On the distribution of the Picard ranks of the reductions of a $K3$ surface

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joint work with
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Étale cohomology

Consider (smooth proper) varieties over a field of characteristic p [or 0].

The l-adic (étale) cohomology theory shares many properties of the usual (topological) cohomology of varieties over $\mathbb{C}$. *Differences:*

- $\mathbb{Z}$ or $\mathbb{Q}$ may not be used as coefficients. Only $\mathbb{Z}_l$ or $\mathbb{Q}_l$ for $l \neq p$.
- There is an operation of Frob on $H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(j))$.

There is even an operation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^i_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(j))$, for S a over $\mathbb{Q}$ [although the operation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S_{\mathbb{C}}$ is far from continuous].

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The characteristic polynomial $\Phi_j^{(i)}$ of Frob is independent of $l \neq p$ and has coefficients in $\mathbb{Q}$.

Functional equation, sign, det Frob

Theorem (Deligne, Suh)

Let S be a proper and smooth scheme over a finite field $\mathbb{F}_q$ of characteristic $p > 0$.

1. *The polynomial $\Phi_j^{(i)} \in \mathbb{Q}[T]$ fulfils the functional equation*
$$T^N \Phi(q^{i-2j}/T) = \pm q^{\frac{N}{2}(i-2j)} \Phi(T)\,, \qquad (1)$$
for $N := \operatorname{rk} H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(j))$.
2. *The sign in the functional equation is that of*
$$\det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q},\mathbb{Q}_l(j)) \circlearrowleft) = (-1)^N \det(\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \circlearrowleft)\,.$$
It is independent of the Tate twist, i.e., of the choice of j.
3. *If i is even then* $\det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(i/2)) \circlearrowleft)$ *is either $(+1)$ or (-1). I.e., it gives the sign in (1) exactly.*
4. *If i is odd then N is even and the plus sign always holds.*

A twofold étale covering

Goal

We want to study the behaviour of the sign in the functional equation

$$[= \det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(i/2)) \circlearrowleft)]$$

within families, thereby varying S and p.

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Theorem (det Frob in families – T. Saito 2012)

Let K be a number field, $\mathscr{O}_K$ its ring of integers, X an irreducible $\mathscr{O}_K$-scheme, and $\pi\colon F \to X$ a smooth and proper family of schemes. Assume that π is pure of even *relative dimension i.*

Then there exists naturally a [unique] twofold étale covering $\varrho\colon Y \to X$ such that, for every closed point $x \in X$, the determinant of Frob *on $H^i_{\text{ét}}(F_{\overline{x}}, \mathbb{Q}_l(i/2))$ is $(+1)$ if and only if x splits under ϱ.*

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Remark

In the projective case, the same is true for non-middle cohomology.

Idea of proof. The higher direct image sheaf $R^i\pi_*\mathbb{Z}_l(i/2)$ on X is twisted constant [according to smooth base change]. Hence,

$$\Lambda^{\max} R^i\pi_*\mathbb{Z}_l(i/2)$$

is twisted constant of rank one. It is therefore given by a representation

$$r\colon \pi_1^{\text{ét}}(X,\overline{\eta}) \longrightarrow \mathbb{Z}_l^* \,,$$

for $\overline{\eta}$ any geometric point on X.

Moreover, Poincaré duality yields a perfect pairing

$$\Lambda^{\max} R^i\pi_*\mathbb{Z}_l(i/2) \times \Lambda^{\max} R^i\pi_*\mathbb{Z}_l(i/2) \longrightarrow \mathbb{Z}_l \,.$$

As this must be compatible with the operation of $\pi_1^{\text{ét}}(X,\overline{\eta})$, the image of r is actually contained in $\{\pm 1\}$. Thus, r gives rise to a twofold étale covering $\varrho\colon Y \to X$.

[Technical issues: The argument works only away from the prime l. The higher direct image sheaf $R^i\pi_*\mathbb{Z}_l(i/2)$ might have torsion, ...]

Concrete description

Lemma

$(*)$ *Let $X := P \backslash D$, for P a non-singular, integral, separated, and Noetherian scheme and $D \subset P$ a closed subscheme. Furthermore, let a twofold étale covering $Y \to X$ be given.*

Then there exist an invertible sheaf $\mathscr{D} \in \operatorname{Pic}(P)$ being divisible by 2 and a global section $\Delta \in \Gamma(P, \mathscr{D})$ such that $\operatorname{div} \Delta$ is a reduced divisor, $\operatorname{supp} \operatorname{div} \Delta \subseteq D$, and $Y \to X$ is described by the equation

$$w^2 = \Delta \,.$$

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For a closed point $x \in X$ with finite residue field $k(x)$, the following statements are equivalent.

- $\det(\text{Frob}\colon H^i_{\text{ét}}(F_{\overline{x}}, \mathbb{Q}_l(i/2)) \circlearrowleft) = 1$,
- $\Delta(x) \in (k(x)^*)^2$.

A criterion for non-triviality

Theorem (Non-triviality criterion – Costa/Elsenhans/J. 2015)

Let K be a number field, $\mathscr{O}_K$ its ring of integers, P a non-singular, irreducible scheme that is flat over $\mathscr{O}_K$, $D \subset P$ a closed subscheme, and $X := P \setminus D$.

As above, let $\pi' \colon F' \to X$ be a smooth and proper family of schemes. Suppose, moreover, that π' extends to a proper and flat family $\pi \colon F \to P$ of even *relative dimension i, in which F is still non-singular.*

(∗∗) *Furthermore, assume that, for some geometric point $\overline{z} \colon \overline{K} \to D$, the fibre $F_{\overline{z}}$ has exactly one singular point, which is an ordinary double point.*

Then the twofold étale covering $\varrho \colon Y \to X$, associated with π, is obstructed at D. [In particular, it is non-trivial.]

Idea of proof. The Picard-Lefschetz formula [SGA7] describes the monodromy operation around singular fibres of $R^i\pi_*\mathbb{Z}_l(i/2)$. One ordinary double point in the fibre leads to one eigenvalue (-1). □

The normalised discriminant (model case)

Definition

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Assume that the discriminant locus $D = D_1 \cup \ldots \cup D_m$ is a union of divisors. If the non-triviality criterion applies to every divisor D_i then

$$\operatorname{div} \Delta = (D_1) + \ldots + (D_m)\,.$$

Classically, every section Λ such that $\operatorname{div} \Lambda = (D_1) + \ldots + (D_m)$ is called a *discriminant*.

- If P is proper over a field K then the discriminant is thus unique up to a scaling factor from K^*.
- If P is proper over $\mathbb{Z}$ then the discriminant is unique up to sign.

The normalised discriminant (model case) II

Definition (The normalised discriminant)

Let P be a non-singular, integral, and proper $\mathbb{Z}$-scheme and $X := P \backslash D$, for $D \subset P$ a closed subscheme. Furthermore, let $\pi\colon F \to X$ be a smooth and proper family of schemes, which is pure of *even* relative dimension i.

Then, the property

$$\Delta(x) \in (k(x)^*)^2 \iff \det(\mathrm{Frob}\colon H^i_{\text{ét}}(F_{\overline{x}}, \mathbb{Q}_l(i/2)) \circlearrowleft) = 1 \qquad (2)$$

provides a unique section Δ. We call Δ the *normalised discriminant* of the family π.

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The assumptions are not hard to fulfil.

Complete intersections

$$V := \operatorname{Proj} \operatorname{Sym} \bigoplus_{1 \leqslant i \leqslant c} H^0(\mathbf{P}^n_{\mathbb{Z}}, \mathscr{O}(d_i))^\vee$$

is a naive parameter space for complete intersections in $\mathbf{P}^n$ of multidegree $(d_1, \ldots, d_c)$. V is smooth over $\mathbb{Z}$.

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Lemma

1. *There is an irreducible closed subscheme $D \subset V$ of codimension 1 such that the fibre F_x is non-singular of dimension $n - c$ if and only if $x \notin D$. The restriction of π to $\pi^{-1}(V \backslash D)$ is smooth.*
2. *There is a closed subscheme $Z \subset D$ such that $\dim F_x = n - c$ if and only if $x \notin Z$. The restriction of π to $\pi^{-1}(V \backslash Z)$ is flat.*
3. *There exists a closed point $z \in (D \backslash Z)_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ such that F_z has exactly one singular point, which is an ordinary double point.*

Idea of proof. This is mostly standard algebraic geometry. Part 3 and the fact that D is irreducible of codimension 1 are due to O. Benoist (2012). □

Complete intersections II

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Theorem (Normalised discriminant for complete intersections, Costa/Elsenhans/J. 2015)

Let $i = n - c$ *be even. Then the normalised discriminant* Δ *is a section* $\Delta \in \mathscr{O}_V(D)$ *such that* $\operatorname{div} \Delta = (D)$. *It has the property below.*

Let K *be a number field and* $x \in (V \backslash D)(K)$ *be any* K*-rational point. Then, for any prime* $\mathfrak{p} \subset \mathscr{O}_K$ *of good reduction,*

$$\det(\mathrm{Frob}\colon H^i_{\text{ét}}((F_x)_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathbb{Q}_l(i/2)) \circlearrowleft) = \left(\frac{\Delta(x)}{\mathfrak{p}}\right).$$

A particular case: Hypersurfaces

In this case, several simplifications occur.

- the subscheme $Z \subset V$ is empty.
- An explicit example of a hypersurface of degree d with exactly one singular point, which is an ordinary double point, is provided by the equation

$$X_0^{d-2}(X_1^2 + \cdots + X_n^2) + X_1^d + \cdots + X_n^d = 0 \,.$$

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Theorem (G. Boole 1841/45)

The discriminant of degree d hypersurfaces in $\mathbf{P}^n$ is of degree $(d-1)^n(n+1)$.

Double covers

Let d be even. Then

$$W := \operatorname{Proj}\operatorname{Sym}(\mathbb{Z} \oplus H^0(\mathbf{P}^n_{\mathbb{Z}}, \mathscr{O}(d))^\vee)$$

is a naive parameter space for double covers $tw^2 = s$ of $\mathbf{P}^n$ ramified at a degree d hypersurface. W is smooth over $\mathbb{Z}$.

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Lemma

The closed subset $D \subset W$ parametrising singular double covers of $\mathbf{P}^n$ is the union of three irreducible components. These are

- *the cone C_{D_d} over the locus $D_d \subset V$ parametrising singular hypersurfaces in $\mathbf{P}^n$ of degree d [i.e., the ramification locus is singular],*
- *the hyperplane H_0 [corresponding to the case $t = 0$], and*
- *the special fibre W_2.*

Double covers II

Theorem (Normalised discriminant for double covers, Costa/Elsenhans/J. 2016)

Let $i = n$ be even. Then the normalised discriminant Δ is a section $\Delta \in \mathscr{O}_W(D)$ such that $\operatorname{supp}(\operatorname{div}\Delta)_{\mathbb{Q}} = (C_{D_d} \cup H_0)_{\mathbb{Q}}$. It has the property below.

Let K be a number field and $x \in (W \backslash (C_{D_d} \cup H_0))(K)$ be any K-rational point. Then, for any prime $\mathfrak{p} \subset \mathscr{O}_K$ of good reduction,

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Idea of proof. Everything except for $\operatorname{supp}(\operatorname{div}\Delta)_\mathbb{Q} = (C_{D_d} \cup H_0)_\mathbb{Q}$ just follows from the model case. The double cover, given by

$$w^2 = X_0^{d-2}(X_1^2 + \cdots + X_n^2) + X_1^d + \cdots + X_n^d$$

has exactly one singular point, which is an ordinary double point. Thus, the non-triviality criterion applies to C_{D_d}. As $\deg C_{D_d} = (d-1)^n(n+1)$ is odd, $\operatorname{supp}\operatorname{div}\Delta \supseteq H_0$ is enforced, too.

The non-occurrence of W_2 is slightly more subtle.

Double covers III

Remarks

1. The result says, in particular, that the relationship between the normalised discriminant of the double cover $tw^2 = s$ and the discriminant of the hypersurface $s = 0$ is given by the formula

$$\Delta(t, s) = \pm t\Delta_{\text{hyp}}(s)\,.$$

However, as $(n-1)$ is odd, for the discriminant of a hypersurface in $\mathbf{P}^n$, we do not have a canonical choice of sign, anyway.

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However, as $(n-1)$ is odd, for the discriminant of a hypersurface in $\mathbf{P}^n$, we do not have a canonical choice of sign, anyway.
2. If we adopt Demazure's convention that $X_0^d + \cdots + X_n^d = 0$ has positive discriminant then the sign is $(-1)^{\frac{nd}{4}}$.

$K3$ surfaces–Generalities

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The theory above shows that

- $$\det(\text{Frob}\colon H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathbb{Q}_l(1)) \circlearrowleft)$$
  extends to a quadratic character $\text{Gal}(\overline{K}/K) \to \{1, -1\}$. I.e.,
  $$\det(\text{Frob}\colon H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathbb{Q}_l(1)) \circlearrowleft) = \left(\frac{\Delta_{H^2}(S)}{\mathfrak{p}}\right)$$
  for some $\Delta_{H^2}(S) \in K^*$, unique up to squares.
- If $S = F_x$ is a member of a "reasonable" family of $K3$ surfaces then $\Delta_{H^2}(S) = (\Delta(x) \bmod (K^*)^2)$, for $\Delta(x)$ the normalised discriminant.

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There are the algebraic part $H_{\text{alg}} := \text{im}(c_1\colon \text{Pic}(S_{\overline{K}}) \to H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)))$ and its orthogonal complement $T := (H_{\text{alg}})^\perp$, the transcendental part of the cohomology $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)))$.

$K3$ surfaces–Generalities II

The field of definition of $\operatorname{Pic}(S_{\overline{K}})$ is a finite Galois extension L of K.

Lemma

L/K is unramified at every prime, where S has good reduction.

Idea of proof. This follows directly from smooth base change. □

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Fact

One has

$$\det(\mathrm{Frob}_{\mathfrak{p}} \colon H_{\mathrm{alg}} \circlearrowleft) = \left(\frac{\Delta_{\mathrm{alg}}(S)}{\mathfrak{p}}\right).$$

Application to $K3$ surfaces: Rank jumps

Proposition (Rank jumps)

Let S be a $K3$ surface over a number field K and $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of good reduction of residue characteristic $\neq 2$.

1. *Then* $\operatorname{rk} \operatorname{Pic} S_{\overline{\mathbb{F}}_\mathfrak{p}} \geqslant \operatorname{rk} \operatorname{Pic} S_{\overline{K}}$.
2. *Assume that* $\operatorname{rk} \operatorname{Pic} S_{\overline{K}}$ *is even. Then the following is true.*
   If $\det \operatorname{Frob}_\mathfrak{p}|_T = -1$ *then* $\operatorname{rk} \operatorname{Pic} S_{\overline{\mathbb{F}}_\mathfrak{p}} \geqslant \operatorname{rk} \operatorname{Pic} S_{\overline{K}} + 2$.

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Idea of proof. 1. Upon $\operatorname{Pic} S_{\overline{K}}$, $\operatorname{Gal}(\overline{K}/K)$ operates via the finite quotient $\operatorname{Gal}(L/K)$. Hence, some power of $\operatorname{Frob}_\mathfrak{p}$ acts as the identity. Tate's conjecture [proven for $K3$ surfaces by Charles, Madapusi Pera, and Lieblich/ Maulik/Snowden 2013/15] implies the claim.

2. According to the Weil conjectures [Deligne 1973], every eigenvalue of Frob on T has absolute value 1. Those different form 1 and (-1) come in pairs of conjugates. Thus, to have determinant (-1), at least one eigenvalue must be (-1) and at least one must be 1. Use the Tate conjecture. □

Application to $K3$ surfaces: Rank jumps II

Theorem (Costa/Elsenhans/J. 2015)

Let S be a $K3$ surface over a number field K and $\mathfrak{p} \subset \mathscr{O}_K$ be any prime of good reduction.

1. *Then one has*

$$\det(\mathrm{Frob}_{\mathfrak{p}} \colon T \circlearrowleft) = \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{alg}}(S)}{\mathfrak{p}} \right).$$

2. *Assume that* $\mathrm{rk}\,\mathrm{Pic}\, S_{\overline{K}}$ *is even. If* $\left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{alg}}(S)}{\mathfrak{p}} \right) = -1$ *then one has*

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$$\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}}_{\mathfrak{p}}} \geqslant \operatorname{rk}\operatorname{Pic} S_{\overline{K}} + 2\,.$$

Remark

Thus, unless $\Delta_{H^2}(S)\Delta_{\mathrm{alg}}(S)$ is a square in K, the Picard rank jumps for at least half the primes. We call $\left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{alg}}(S)}{\mathfrak{p}}\right)$ the *jump character* of S.

Application to $K3$ surfaces: Rank jumps III

Remarks

- $\Delta_{H^2}(S)$ is a product of only bad primes. Thus, for a given surface, it can be computed by just counting points.
- To compute $\Delta_{\mathrm{alg}}(S)$, some information on $\mathrm{Pic}(S_{\overline{K}})$ is necessary.

Application to $K3$ surfaces: Rank jumps III

Remarks

- $\Delta_{H^2}(S)$ is a product of only bad primes. Thus, for a given surface, it can be computed by just counting points.
- To compute $\Delta_{\mathrm{alg}}(S)$, some information on $\mathrm{Pic}(S_{\overline{K}})$ is necessary.

Example

For the diagonal surface $S\colon X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$ over $\mathbb{Q}$, one has

- $\Delta_{H^2}(S) = 1$ and
- $\Delta_{\mathrm{alg}}(S) = -1$.

Idea of proof. 2 is the only bad prime of S. To exclude the options that $\Delta_{H^2}(S)$ might be (-1) or ± 2, it suffices to count points on the reductions S_3 and S_5.

On the other hand, the Galois operation on $\mathrm{Pic}\, S_{\overline{\mathbb{Q}}}$ is completely described in the Ph.D. thesis of M. Bright. □

Special space quartics

Theorem (Costa/Elsenhans/J. 2016)

Let K be a number field and S the space quartic

$$S\colon cX_3^4 + f_2(X_0,X_1,X_2)X_3^2 + f_4(X_0,X_1,X_2) = 0\,.$$

Then $\operatorname{rk}\operatorname{Pic} S_{\overline{K}} \geqslant 8$*. Assuming* $\operatorname{rk}\operatorname{Pic} S_{\overline{K}} = 8$*, one has*

1. *one has* $\Delta_{\mathrm{alg}}(S) = \delta(f_2^2 - 4cf_4)$*.*
2. *The jump character is* $\left(\frac{c\delta(f_4)\delta(f_2^2-4cf_4)}{.}\right)$*.*

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Idea of proof. The surface $cw^2 + f_2(X_0,X_1,X_2)w + f_4(X_0,X_1,X_2) = 0$, which is Del Pezzo of degree 2, is covered $2:1$ by S. This shows that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}} \geqslant 8$ and claim 1.

The discriminant splits on this subfamily into c, $\delta(f_4)$, and a third factor that enters quadratically. □

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Remark

The discriminant δ of ternary quartics is of degree 27 and [thanks to the efforts of A.-S. Elsenhans] easily computable using `magma`.

Special space quartics II

Example (Surfaces with CM by $\mathbb{Q}(i)$)

Let S be a space quartic of type

$$X_3^4 + f_4(X_0, X_1, X_2) = 0\,,$$

which is of geometric Picard rank 8. Then the jump character is $(\frac{-1}{\cdot})$.

Idea of proof. $\delta(f_4)\delta(-4f_4) = (-4)^{27}\delta(f_4)^2$. □

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Example (A surface with with trivial jump character)

Consider the space quartic $S\colon X_3^4 + f_2(X_0, X_1, X_2)X_3^2 + f_4(X_0, X_1, X_2) = 0$, for
$$f_2(X_0, X_1, X_2) := X_0^2 - X_0X_1 - X_0X_2 - X_1X_2 \qquad \text{and}$$
$$f_4(X_0, X_1, X_2) := -X_0^3X_2 + X_0X_1^2X_2 - X_1^4 - X_2^4\,.$$
Then the geometric Picard rank of S is 8 and the jump character of S is trivial.

Idea of proof. The reduction modulo, e.g., 19 has geometric Picard rank 8. Moreover, $\delta(f_4) = -2^8 3^3 431^2$ and $\delta(f_2^2 - 4f_4) = -2^{60}3^3 47^2$. □

Another application to $K3$ surfaces: Infinitely many rational curves

Conjecture

Every $K3$ surface S over an algebraically closed field K contains infinitely many rational curves.

Evidence. Odd rank case was proven by Li/Liedtke (2012), based on ideas of Bogomolov, Hassett, and Tschinkel. Further sufficient conditions include that S has infinitely many automorphisms or that S is elliptic.

Another application to $K3$ surfaces: Infinitely many rational curves II

Theorem (Costa/Elsenhans/J. 2016)

Let S be a $K3$ surface over a number field K. Assume that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ is even,

- *that $S_{\overline{K}}$ has neither real nor complex multiplication, and*
- *that $\Delta_{H^2}(S)\Delta_{\mathrm{alg}}(S)$ is a non-square in K.*

Then $S_{\overline{K}}$ contains infinitely many rational curves.

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Idea of proof. The approach of Li/Liedtke shows that one needs infinitely jump primes $\mathfrak{p}$ such that $S_{\mathfrak{p}}$ is not supersingular. Infinitely many jump primes are provided by the second assumption. The first one assures that only modulo a small subset of these, the reduction is supersingular. □

Thanks

Thank you!!