consider the induced homomorphism

$$\pi_1^{\text{ét}}(x)\colon \widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{F}_{q'}) \longrightarrow \pi_1^{\text{ét}}(S) \twoheadrightarrow \pi_1^t(S) \twoheadrightarrow \pi_1^t(S)^{\text{ab}} .$$

Send $[x]$ to $\pi_1^{\text{ét}}(x)(1)$. This defines a homomorphism $\iota'_S\colon h_0(S) \to \pi_1^t(S)^{\text{ab}}$.

ii) The degree map $\deg\colon h_0(S) \to \mathbb{Z}$ is compatible with the homomorphism $\pi_1^t(S)^{\text{ab}} \to \pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{F}_q) = \widehat{\mathbb{Z}}$ induced by the structural morphism.

iii) The homomorphism ι_S is exactly the restriction of ι'_S to $\ker(\deg)$.

b) The map ι'_S defines an isomorphism $\widehat{h_0(S)} \to \pi_1^t(S)^{\text{ab}}$.

3.2.4. Lemma. — *Let V be a cubic ruled surface defined over the finite field $\mathbb{F}_q$. Then $\pi_1^{t,\text{geo}}(V^{\text{reg}}) = 0$.*

Proof. It will suffice to show $\pi_1^t(V^{\text{reg}}_{\overline{\mathbb{F}}_q}) = 0$. In the present situation, a smooth compactification of $V^{\text{reg}}_{\overline{\mathbb{F}}_q}$ is given by a projective plane, blown up in one point. The preimage of the singular locus is a (double) line through the point blown up. Consequently, $V^{\text{reg}}_{\overline{\mathbb{F}}_q}$ is a ruled surface over $\mathbf{A}^1$. This yields $\pi_1^t(V^{\text{reg}}_{\overline{\mathbb{F}}_q}) = 0$. $\square$

The Picard group.

3.2.5. Proposition. —— *Let V be a geometrically irreducible cubic surface over $\mathbb{F}_q$, $q = p^r$, that is not a cone. Suppose V is normal, i.e., of one of the types I.i). Then*

$$\pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}})^{\mathrm{ab}} = [(\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_{\text{prime to } p} \otimes_{\mathbb{Z}} \mu_\infty^\vee)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}]^\vee .$$

Here, $^\vee$ denotes the Pontryagin dual, given by the functor $\mathrm{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$.

Proof. *First step.* p-torsion.

We know a smooth compactification $\overline{V}$ of V^{reg}, explicitly. $\overline{V}_{\overline{\mathbb{F}}_q}$ is isomorphic to $\mathbf{P}^2$ blown-up in six points. In particular, we have $\pi_1^{\text{ét}}(\overline{V}_{\overline{\mathbb{F}}_q}) = 0$. This suffices for $\pi_1^t(V_{\overline{\mathbb{F}}_q})^{\mathrm{ab}}_{p\text{-power}} = 0$ and $\pi_1^{t,\mathrm{geo}}(V)^{\mathrm{ab}}_{p\text{-power}} = 0$.

Second step. The Pontryagin dual.

Let us compute the Pontryagin dual $(\pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}})^{\mathrm{ab}})^\vee$. For l prime to p, we have

$$\begin{aligned}
\mathrm{Hom}(\pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}})^{\mathrm{ab}}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z}) &= \mathrm{Hom}(\pi_1^t(V^{\mathrm{reg}})^{\mathrm{ab}}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z})/\,\mathrm{Hom}(\pi_1(\mathrm{Spec}\,\mathbb{F}_q), \tfrac{1}{l}\mathbb{Z}/\mathbb{Z}) \\
&= \mathrm{Hom}(\pi_1(V^{\mathrm{reg}})^{\mathrm{ab}}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z})/\,\mathrm{Hom}(\pi_1(\mathrm{Spec}\,\mathbb{F}_q), \tfrac{1}{l}\mathbb{Z}/\mathbb{Z}) \\
&= H^1_{\text{ét}}(V^{\mathrm{reg}}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z})/H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \tfrac{1}{l}\mathbb{Z}/\mathbb{Z}) .
\end{aligned}$$

According to the Hochschild-Serre spectral sequence

$$H^p(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), H^q_{\text{ét}}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z})) \Longrightarrow H^{p+q}_{\text{ét}}(V^{\mathrm{reg}}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z}) ,$$

the latter quotient is nothing but $H^1_{\text{ét}}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z})^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$.

Third step. The torsion part of the Picard group.

We have $\Gamma(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q}, \mathbb{G}_m) = \overline{\mathbb{F}}_q^*$. In fact, the A-, D-, and E-configurations do not contain any principal divisor. This immediately yields $H^1_{\text{ét}}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q}, \mu_l) = \mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_l$ for any l prime to p. On $V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q}$, the étale sheaves μ_l and $\frac{1}{l}\mathbb{Z}/\mathbb{Z}$ coincide up to the Galois operation. We therefore have

$$\begin{aligned}
\mathrm{Hom}(\pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}})^{\mathrm{ab}}, \tfrac{1}{l}\mathbb{Z}/\mathbb{Z}) &= (H^1_{\text{ét}}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q}, \mu_l) \otimes_{\mathbb{Z}} \mu_l^\vee)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)} \\
&= (\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_l \otimes_{\mathbb{Z}} \mu_l^\vee)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)} .
\end{aligned}$$

Summing this up over all l, we see that

$$(\pi_1^{t,\mathrm{geo}}(V^{\mathrm{reg}})^{\mathrm{ab}})^\vee = (\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_{\text{prime to } p} \otimes_{\mathbb{Z}} \mu_\infty^\vee)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)} ,$$

which is equivalent to the assertion. □

3.2.6. Corollary. —— *Let V be a cubic surface over a finite field* $\mathbb{F}_q$.

a) *If V is geometrically ruled then* $\mathrm{A}_0(V^{\mathrm{reg}}) = 0$.

b) *If V is geometrically of one of the types I.i) then* $\mathrm{A}_0(V^{\mathrm{reg}}) \cong [(\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_{\text{prime to } p} \otimes_{\mathbb{Z}} \mu_\infty^\vee)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}]^\vee$.

Proof. a) summarizes Theorem 3.2.2.ii) and Lemma 3.2.4, while b) follows from Theorem 3.2.2.ii) together with Proposition 3.2.5. □

Thus, in order to compute $\mathrm{A}_0(V^{\mathrm{reg}})$, we only need to know $\mathrm{Pic}(V^{\mathrm{reg}})_{\mathrm{tors}}$ for each of the 21 types of cubic surfaces summarized in I.i).

The 21 types of normal cubic surfaces not being cones.

3.2.7. Lemma. — *Let V be a normal, proper surface over an algebraically closed field and $\widetilde{V}$ its desingularization. Then*

$$\operatorname{Pic}(V^{\mathrm{reg}}) = \operatorname{Pic}(\widetilde{V})/\langle E_1, \ldots, E_k \rangle\,,$$

where $E_1, \ldots, E_k$ denote the irreducible components of the preimages of the singularities on $\widetilde{V}$.

Proof. As $\widetilde{V}$ is non-singular, the restriction homomorphism $\operatorname{Pic}(\widetilde{V}) \to \operatorname{Pic}(V^{\mathrm{reg}})$ is surjective. Its kernel consists of these invertible sheaves that allow a trivialization on V^{reg}. $\square$

3.2.8. Theorem. — *Let V be an irreducible cubic surface over $\overline{\mathbb{F}}_q$, not being a cone. Suppose that V is normal, i.e., of one of the 21 types I.i).*

Then the Picard group $\operatorname{Pic}(V^{\mathrm{reg}})$ *is torsion-free for 17 of the 21 types. For the four remaining types, the torsion is given in the table below.*

type	singularities	$\operatorname{Pic}(V^{\mathrm{reg}})_{\mathrm{tors}}$
XVI	$4A_1$	$\mathbb{Z}/2\mathbb{Z}$
XVIII	$A_3 + 2A_1$	$\mathbb{Z}/2\mathbb{Z}$
XIX	$A_5 + A_1$	$\mathbb{Z}/2\mathbb{Z}$
XXI	$3A_2$	$\mathbb{Z}/3\mathbb{Z}$

Proof. We distinguish the cases systematically. Each time, we apply Lemma 3.2.7.

One has $\operatorname{Pic}(\widetilde{V}) \cong \mathbb{Z}^7$. The signature is $(1,-1,-1,-1,-1,-1,-1)$. I.e., we have torsion-freeness in the case of a smooth cubic surface.

Otherwise, the A-, D-, or E-configuration of (-2)-curves generates a sublattice of $\operatorname{Pic}(\widetilde{V})$. The quotient has torsion if and only if this sublattice can be refined in $\mathbb{Z}^7$ without enlarging the rank. This immediately shows torsion-freeness in the cases A_n for $n \neq 3$ and E_6 as the lattice discriminants are square-free.

For the other cases, the constructions described after [Do, Lemma 9.2.5] yield explicit generators for sublattices of $\mathbb{Z}^7$. We summarize them in the following table.

$2A_1$		$(2,-1,-1,-1,-1,-1,-1)$
		$(0,0,0,0,0,1,-1)$
A_3		$(0,0,0,0,0,1,-1)$
		$(0,0,0,0,1,-1,0)$
		$(0,0,0,1,-1,0,0)$
$A_2 + A_1$	A_1 :	$(2,-1,-1,-1,-1,-1,-1)$
	A_2 :	$(0,0,0,0,0,1,-1)$
	A_2 :	$(0,0,0,0,1,-1,0)$
$3A_1$		$(2,-1,-1,-1,-1,-1,-1)$
		$(0,0,0,0,0,1,-1)$
		$(0,0,0,1,-1,0,0)$
$2A_2$	1. A_2 :	$(0,0,0,0,0,1,-1)$
	1. A_2 :	$(0,0,0,0,1,-1,0)$
	2. A_2 :	$(0,0,1,-1,0,0,0)$
	2. A_2 :	$(0,1,-1,0,0,0,0)$
$A_3 + A_1$	A_1 :	$(2,-1,-1,-1,-1,-1,-1)$
	A_3 :	$(0,0,0,0,0,1,-1)$
	A_3 :	$(0,0,0,0,1,-1,0)$
	A_3 :	$(0,0,0,1,-1,0,0)$
D_4		$(1,-1,-1,-1,0,0,0)$
		$(0,1,0,0,-1,0,0)$
		$(0,0,1,0,0,-1,0)$
		$(0,0,0,1,0,0,-1)$
$A_2 + 2A_1$	1. A_1 :	$(2,-1,-1,-1,-1,-1,-1)$
	2. A_1 :	$(0,0,0,0,0,1,-1)$
	A_2 :	$(0,0,0,1,-1,0,0)$
	A_2 :	$(0,0,1,-1,0,0,0)$

$A_4 + A_1$	$A_1:$	$(2,-1,-1,-1,-1,-1,-1)$	
	$A_4:$	$(0,\ 0,\ 0,\ 0,\ 0,\ 1,-1)$	
	$A_4:$	$(0,\ 0,\ 0,\ 0,\ 1,-1,\ 0)$	
	$A_4:$	$(0,\ 0,\ 0,\ 1,-1,\ 0,\ 0)$	
	$A_4:$	$(0,\ 0,\ 1,-1,\ 0,\ 0,\ 0)$	
D_5		$(1,-1,-1,\ 0,\ 0,\ 0,-1)$	
		$(0,\ 1,-1,\ 0,\ 0,\ 0,\ 0)$	
		$(0,\ 0,\ 1,-1,\ 0,\ 0,\ 0)$	
		$(0,\ 0,\ 0,\ 1,-1,\ 0,\ 0)$	
		$(0,\ 0,\ 0,\ 0,\ 1,-1,\ 0)$	
$4A_1$		$(2,-1,-1,-1,-1,-1,-1)$	
		$(0,\ 0,\ 0,\ 0,\ 0,\ 1,-1)$	
		$(0,\ 0,\ 0,\ 1,-1,\ 0,\ 0)$	
		$(0,\ 1,-1,\ 0,\ 0,\ 0,\ 0)$	
$2A_2 + A_1$	$A_1:$	$(2,-1,-1,-1,-1,-1,-1)$	
	$1.\,A_2:$	$(0,\ 0,\ 0,\ 0,\ 0,\ 1,-1)$	
	$1.\,A_2:$	$(0,\ 0,\ 0,\ 0,\ 1,-1,\ 0)$	
	$2.\,A_2:$	$(0,\ 0,\ 1,-1,\ 0,\ 0,\ 0)$	
	$2.\,A_2:$	$(0,\ 1,-1,\ 0,\ 0,\ 0,\ 0)$	
$A_3 + 2A_1$	$1.\,A_1:$	$(2,-1,-1,-1,-1,-1,-1)$	
	$2.\,A_1:$	$(0,\ 0,\ 0,\ 0,\ 0,\ 1,-1)$	
	$A_3:$	$(0,\ 0,\ 0,\ 1,-1,\ 0,\ 0)$	
	$A_3:$	$(0,\ 0,\ 1,-1,\ 0,\ 0,\ 0)$	
	$A_3:$	$(0,\ 1,-1,\ 0,\ 0,\ 0,\ 0)$	
$A_5 + A_1$	$A_1:$	$(2,-1,-1,-1,-1,-1,-1)$	
	$A_5:$	$(0,\ 0,\ 0,\ 0,\ 0,\ 1,-1)$	
	$A_5:$	$(0,\ 0,\ 0,\ 0,\ 1,-1,\ 0)$	
	$A_5:$	$(0,\ 0,\ 0,\ 1,-1,\ 0,\ 0)$	
	$A_5:$	$(0,\ 0,\ 1,-1,\ 0,\ 0,\ 0)$	
	$A_5:$	$(0,\ 1,-1,\ 0,\ 0,\ 0,\ 0)$	
$3A_2$	$1.\,A_2:$	$(1,-1,-1,-1,\ 0,\ 0,\ 0)$	$=: v_1$
	$1.\,A_2:$	$(1,\ 0,\ 0,\ 0,-1,-1,-1)$	$=: v_2$
	$2.\,A_2:$	$(0,\ 1,-1,\ 0,\ 0,\ 0,\ 0)$	$=: v_3$
	$2.\,A_2:$	$(0,\ 0,\ 1,-1,\ 0,\ 0,\ 0)$	$=: v_4$
	$3.\,A_2:$	$(0,\ 0,\ 0,\ 0,\ 1,-1,\ 0)$	$=: v_5$
	$3.\,A_2:$	$(0,\ 0,\ 0,\ 0,\ 0,\ 1,-1)$	$=: v_6$

Table 1: Sublattices in $\mathbb{Z}^7$ generated by the A-, D-, and E-configurations

The assertions now follow from mechanical calculations.

In the cases where torsion-freeness is claimed, one may easily extend the basis of the sublattice given to a basis of $\mathbb{Z}^7$. For example, consider the types $A_n + A_1$. Then we have subsets of the lattice base consisting of $2e_1 - e_2 - \cdots - e_7$, $e_i - e_{i+1}$ for $i = 3, \ldots, 6$, e_1, and e_7.

In the cases $4A_1$, $A_3 + 2A_1$, and $A_5 + A_1$, the lattices may indeed be extended by the vector $(1, 0, -1, 0, -1, 0, -1)$ without changing the ranks. The lattices obtained in this way are then maximal.

In the case $3A_2$, the vector $(v_1 + v_3 - v_4) - (v_2 + v_5 - v_6) = -3e_3 + 3e_6$ is obviously 3-divisible. The refined lattice has discriminant 3 and is, therefore, not refinable any further. $\square$

3.2.9. Corollary. —— *Let V be a cubic surface over $\mathbb{F}_q$ that is geometrically of type* $3A_2$*. If* $q \equiv 1 \pmod 3$ *and* Frob *acts on the singular points by an even permutation or* $q \equiv 2 \pmod 3$ *and* Frob *acts by an odd permutation then* $\mathrm{A}_0(V^{\mathrm{reg}}) \cong \mathbb{Z}/3\mathbb{Z}$*. Otherwise,* $\mathrm{A}_0(V^{\mathrm{reg}}) = 0$.

Proof. If $3|q$ then Corollary 3.2.6.ii) immediately shows $\mathrm{A}_0(V^{\mathrm{reg}}) = 0$. Otherwise, $\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_{\text{prime to } p} \cong \mathbb{Z}/3\mathbb{Z}$ by Theorem 3.2.8. Further, Table 1 shows that permuting the second and third singularities changes $-e_3+e_6$ into $e_3 - e_6$. Thus, Frob operates on $\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})_{\text{prime to } p}$ via the sign of the permutation of the three singular points. On the other hand, on $\mu_3^\vee$, Frob acts as 1 or (-1) according to whether $\mu_3 \subseteq \mathbb{F}_q$ or not. But $\mu_3 \subseteq \mathbb{F}_q$ is equivalent to $q \equiv 1 \pmod 3$. The assertion follows. $\square$

3.2.10. Theorem. —— *Let V be a cubic surface over a finite field $\mathbb{F}_q$. Suppose that V is geometrically ruled or of one of the 21 types in I.i).*

a) *Let V be geometrically of type* $4A_1$, A_3+2A_1, *or* A_5+A_1. *If* $\operatorname{char}\mathbb{F}_q \neq 2$ *then* $\mathrm{A}_0(V^{\mathrm{reg}}) = \mathbb{Z}/2\mathbb{Z}$.

b) *Let V is geometrically of type* $3A_2$. *If* $q \equiv 1 \pmod 3$ *and* Frob *acts on the singular points by an even permutation or* $q \equiv 2 \pmod 3$ *and* Frob *acts by an odd permutation then* $\mathrm{A}_0(V^{\mathrm{reg}}) \cong \mathbb{Z}/3\mathbb{Z}$.

c) *Otherwise,* $\mathrm{A}_0(V^{\mathrm{reg}}) \cong 0$.

Proof. For the geometrically ruled case, see Corollary 3.2.6.i). Otherwise, we use Corollary 3.2.6.ii). This proves, in particular, $\mathrm{A}_0(V) \cong 0$ for the 17 types where $\mathrm{Pic}(V^{\mathrm{reg}}_{\overline{\mathbb{F}}_q})$ is torsion-free.

If V is of type $4A_1$, A_3+2A_1, or A_5+A_1 then Corollary 3.2.6.ii) immediately shows $\mathrm{A}_0(V^{\mathrm{reg}}) = \mathbb{Z}/2\mathbb{Z}$ unless the characteristic of the base field is two. Indeed, the Galois operation on $\mathbb{Z}/2\mathbb{Z}$ is automatically trivial. When $\operatorname{char}\mathbb{F}_q = 2$, the same formula yields $\mathrm{A}_0(V^{\mathrm{reg}}) = 0$. The type $3A_2$ was dealt with above. □

4. Experiments

4.1. Description of the sample. — We let p run through the prime numbers form 5 through 101. For each of the primes, we followed the classification of cubic surfaces as described in 2.1.2 and 2.1.3. Recall that, as the base field is finite, there are 34 types of surfaces with finitely many double points.

For each type, equations for the corresponding surfaces are suggested in [Do, Section 9.2]. We selected coefficients by help of a random number generator and worked with ten examples per type. For those types which clearly have no moduli [BW], we took only one example. We avoided the surfaces decomposing into three planes over a proper extension of $\mathbb{F}_p$ as, for these, $\mathrm{MW}(V)$ is known to degenerate. All in all, we worked with 330 cubic surfaces per prime.

4.2. The results. — For each surface in the sample, we run Algorithm A.3 to determine the partition of $V^{\mathrm{reg}}(\mathbb{F}_p)$ into equivalence classes. A priori, the algorithm might produce a partition that is too fine. It turned out, however, that it could find the exact equivalence relation in each case.

Recall that in each case, according to the theory described in sections 2 and 3, we have a lower bound for $\mathrm{MW}(V)$. More precisely, we know an abelian group, $\mathrm{MW}(V)$ naturally maps to. On the other hand, Algorithm A.3 provides an upper bound. I.e., another abelian group mapping onto $\mathrm{MW}(V)$.

Astonishingly, both turned out to be the same with only one exception. The partition of the points found allowed us to determine $\mathrm{MW}(V)$ for every surface in the sample.

Case I.i) Among the normal cubic surfaces having only double points, we always found $\mathrm{MW}(V) = 0$ except for the cases $4A_1$, A_3+2A_1, A_5+A_1, and $3A_2$. In the first three of these cases, we have $\mathrm{MW}(V) = \mathbb{Z}/2\mathbb{Z}$.

Finally, in the case $3A_2$, we indeed found that $\mathrm{MW}(V) = \mathbb{Z}/3\mathbb{Z}$ for $p \equiv 1 \pmod 3$ and Frob acting on the singular points by an even permutation and for $p \equiv 2 \pmod 3$ and Frob acting by an odd permutation. Otherwise, $\mathrm{MW}(V) = 0$.

Cases I.ii) and II.ii) Ignoring the exception described in Example 4.3 below, for the cones, $\mathrm{MW}(V)$ was always equal to the Mordell-Weil group of the underlying curve.

Case II.i) The cubic ruled surfaces always fulfilled $\mathrm{MW}(V) = 0$.

Two components. When V consisted of a non-degenerate quadric and a plane, we always found that $\mathrm{MW}(V) = \mathbb{Z}$, two points being equivalent if and only if they belonged to the same component. When the quadric was a cone and the plane did not meet the cusp, it turned out that $\mathrm{MW}(V) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, the surjection described in Example 2.3.2 being bijective.

A cubic surface consisting of a cone and a plane through the cusp is a cone over a reducible cubic curve. Here, $\mathrm{MW}(V)$ was always isomorphic to the Mordell-Weil group of the curve.

Three components. A cubic surface consisting of three planes meeting in a point is the cone over a triangle. $\mathrm{MW}(V)$ was always equal to the Mordell-Weil group of the triangle.

Finally, three planes meeting in a line form a cone in may ways. Hence, two distinct points are never equivalent to each other. We have $\mathrm{MW}(V) \cong (K^+)^2 \oplus \mathbb{Z}^2$.

4.3. Example. —– Consider the cone C over the elliptic curve given by $y^2 = x^3 + 2x$ over $\mathbb{F}_5$. This elliptic curve has only two $\mathbb{F}_5$-rational points, $P = (0, 0)$ and Q, the point at infinity. Correspondingly, C^{reg} has ten points, $P_1, \ldots, P_5, Q_1, \ldots, Q_5$. The lines define the relations $[P_i, P_i, Q_j]$ and $[Q_j, Q_j, Q_j]$ for $i, j = 1, \ldots, 5$. Consequently, $\mathrm{MW}(C) \cong (\mathbb{Z}/2\mathbb{Z})^5$, generated by all $(P_i - Q_j)$, subject to the relations $(Q_i - Q_j) = 0$.

Here, the construction of $\mathrm{MW}(V)$ degenerates as, on C^{reg}, there are too few $\mathbb{F}_5$-rational points. This effect clearly becomes worse for $p = 2$ or 3. This is one of the reasons why these primes were excluded from the experiments.

4.4. Remark. —– The case of a cubic surface consisting of three planes with a line in common is the easiest from the theoretical point of view. For Algorithm A.3, it is, however, the most complicated one. No simplification occurs as no equivalent points may be found. The running time is dominated by steps iv) and v), which are otherwise negligible. For $p > 70$, we excluded this case from the experiments.

4.5. Remark. —– We run an implementation of Algorithm A.3 in `magma`. On a Quad-Core AMD Opteron Processor 2356, the average CPU time per surface was 1.5 seconds for $p = 5$, 11 seconds for $p = 37$, 38 seconds for $p = 71$, and 1:55 minutes for $p = 101$.

Some observations.

The facts presented in this subsection were obtained after having seen the results of the experiments to compute Mordell-Weil groups. Lemma 4.7.i) is important for the application given in Proposition 5.2. Although we have no applications for the other results, we think that they are nevertheless of interest.

4.6. Definition. —– Let V be a cubic surface over the finite field $\mathbb{F}_q$ and suppose there is a surjection $\pi\colon \mathrm{MW}(V) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. Then, with respect to π, $V^{\mathrm{reg}}(\mathbb{F}_q)$ decomposes into exactly two equivalence classes. We will call the equivalence class *odd* that occurs an even number of times on each line, the other class *even*.

4.7. Lemma. —– *Let V be a cubic surface over the finite field $\mathbb{F}_q$. Fix a surjection $\pi\colon \mathrm{MW}(V) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$. Suppose that, in every equivalence class of $V^{\mathrm{reg}}(\mathbb{F}_q)$ according to π, there is a point not contained in any of the lines lying on V.*

i) *Suppose $n = 2$. Assume further that not all points of $V^{\mathrm{reg}}(\mathbb{F}_q)$ are contained in a plane. Then for the two equivalence classes M_0, M_1 of $V^{\mathrm{reg}}(\mathbb{F}_q)$, we have the relation $\#M_+ - \#M_- = q$.*

ii) *Suppose $n = 3$. Then the three equivalence classes M_0, M_1, M_2 of $V^{\mathrm{reg}}(\mathbb{F}_q)$ are of the same size.*

Proof. i) Fix a point $X \in M_-$ not contained in a line lying on V. By assumption, there is some $X' \in V^{\mathrm{reg}}(\mathbb{F}_q)$ outside the plane tangent at X. The line g connecting X and X' meets V in two distinct points $X, Y \in M_-$ and in $Z \in M_+$. (We have either $Y = X'$ or $Z = X'$.)

Now, we intersect V with the pencil of planes containing g. We assert that each of the $(q + 1)$ curves C_t arising contains as many points from M_+ as from M_-. This immediately implies the assertion. Indeed, equinumerosity occurs as soon as we count the points X, Y, and Z multiply.

To verify the assertion, let C_t be any of the intersection curves. We first observe that $X \in C_t$ is a smooth point. In fact, we do not intersect V with the tangent plane at X since that does not contain g. C_t may be reducible. However, X is, by assumption, not contained in a line. Therefore, for every $P \in C_t(\mathbb{F}_q)$, there is a unique $P' \in C_t(\mathbb{F}_q)$ such that X, P, and P' are collinear. As P and P' are in different classes, the assertion follows.

ii) Here, there are two cases.

First case. If $X \in M_i$, $Y \in M_j$, and $Z \in M_k$ are the three points of intersection of a line with V then $i+j+k \equiv 0 \pmod 3$.

We choose a point $X \in M_0$ which is not contained in any of the lines on V. Then for every $P \in M_1$, there is a unique $P' \in M_2$ such that X, P, and P' are collinear. As this assignment is invertible, one has $\#M_1 = \#M_2$. Analogously, a starting point $X \in M_1$ yields the equality $\#M_2 = \#M_0$.

Second case. If $X \in M_i$, $Y \in M_j$, and $Z \in M_k$ are the three points of intersection of a line with V then $i + j + k \not\equiv 0 \pmod 3$.

We may assume without restriction that $i+j+k \equiv 1 \pmod 3$. Choose a point $X \in M_0$ which is not contained in any of the lines on V. The tangent plane T_X contains, besides X, only points from M_1. Further, there are exactly $q + 1$ of them, as, by the assumption of this case, there is no line tangent at X of order three. On the other hand, outside T_X, the sets M_0 and M_1 are equinumerous since the lines through X cause a bijection. Consequently, $\#M_1 = \#M_0 + q$.

Analogously, we obtain $\#M_2 = \#M_1 + q$ and $\#M_0 = \#M_2 + q$ when starting with a point $X \in M_1$ or $X \in M_2$, respectively. Thus, the second case is contradictory. □

4.8. Corollary. — *Let V be a cubic surface over the finite field $\mathbb{F}_q$ having at most finitely many double points. Suppose $q > 101$. Fix a surjection $\pi\colon \mathrm{MW}(V) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. Then every point $x \in V^{\mathrm{reg}}(\mathbb{F}_q)$ lying on a line contained in V is even.*

Proof. Suppose, to the contrary, that x is odd. Being a smooth point on an irreducible cubic surface, x lies on $n \leq 3$ lines $l_1, \ldots, l_n$ contained in V. As x is odd, outside $l_1, \ldots, l_n$ there are the same numbers of odd and even points.

V is geometrically irreducible. Hence, $\#V^{\mathrm{reg}}(\mathbb{F}_q) \geq q^2 - 5q - 3$. Since a plane cubic curve has at most $3q + 1$ points, not all points of $V^{\mathrm{reg}}(\mathbb{F}_q)$ are contained in a plane. Further, by the argument above, there are at least $\frac{q^2-8q-4}{2}$ even and $\frac{q^2-8q-4}{2}$ odd points. As the number of points on lines is $\leq 27(q + 1)$ and $q > 101$, points of both kinds occur outside the lines.

Therefore, by Lemma 4.7.i), $\#M_0 - \#M_1 = q$. The same must apply for the numbers lying on the n lines. But n lines with a point in common contain $nq + 1$ points, which is an even number for $n = 1, 3$. Hence, we necessarily have $n = 2$. In particular, all points lying on exactly one line are even.

As the number of lines is bounded by 27, the two lines contain at most $2 \cdot 25 + 1 = 51$ points that lie on more than one line. Thus, at least $2q - 50$ points are even and most 51 points are odd. This implies $2q - 101 = (2q - 50) - 51 \leq q$, a contradiction. □

4.9. Lemma (Connection to the Hessian). —
Let V, given by $F(X_0, \ldots, X_3) = 0$, be a cubic surface over the finite field $\mathbb{F}_q$ of characteristic $\neq 2$. Suppose, there is a surjection $\mathrm{MW}(V) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. Then the following is true.

If $P \in V^{\mathrm{reg}}(\mathbb{F}_q)$ is a odd point not lying on a line contained in V then the Hessian

$$\det \frac{\partial^2 F}{\partial X_i \partial X_j}(P)$$

is a non-square in $\mathbb{F}_q$.

Proof. Consider the tangent plane T_P at P. The intersection $C_P := V \cap T_P$ is a cubic curve with a singularity at P. Thus, in affine coordinates and locally near P, the equation of C_P is of the form $Q(x, y) + G(x, y) = 0$ for a quadratic form Q and a cubic form G.

By assumption, there is no line in T_P meeting P with multiplicity 3. This means, in particular, that $P \in C_P$ is a double point, not a triple point. Further, the two tangent directions at P are not defined over $\mathbb{F}_q$. In other words, the binary quadratic form Q does not represent zero over $\mathbb{F}_q$. This exactly means that minus the discriminant of Q is a non-square in $\mathbb{F}_q$. It is a direct calculation to show that $(-\operatorname{disc} Q)$ coincides, up to square factors, with the Hessian of F at P. $\square$

5. Applications

The Mordell-Weil problem.

5.1. —— The Mordell-Weil group is related to the famous *Mordell-Weil problem* [Ma2]. This may be formulated as to determine the minimal number of generators of $\mathrm{MW}(V)$, most notably in the case that V is a cubic surface over a number field.

In the main part this article, we analyzed the Mordell-Weil group for cubic surfaces over finite fields. Of particular interest are the types for which $\mathrm{MW}(V) \neq 0$. The point is that there is the following application to the number field case.

5.2. Proposition. —— *Let $\mathscr{V}$ be a cubic surface over $\mathbb{Q}$ and $p_1, \ldots, p_t$ be primes satisfying the following conditions.*

i) *For every i, the reduction $\mathscr{V}_{p_i}$ has at most finitely many double points. Further, $p_i > 101$.*

ii) *For each i, there is a surjection $\pi_{p_i}\colon \mathrm{MW}(\mathscr{V}_{p_i}) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$.*

iii) *No singularity of $\mathscr{V}_{p_i}$ lifts to a smooth $\mathbb{Q}$-rational point on $\mathscr{V}$.*

Then the specialization maps induce a natural homomorphism

$$\pi\colon \mathrm{MW}(\mathscr{V}) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^t .$$

If $\mathscr{V}$ has weak approximation then π is a surjection. Then at least t elements are necessary in order to generate $\mathrm{MW}(\mathscr{V})$.

Proof. We fix $i \in \{1, \ldots, t\}$ and will construct the corresponding homomorphism $\pi_i\colon \mathrm{MW}(\mathscr{V}) \to \mathbb{Z}/2\mathbb{Z}$. By assumption i), each $X \in \mathscr{V}(\mathbb{Q})$ specializes to a point $X_i \in \mathscr{V}_{p_i}^{\mathrm{reg}}(\mathbb{F}_{p_i})$. Thus, π_i is supposed to send $[X]$ to $\pi_{p_i}([X_i])$, a condition that determines π_i uniquely.

To show that π_i is a well-defined homomorphism, we need

$$\pi_{p_i}([X_i^{(1)}]) + \pi_{p_i}([X_i^{(2)}]) + \pi_{p_i}([X_i^{(3)}]) - \pi_{p_i}([X_i'^{(1)}]) - \pi_{p_i}([X_i'^{(2)}]) - \pi_{p_i}([X_i'^{(3)}]) = 0$$

for X_1, X_2, X_3 the intersection points of $\mathscr{V}$ with a line l and X_1', X_2', X_3' the intersection points with another line l'. If both l and l' specialize to lines, not contained in $\mathscr{V}_{p_i}$, then this assertion follows from the definition of $\mathrm{MW}(\mathscr{V}_{p_i})$ and the fact that π_{p_i} is a homomorphism of groups. On the other hand, if l specializes to a line λ contained in $\mathscr{V}_{p_i}$ then $[X_1], [X_2], [X_3] \in \lambda$ are even by virtue of Corollary 4.8.

The final assertions are evident. $\square$

5.3. Theorem. —— *For every $t > 0$, there exists a smooth cubic surface $\mathscr{V}$ over $\mathbb{Q}$ such that at least t elements are necessary in order to generate* $\mathrm{MW}(\mathscr{V})$.

Proof. We choose t distinct prime numbers $p_1, \ldots, p_t$ such that $p_i \equiv 2, 3 \pmod 5$ and $p_i > 101$ for every i. The first assumption implies $\mathbb{F}_{p_i}(\zeta_5) = \mathbb{F}_{p_i^4}$. Further, the four lines in $\mathbf{P}^2_{\overline{\mathbb{F}}_{p_i}}$ defined by

$$x + \zeta_5^k y + \zeta_5^{2k} z = 0\,, \qquad k = 1, \ldots, 4\,,$$

are in general position. Indeed, intersecting three of the four lines leads to the homogeneous system of linear equations, the coefficient matrix of which is a non-trivial Vandermonde. The six intersection points are $(x_p : y_p : z_p)$, $(x'_p : y'_p : z'_p)$ and conjugates for $x_p, y_p, z_p \in \mathbb{F}_{p_i}(\zeta_5)$ and $x'_p, y'_p, z'_p \in \mathbb{F}_{p_i}(\sqrt{5})$.

Further, choose six $\mathbb{F}_{11}$-rational points $(x_{11}^{(1)} : y_{11}^{(1)} : z_{11}^{(1)}), \ldots, (x_{11}^{(6)} : y_{11}^{(6)} : z_{11}^{(6)})$ in general position on $\mathbf{P}^2$. This is possible, for instance $(1 : 0 : 1)$, $(0 : 1 : 1)$, $(3 : 4 : 1)$, $(4 : 3 : 1)$, $(10 : 0 : 1)$, and $(0 : 0 : 1)$ will do. Indeed, the first five points lie on the irreducible quadric, given by $x^2 + y^2 - z^2 = 0$, and no line through two of these five points meets the last.

Choose $x, y, z \in \mathbb{Z}[\zeta_5]$ and $x', y', z' \in \mathbb{Z}[\sqrt{5}]$ such that, for every i,

$$x \equiv x_p \pmod{p_i}, \quad \ldots, \quad z' \equiv z'_p \pmod{p_i}\,.$$

Further, make sure that under the homomorphisms $\mathbb{Z}[\zeta_5] \to \mathbb{F}_{11}$, $(x : y : z)$ defines four of the six $\mathbb{F}_{11}$-rational points chosen and that, under the homomorphisms $\mathbb{Z}[\sqrt{5}] \to \mathbb{F}_{11}$, $(x' : y' : z')$ defines the other two.

The points $(x : y : z)$, $(x' : y' : z')$ and their conjugates define a subscheme B of $\mathbf{P}^2_{\mathbb{Q}}$ of length six. Blow up $\mathbf{P}^2_{\mathbb{Q}}$ in B and obtain a scheme $\mathscr{V}$ over $\mathbb{Q}$. The scheme B defines six points on $\mathbf{P}^2_{\overline{\mathbb{Q}}}$. These points are in general position as their reduction modulo 11 already is. Therefore, $\mathscr{V}$ is a smooth cubic surface. As its $\mathbb{Q}$-rational points are in bijection with those on $\mathbf{P}^2$, $\mathscr{V}$ has weak approximation.

At any of the primes $p_i, \ldots, p_t$, the reduction of $\mathscr{V}$ is a singular cubic surface. Its resolution of singularities is a weak Del-Pezzo surface obtained from blowing up $\mathbf{P}^2_{\mathbb{F}_{p_i}}$ in the subscheme of length six defined by $(x_p : y_p : z_p)$ and $(x'_p : y'_p : z'_p)$. Hence, $\mathscr{V}_{p_i}$ is a cubic surface with finitely many double points.

By [Do, list after Lemma 9.2.5], $\mathscr{V}_{p_i}$ is of type $4A_1$. The four singularities correspond to the four lines through three blow-up points. Hence, none of them is defined over $\mathbb{F}_{p_i}$. In particular, they do not lift to $\mathbb{Q}$-rational points. Thus, assumptions i) through iii) of Proposition 5.2 are thus fulfilled. The proof is complete. $\square$

5.4. Remarks. — i) Conjecturally, $\mathrm{MW}(\mathscr{V})$ is finitely generated for every smooth cubic surface $\mathscr{V}$ over a number field. Recall from Example 1.6 that the same is wrong in the singular case.

ii) Recently, S. Siksek [Sik] announced that he can prove $\mathrm{MW}(V) = 0$ for a certain class of smooth cubic surfaces. Observe, however, that he works with the definition of the Mordell-Weil group as suggested by [Ma1], cf. Remark 1.5.

Brauer equivalence.

5.5. — There is another application, which is related to the so-called Brauer-Manin obstruction. This is a method, invented by Yu. I. Manin [Ma1, Chapter VI], to explain the failure of the Hasse principle or weak approximation in certain cases. It is based on the consideration of a non-trivial Brauer class $\alpha \in \mathrm{Br}(\mathscr{V})$ and the corresponding p-adic evaluation maps

$$\mathrm{ev}_{\alpha,p}\colon \mathscr{V}(\mathbb{Q}_p) \longrightarrow \mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \alpha|_x\,.$$

Proposition. *Let $p \neq 2, 3$ be a prime number and $F \in \mathbb{Z}_p[X_0, X_1, X_2, X_3]$ a cubic form defining a smooth cubic surface $\mathscr{V}$ over $\mathbb{Q}_p$. Suppose that all $x \in \mathscr{V}(\mathbb{Q}_p)$ specialize to $\mathscr{V}_p^{\mathrm{reg}}$ and that* $\mathrm{MW}(\mathscr{V}_p) = 0$.
Then $\mathrm{ev}_{\alpha,p}$ is constant for every $\alpha \in \mathrm{Br}(\mathscr{V})$.

Proof. It is known that $\mathrm{ev}_{\alpha,p}(P)$ depends only on the reduction of P modulo p [Br, Theorem 1]. Further, $\mathrm{MW}(\mathscr{V}_p)$ is the quotient of zero cycles modulo the relation generated by rational equivalence on cubic curves. Hence, an application of Lichtenbaum duality [Li, Corollary 1] proves that $\mathrm{ev}_{\alpha,p}$ is induced by a group homomorphism $\mathrm{MW}(\mathscr{V}_p) \to \mathbb{Q}/\mathbb{Z}$. As $\mathrm{MW}(\mathscr{V}_p) = 0$, the assertion follows. □

5.6. Remark. — In [EJ], we studied explicit examples of cubic surfaces for which the Brauer-Manin obstruction works. I.e., such that there are a Brauer class α and certain primes p leading to non-constant p-adic evaluation maps $\mathrm{ev}_{\alpha,p}$. It was noticeable in the experiments that the reduction types at the relevant primes were distributed in an unusual way. Reducible reductions and reductions to the Cayley cubic occurred frequently. This observation was actually the starting point of our investigations on the Mordell-Weil group.

As our theory gives only lower bounds for $\mathrm{MW}(\mathscr{V}_p)$, Proposition 5.5 does not have immediate consequences for a large number of cases. Nevertheless, one is led to conjecture that $\mathrm{ev}_{\alpha,p}$ may distinguish points specializing to $\mathscr{V}_p^{\mathrm{reg}}$ only when there is reducible reduction, reduction to a cone, or reduction to one of the types $4A_1$, $A_3 + 2A_1$, $A_5 + A_1$, or $3A_2$.

5.7. Remark (Comparison with other concepts of equivalence.)**. —**
Equivalence with respect to $\mathrm{MW}(V)$ clearly implies rational equivalence, cf. Proposition 3.1.2. For this reason, it implies Brauer equivalence. We do not know whether it implies R-equivalence [Ma1, Definition 14.1].

On the other hand, R-equivalence certainly does not imply equivalence with respect to $\mathrm{MW}(V)$. An obvious counterexample is provided by the cone over an elliptic curve with only one K-rational point. Here, $\mathrm{MW}(V) = \ker(\mathrm{sum}\colon \mathbb{Z}[V(K)] \to \mathbb{Z})$, no two points being equivalent although any two are R-equivalent. Example 2.3.2 provides another case that is more interesting.

A. Algorithms

A.1. Algorithm (Two equivalent points)**. —** i) Using a random number generator, choose four distinct points $X_{11}, X_{12}, X_{21}, X_{22} \in V^{\mathrm{reg}}(\mathbb{F}_q)$.

ii) Determine four points $X_{13}, X_{23}, X_{31}, X_{32} \in V^{\mathrm{reg}}(\mathbb{F}_q)$ such that the relations $[X_{11}, X_{12}, X_{13}]$, $[X_{21}, X_{22}, X_{23}]$, $[X_{11}, X_{21}, X_{31}]$, and $[X_{12}, X_{22}, X_{32}]$ are fulfilled. If this turns out to be impossible as (X_{11}, X_{12}), (X_{21}, X_{22}), (X_{11}, X_{21}), or (X_{12}, X_{22}) are lying on a line contained in V then output FAIL and terminate prematurely.

iii) Determine points X_{33} and X'_{33} such that $[X_{13}, X_{23}, X_{33}]$ and $[X_{31}, X_{32}, X'_{33}]$. If this turns out to be impossible as (X_{13}, X_{23}) or (X_{31}, X_{32}) are lying on a line contained in V then output FAIL and terminate prematurely.

iv) Output "X_{33} and X'_{33} are equivalent."

A.2. Algorithm (A point being equivalent to a given $X_0 \in V^{\mathrm{reg}}(\mathbb{F}_q)$)**. —**

i) Execute Algorithm A.1 in order to find two mutually equivalent points X_1 and X_2.

ii) Determine a point X'_1 such that $[X_1, X_0, X'_1]$. If this turns out to be impossible as (X_1, X_0) are lying on a line completely contained in V then output FAIL and terminate prematurely.

iii) Now, determine a point X'_0 such that $[X'_1, X_2, X'_0]$. If this turns out to be impossible as (X'_1, X_2) are lying on a line completely contained in V then output FAIL and terminate prematurely.

iv) Output "X'_0 is equivalent to X_0."

A.3. Algorithm (Partition of the points)**. —**

i) Choose a natural number N.

ii) Decompose $V^{\mathrm{reg}}(\mathbb{F}_q)$ into a set $\mathfrak{M} = \{N_1, \dots, N_m\} = \{\{X_1\}, \dots, \{X_m\}\}$ of singletons.

iii) Execute Algorithm A.1, Nq^2 times. When two equivalent points $X_1 \in M_k$ and $X_2 \in M_l$ for $k \neq l$ are found, unite M_k with M_l and reduce m by 1.

iv) List the singletons still contained in $\mathfrak{M}$, i.e., the points that were never met in step iii). For each element in the list obtained, execute Algorithm A.2 N times. When two equivalent points $X_1 \in M_k$ and $X_2 \in M_l$ for $k \neq l$ are found, unite M_k with M_l and reduce m by 1.

v) If sets of size less than q remain in $\mathfrak{M}$ then choose a single element from each of these sets. For each element in the list obtained, execute Algorithm A.2 N times. When two equivalent points $X_1 \in M_k$ and $X_2 \in M_l$ for $k \neq l$ are found, unite M_k with M_l and reduce m by 1.

vi) Output the partition of $V^{\mathrm{reg}}(\mathbb{F}_q)$ found.

A.4. Remarks. — i) Algorithm A.3 finds a partition which is possibly too fine in comparison with the actual partition into equivalence classes.

ii) In practice, the value $N = 7$ seems to work perfectly, for $p = 5$ as well as for the biggest primes for which such an algorithm seems reasonable.

References

[Br] Bright, M. J.: *Efficient evaluation of the Brauer-Manin obstruction,* Math. Proc. Cambridge Philos. Soc. **142** (2007), 13–23

[BW] Bruce, J. W. and Wall, C. T. C.: *On the classification of cubic surfaces,* J. London Math. Soc. **19** (1979), 245–256

[Ca] Cayley, A.: *A memoir on cubic surfaces,* Phil. Trans. Royal Soc. **159** (1869), 231–326

[Do] Dolgachev, I. V.: *Classical algebraic geometry: a modern view,* To appear at Cambridge University press, Available at: `http://www.math.lsa.umich.edu/~idolga/CAG.pdf`

[EJ] Elsenhans, A.-S. and Jahnel, J.: *On the Brauer–Manin obstruction for cubic surfaces,* Journal of Combinatorics and Number Theory **2** (2010)

[Li] Lichtenbaum, S.: *Duality theorems for curves over p-adic fields,* Invent. Math. **7** (1969), 120–136

[Ma1] Manin, Yu. I.: *Cubic forms, algebra, geometry, arithmetic,* North-Holland Publishing Co. and American Elsevier Publishing Co., Amsterdam, London, and New York 1974

[Ma2] Manin, Yu. I.: *Mordell-Weil problem for cubic surfaces,* in: Advances in mathematical sciences: CRM's 25 years (Montreal 1994), 313–318

[Schl] Schläfli, L.: *An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface,* Quart. J. Math. **2** (1858), 110–120

[Sik] Siksek, S.: *On the number of Mordell-Weil generators for cubic surfaces,* `arXiv:1012.1838v4`

[Sil] Silverman, J. H.: *The arithmetic of elliptic curves,* Graduate Texts in Mathematics 106, Springer, New York 1986

[Schm] Schmidt, A.: *Singular homology of arithmetic schemes,* Algebra & Number Theory **1** (2007), 183–222

[SchS] Schmidt, A. and Spieß, M.: *Singular homology and class field theory of varieties over finite fields,* J. Reine Angew. Math. **527** (2000), 13–36

[SGA1] Grothendieck, A.: *Revêtements étales et groupe fondamental,* Lecture Notes in Math. 224, Springer, Berlin 1971

[SV] Suslin, A. and Voevodsky, V.: *Singular homology of abstract algebraic varieties,* Invent. Math. **123** (1996), 61–94

[VY] Veblen, O. and Young, J. W.: *Projective geometry,* Vol. 2, Blaisdell Publishing Co., New York, Toronto, and London 1946