$K3$ surfaces and their Picard groups

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Intense collaboration workshop on
Rational points

MSRI, Berkeley
October 11, 2012

joint work with
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$K3$ surfaces

Definition

A $K3$ *surface* is a simply connected, projective algebraic surface having a global (algebraic, holomorphic) 2-form without zeroes or poles.

Examples

1. A double cover of $\mathbf{P}^2$, ramified at a smooth sextic curve.
2. A smooth quartic in $\mathbf{P}^3$.
3. A smooth complete intersection of a quadric and a cubic in $\mathbf{P}^4$.
4. A smooth complete intersection of three quadrics in $\mathbf{P}^5$.

Remark

Resolutions of singular quartics in $\mathbf{P}^3$ are $K3$ surfaces, too, when the singularities are rational.

$K3$ surfaces as complex algebraic surfaces

Properties of $K3$ surfaces

Betti numbers: $1, 0, 22, 0, 1.$

Hodge diamond:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}.$$

Picard group: $\mathbb{Z}^n$ for $n \in \{1, \ldots, 20\}$.

Question

Given a concrete $K3$ surface, defined over $\mathbb{Q}$, can one compute its geometric Picard group (really, in practice, ...)?

Reduction modulo p

Fact

Let S be a $K3$ surface over $\mathbb{Q}$ and p a prime of good reduction. Then the homomorphism

$$\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \to \operatorname{Pic}(S_{\overline{\mathbb{F}}_p}),$$

given by reduction, is injective.

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2. Alternative approach (Idea due to Ronald van Luijk): Choose two primes p_1 and p_2 of good reduction. Verify

$$\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_{p_1}}) = \operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_{p_2}}) = 2.$$

Prove, in addition, that the two Picard lattices are incompatible. (I.e., show that the discriminants differ by a factor being a non-square.)

$K3$ surfaces over finite fields

Facts

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Remarks (The Galois operation)

- The Galois group operates on the Picard group and on étale cohomology. We have two $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-representations.

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Remarks (The Galois operation)

- The Galois group operates on the Picard group and on étale cohomology. We have two $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-representations.
- The operations are compatible with the Chern class map. The Picard group is a sub-representation of the cohomology.

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The arithmetic Picard group injects into to second étale cohomology. The image is contained in the eigenspace for the eigenvalue 1.

Consequence

The number of eigenvalues that are roots of unity, counted with multiplicity, is an upper bound for the geometric Picard rank.

The Tate conjecture

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- Consequently, for every $K3$ surface over $\overline{\mathbb{F}}_p$, the Picard rank is predicted to be even. (This implies $\operatorname{rk}\operatorname{Pic}(V) \geq 2$ for every $K3$ surface over $\overline{\mathbb{F}}_p$.)

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- Consequently, for every $K3$ surface over $\overline{\mathbb{F}}_p$, the Picard rank is predicted to be even. (This implies $\operatorname{rk}\operatorname{Pic}(V) \geq 2$ for every $K3$ surface over $\overline{\mathbb{F}}_p$.)
- The Tate conjecture is proven for most $K3$ surfaces. (For example, in characteristic ≥ 5, Nygaard 1983+Charles 2012.)

The Galois operation on étale cohomology

Question

Can we compute the Galois operation on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$?

As the Galois group is generated by the Frobenius, we had to compute the action of the Frobenius. This would mean to give a 22×22-matrix and a basis of the étale cohomology group. It seems hard to give an explicit basis.

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Easier problem

Compute the characteristic polynomial Φ of the Frobenius.

The characteristic polynomial Φ of the Frobenius is independent of the choice of a basis.

Computing Φ

Theorem (Lefschetz' Trace Formula)

For a $K3$ surface V over $\mathbb{F}_p$, one has

$$\#V(\mathbb{F}_{p^e}) = 1 + p^{2e} + \mathrm{Tr}(\mathrm{Frob}^e)\,.$$

Here, Frob *denotes the operation of Frobenius on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$.*

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Theorem (Newton's identities)

Let V be a K3 surface over $\mathbb{F}_p$ and Φ be the characteristic polynomial of Frob *on* $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$.

Then the coefficient of Φ at T^{22-e} may be computed from the traces of $\mathrm{Frob}, \mathrm{Frob}^2, \ldots, \mathrm{Frob}^e$.

Interlude: Two versions of the characteristic polynomial

- The Picard group injects into $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$.
- However, $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$ appears to be more natural. And it occurs in the Lefschetz trace formula.
- Both differ only in the operation of Frob.

  The operation of Frob on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$ is the operation on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$ divided by p.

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Fact

We therefore have

$$\Phi^{(1)}(T) = \frac{1}{p^{22}}\Phi(pT)\,.$$

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Remark

From now on, in this talk, we will prefer $\Phi^{(1)}$ versus Φ.

Restrictions on $\Phi^{(1)}$

Not every polynomial of degree 22 may appear as the characteristic polynomial of Frobenius for a $K3$ surface over $\mathbb{F}_p$. There are the following restrictions, which were established in the Grothendieck era.

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Observation (Hyperplane section)

Generally, for projective varieties, we also have $\Phi^{(1)}(1) = 0$.

Computing Φ II

Algorithm (Candidates for the characteristic polynomial)

1. Count $V(\mathbb{F}_q)$, $V(\mathbb{F}_{q^2})$,..., $V(\mathbb{F}_{q^{10}})$.
2. Compute the coefficients of $T^{21}, \dots, T^{12}$. (Newton's identities)
3. Determine the coefficients of T^0,...,T^{10} up to a common sign. (Functional equation)
4. Calculate the coefficient of T^{11} using $\Phi^{(1)}(1) = 0$.

The result are two candidates for $\Phi^{(1)}$. One for each sign in the functional equation.

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Computing Φ III

Algorithm (A naive method to determine the sign)

- Count $V(\mathbb{F}_{q^{11}})$, $V(\mathbb{F}_{q^{12}}), \ldots$ until the sign is determined.

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Algorithm (A better algorithm)

- For both candidates, calculate the absolute values of their zeroes.
- If that excludes neither candidate then count $V(\mathbb{F}_{q^{11}})$, $V(\mathbb{F}_{q^{12}}), \ldots$ until the sign is determined.

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Question

Can we do better? I.e., can we exclude a candidate in another way?

Unfortunately, the Theorem of Mazur-Ogus never excludes any of the candidates.

The Artin-Tate conjecture

Notation

- V – a $K3$ surface over $\mathbb{F}_q$,
- $\Phi^{(1)}$ – the characteristic polynomial of Frob on $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(1))$,
- ρ – the rank of the arithmetic (i.e., over $\mathbb{F}_q$) Picard group,
- Δ – the discriminant of the arithmetic Picard group,
- $\text{Br}(V)$ – the Brauer group. $\# \text{Br}(V)$ is a perfect square (if finite).

Conjecture (Artin-Tate, special case of a $K3$ surface)

In the notation above, one has

$$|\Delta| = \frac{q \cdot \lim\limits_{T \to 1} \frac{\Phi^{(1)}(T)}{(T-1)^\rho}}{\#\text{Br}(V)} .$$

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Theorem (Milne)

The Tate conjecture implies $\#\,\text{Br}(V) < \infty$ and the Artin-Tate conjecture.

Another restriction on $\Phi^{(1)}$

Observation

Let V be a $K3$ surface over $\mathbb{F}_p$. Assume that $\operatorname{rk}\operatorname{Pic}(V) = \operatorname{rk}\operatorname{Pic}(V_{\mathbb{F}_{p^k}})$.
Then, as the Picard lattices are contained in each other, the discriminants differ only by a factor being a perfect square.

Suppose further that V and $V_{\mathbb{F}_{p^k}}$ satisfy the Tate conjecture.
Then, as $\Phi^{(1)}$ determines the polynomial $\Phi^{(1)}_{\mathbb{F}_{p^k}}$, the Artin-Tate formula allows to calculate the square classes of both discriminants.

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Definition

This yields a restriction for $\Phi^{(1)}$, which we call the *field extension condition*.

The field extension condition

Theorem (Elsenhans & J. 2010)

1. *The field extension condition for $\mathbb{F}_{q^2}/\mathbb{F}_q$ implies all other field extension conditions.*
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Remark

For us, it was very surprising that the Artin-Tate formula has the potential to contradict itself under field extensions.

The field extension condition II

Theorem (Elsenhans & J. 2011)

Let X be a smooth, projective variety of even dimension d over a finite field $\mathbb{F}_q$ of characteristic p and $\Phi^{(d/2)} \in \mathbb{Q}[T]$ be the characteristic polynomial of Frob *on $H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(d/2))$.*
Then $(-2)^N \Phi^{(d/2)}(-1)$ is a square in $\mathbb{Q}$, up to a possible factor p.

Idea of proof ($l \neq 2, p$):
$H := H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(d/2))/\text{tors}$ is a unimodular $\mathbb{Z}_l$-lattice.
Frob operates on H as an orthogonal map.
Zarhin (1982) showed that $[H/(\text{id} - \varphi)H]_{\text{tors}}$ carries a nondegenerate alternating pairing, for any orthogonal φ. Apply this to $\varphi := -\text{Frob}$.

Remark

When $p \neq 2$, there is a geometric characterization of the exponent of p.

A $K3$ surface over $\mathbb{Q}$ of geometric Picard rank 1

We want to construct $K3$ surfaces over $\mathbb{Q}$ of prescribed geometric Picard rank. The example below shows the method in its simplest form.

Example

Let V be a $K3$ surface of degree 2, given by

$$w^2 = f_6(x, y, z)$$

for

$$\begin{aligned} f_6(x, y, z) \equiv{} & 4z^6 + 2xy^5 + 3x^2z^4 + x^2y^4 + 2x^3z^3 \\ & + x^3y^3 + 3x^4z^2 + 2x^4y^2 + x^5y + 2x^6 \pmod 5, \\ f_6(x, y, z) \equiv{} & y^6 + 3z^6 + 5xz^5 + 5x^2y^4 + x^2z^4 \\ & + 3x^3y^3 + x^3z^3 + 5x^4y^2 + x^4z^2 + 5x^5y + 2x^6 \pmod 7. \end{aligned}$$

Then the geometric Picard rank of V is equal to 1.

Verifying Picard rank 1

The characteristic polynomials of the Frobenius are

$$\begin{aligned}
\Phi_5^{(1)}(t) &= \frac{1}{5}(t-1)^2\big(5t^{20} - t^{19} + t^{18} + 2t^{17} + 3t^{15} + t^{14} - 2t^{13} + t^{12} - t^{11} \\
&\qquad + 2t^{10} - t^9 + t^8 - 2t^7 + t^6 + 3t^5 + 2t^3 + t^2 - t + 5\big), \\
\Phi_7^{(1)}(t) &= \frac{1}{7}(t-1)(t+1)\big(7t^{20} - 16t^{19} + 27t^{18} - 37t^{17} + 44t^{16} - 52t^{15} + 60t^{14} \\
&\qquad - 68t^{13} + 74t^{12} - 76t^{11} + 75t^{10} - 76t^9 + 74t^8 - 68t^7 \\
&\qquad + 60t^6 - 52t^5 + 44t^4 - 37t^3 + 27t^2 - 16t + 7\big).
\end{aligned}$$

The reductions modulo 5 and 7 are surfaces of geometric Picard rank 2.

The Artin-Tate formula gives us the square classes of (-5) and (-997) for the discriminants.

This yields Picard rank 1 over $\overline{\mathbb{Q}}$.

An improvement using the theory of deformations

Theorem (Elsenhans & J. 2009)

Let $p \neq 2$ *be a prime number and* X *be a scheme, proper and smooth over* $\mathbb{Z}$.

Then the specialization homomorphism $\mathrm{Pic}(X_{\overline{\mathbb{Q}}}) \to \mathrm{Pic}(X_{\overline{\mathbb{F}}_p})$ *has a torsion-free cokernel.*

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Remarks

- The same result is true in a more general relative situation over a discrete valuation ring R with perfect residue field of characteristic p and ramification degree $e < p - 1$.
- The special case that R is complete is due to M. Raynaud (1979).
- The most elementary proof is based on a deformation-theoretic argument, controlling the obstructions to lifting $\mathscr{L} \in \mathrm{Pic}(X_p)$ to $\mathrm{Pic}(X_{\mathbb{Z}/p^2\mathbb{Z}}), \mathrm{Pic}(X_{\mathbb{Z}/p^3\mathbb{Z}}), \ldots$.

An improvement using the theory of deformations II

Example (Elsenhans & J. 2010)

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$$f_6(x,y,z) \equiv x^6 + 2x^5z + 2x^4y^2 + 2x^4z^2 + 2x^3y^3 + 2x^3z^3 + 2x^2y^4 + 2x^2y^3z + x^2z^4 + xy^3z^2 + 2xz^5 + y^6 \pmod 3.$$

Assume further that the coefficient of y^2z^4 is not divisible by 9.

Then the geometric Picard rank of V is equal to 1.

Verifying Picard rank 1

The characteristic polynomial of the Frobenius is

$$\Phi_3^{(1)}(t) = \frac{1}{3}(t-1)^2\big(3t^{20} - 3t^{19} - 3t^{18} + 8t^{17} - 3t^{16} - 4t^{15} + 6t^{14} - 4t^{13} + 2t^{12} + 4t^{11} - 7t^{10} + 4t^{9} + 2t^{8} - 4t^{7} + 6t^{6} - 4t^{5} - 3t^{4} + 8t^{3} - 3t^{2} - 3t + 3\big).$$

The reduction modulo 3 is a surface of geometric Picard rank 2.

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The reduction modulo 3 is a surface of geometric Picard rank 2.

Explicit generators

One has

$$f_6 \equiv f_3^2 + xf_5 \pmod 3$$

for $f_3 = 2x^3 + 2x^2z + xz^2 + 2y^3$ and $f_5 = 2x^3y^2 + x^2z^3 + 2xy^4 + 2z^5$.

Hence, $x = 0$ defines a line ℓ that is a tritangent line to the ramification locus. The pull-back of ℓ splits into two divisors L_1 and L_2.

Verifying Picard rank 1 II

Intersection matrix:

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$$

of determinant (-5).

Verifying Picard rank 1 II

Intersection matrix:

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$$

of determinant (-5).

Observation

As $\operatorname{Pic}(X_{\overline{\mathbb{F}}_p}) / \operatorname{Pic}(X_{\overline{\mathbb{Q}}})$ is torsion-free, for $\operatorname{rk} \operatorname{Pic}(X_{\overline{\mathbb{Q}}}) = 1$, it suffices to find one $\mathscr{L} \in \operatorname{Pic}(X_{\overline{\mathbb{F}}_p})$ that does not lift.

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Explicit obstruction

Put $f_6 \equiv f_3^2 + xf_5 \pmod{p}$. Then the obstruction to lifting $\mathscr{O}(L_1)$ and $\mathscr{O}(L_2)$ to $V_{\mathbb{Z}/p^2\mathbb{Z}}$ is given by $(G \bmod (p, x, f_3, f_5))$ for

$$G(x, y, z) := (f_6 - f_3^2 - xf_5)/p\,.$$

Do we have a practical algorithm to compute the Picard rank for a $K3$ surface given?

Problems

- The method of R. van Luijk gives an upper bound for the Picard rank. The resulting bound depends on the primes used.

  Good primes do not always exist (Charles 2011).
- To verify the rank bound 2 at a place p, we need $\#V(\mathbb{F}_p), \ldots, \#V(\mathbb{F}_{p^{10}})$.

  How to determine these numbers even for medium sized primes?
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To summarize, in general, we don't. Let me nevertheless continue showing

- a few improvements, mainly to save computational time.
- A systematic test on the effect of the Artin-Tate conditions.

Verify rank two using $\#V(\mathbb{F}_q), \ldots, \#V(\mathbb{F}_{q^9})$

In some cases, we can prove an upper bound of 2 for the geometric Picard rank without the most expensive counting step.

Algorithm (Bounding the Picard rank using $\#V(\mathbb{F}_q),\ldots,\#V(\mathbb{F}_{q^9})$)

1. Compute the coefficients for $T^{21}, \ldots, T^{13}, T^9, \ldots, T^0$. Three coefficients remain plus an unknown sign.
2. Assume, there are more than two zeroes that are roots of unity. I.e., assume a Picard rank bigger than 2.

   The order of such a root of unity is not bigger than 66.
3. Compute the characteristic polynomial for each assumption. This means to solve a linear system of equations in each case.
4. Exclude as many of the candidates as possible.

An example

Example

Consider the $K3$ surface of degree 2 over $\mathbb{F}_7$, given by

$$w^2 = y^6+3z^6+5xz^5+5x^2y^4+x^2z^4+3x^3y^3+x^3z^3+5x^4y^2+x^4z^2+5x^5y+2x^6.$$

Point counting up to $\mathbb{F}_{7^9}$ yields 66, 2 378, 118 113, 5 768 710, 282 535 041, 13 841 275 877, 678 223 852 225, 33 232 944 372 654, and 1 628 413 551 007 224.

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Question

Can we prove an upper bound of 2 for the Picard rank?

Hypothetical characteristic polynomials

Assuming that the geometric Picard rank is bigger than 2, we find three candidates,

$$\Phi_i^{(1)}(t) = \frac{1}{7}\big(7t^{22} - 16t^{21} + 20t^{20} - 21t^{19} + 17t^{18} - 15t^{17} + 16t^{16} - 16t^{15} + 14t^{14} - 8t^{13} + a_i t^{12} + b_i t^{11} + c_i t^{10} + (-1)^{j_i}(-8t^9 + 14t^8 - 16t^7 + 16t^6 - 15t^5 + 17t^4 - 21t^3 + 20t^2 - 16t + 7)\big)$$

for

$$j_1 = 0, \qquad (a_1, b_1, c_1) = (4, -4, 4)\,,$$
$$j_2 = 1, \qquad (a_2, b_2, c_2) = (2, 0, -2)\,,$$
$$j_3 = 1, \qquad (a_3, b_3, c_3) = (3, 0, -3)\,.$$

All roots are of absolute value 1.

Application of the Artin-Tate formula

polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
Φ_1	$\mathbb{F}_7$	2	58
	$\mathbb{F}_{49}$	2	4524
Φ_2	$\mathbb{F}_7$	1	4
	$\mathbb{F}_{49}$	2	1996
Φ_3	$\mathbb{F}_7$	1	6
	$\mathbb{F}_{49}$	2	2997

Interpretation

Φ_1 is impossible, in general. Φ_2 and Φ_3 are impossible in degree 2.

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Conclusion

The geometric Picard rank is at most 2.

Using $\mathbb{F}_{7^{10}}$ data

To determine the characteristic polynomial exactly, we have to count the number of points over $\mathbb{F}_{7^{10}}$. The result is

$\#V(\mathbb{F}_{7^{10}}) = 79\,792\,267\,067\,823\,523.$

We find two candidates Φ_4 and Φ_5, one for each sign in the functional equation.

$$\Phi_i^{(1)}(t) = \frac{1}{7}(7t^{22} - 16t^{21} + 20t^{20} - 21t^{19} + 17t^{18} - 15t^{17} + 16t^{16} - 16t^{15} + 14t^{14} - 8t^{13} + t^{12} + a_i t^{11} + (-1)^{j_i}(-t^{10} + 8t^9 - 14t^8 + 16t^7 - 16t^6 + 15t^5 - 17t^4 + 21t^3 - 20t^2 + 16t - 7))$$

for $j_4 = 0$, and $a_4 = 0$, or $j_5 = 1$, and $a_5 = 2$.

All roots are of absolute value 1.

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polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
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Φ_5	$\mathbb{F}_7$	2	55
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Φ_4 is possible for a $K3$ surface of degree 2. Φ_5 is impossible for $K3$ surfaces, in general.

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Interpretation

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Conclusion

Φ_4 is the characteristic polynomial. In the functional equation, the minus-sign is correct.

A statistical test of the conditions

Our sample

	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$d = 2$	1000 rand	1000 rand	1000 dec	1000 dec
$d = 4$	1000 rand	1000 ell		
$d = 6$	1000 rand	1000 ell		
$d = 8$	1000 rand	1000 ell		

dec = decoupled, ell = elliptic, rand = random

Methods for point counting:

- Naive counting.
- Using the elliptic fibration (if existing).
- Calculating a convolution (Decoupled case).

Proving geometric Picard rank ≤ 2 using data up to $\mathbb{F}_{q^9}$

	Number of polynomials	0	1	2	3	4	5	6
$d = 2,\ p = 2$	without	84	479	312	89	21	12	3
	with A-T conditions	149	598	218	28	7	0	0
$d = 2,\ p = 3$	without	116	480	285	88	24	4	3
	with A-T conditions	214	573	193	20	0	0	0
$d = 2,\ p = 5$	without	85	581	209	96	25	4	0
	with A-T conditions	158	651	169	20	2	0	0
$d = 2,\ p = 7$	without	92	534	232	98	37	7	0
	with A-T conditions	214	611	154	21	0	0	0
$d = 4,\ p = 2$	without	40	532	303	87	29	8	1
	with A-T conditions	81	638	249	27	5	0	0
$d = 4,\ p = 3$	without	22	669	242	57	9	1	0
	with A-T conditions	53	785	161	1	0	0	0
$d = 6,\ p = 2$	without	39	549	312	70	22	6	2
	with A-T conditions	83	645	257	14	1	0	0
$d = 6,\ p = 3$	without	16	713	217	47	7	0	0
	with A-T conditions	50	797	148	5	0	0	0
$d = 8,\ p = 2$	without	25	657	268	38	8	4	0
	with A-T conditions	29	723	239	5	4	0	0
$d = 8,\ p = 3$	without	12	720	236	27	4	1	0
	with A-T conditions	20	803	175	2	0	0	0

Determination of sign using data up to $\mathbb{F}_{q^{10}}$

p	2	3	5	7	2	3	2	3	2	3
d	2	2	2	2	4	4	6	6	8	8
Known signs without A-T	768	843	864	869	761	876	790	888	822	897
Known signs using A-T	863	940	940	961	863	943	868	933	867	944
Remaining unknown signs	137	60	60	39	137	57	132	67	133	56
Data up to $\mathbb{F}_{p^{11}}$ insufficient	84	23	15	12	69	19	77	25	72	21
Data up to $\mathbb{F}_{p^{12}}$ insufficient	41	11	2	1	39	3	42	11	47	7
Data up to $\mathbb{F}_{p^{13}}$ insufficient	22	5	1	0	24	2	20	2	24	2
Data up to $\mathbb{F}_{p^{14}}$ insufficient	13	2	0	0	12	0	13	1	8	0
Data up to $\mathbb{F}_{p^{15}}$ insufficient	7	0	0	0	8	0	7	0	5	0
Data up to $\mathbb{F}_{p^{16}}$ insufficient	4	0	0	0	3	0	2	0	4	0
Data up to $\mathbb{F}_{p^{17}}$ insufficient	4	0	0	0	2	0	2	0	0	0
Data up to $\mathbb{F}_{p^{18}}$ insufficient	4	0	0	0	0	0	1	0	0	0
Data up to $\mathbb{F}_{p^{19}}$ insufficient	2	0	0	0	0	0	1	0	0	0
Data up to $\mathbb{F}_{p^{20}}$ insufficient	0	0	0	0	0	0	0	0	0	0

Proving geometric Picard rank ≤ 2 using data up to $\mathbb{F}_{q^{10}}$

		rank 2 proven not using $\#V(\mathbb{F}_{p^{10}})$	rank 2 proven	rank 2 possible
$p = 2,\ d = 2$	without	84	271	330
	with A-T conditions	149	278	301
$p = 3,\ d = 2$	without	116	397	460
	with A-T conditions	214	409	428
$p = 5,\ d = 2$	without	85	353	425
	with A-T conditions	158	360	382
$p = 7,\ d = 2$	without	92	460	511
	with A-T conditions	214	464	476
$p = 2,\ d = 4$	without	40	132	197
	with A-T conditions	81	138	163
$p = 3,\ d = 4$	without	22	79	114
	with A-T conditions	53	79	81
$p = 2,\ d = 6$	without	39	145	183
	with A-T conditions	83	152	163
$p = 3,\ d = 6$	without	16	74	101
	with A-T conditions	50	74	81
$p = 2,\ d = 8$	without	25	65	93
	with A-T conditions	29	65	74
$p = 3,\ d = 8$	without	12	23	47
	with A-T conditions	20	23	25

Existence of good primes

Definition

By a *good* prime, we mean a prime p such that $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}})+1 \geq \operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{F}}_p})$.

Theorem (Charles 2011)

There exists a $K3$ surface such that there are no good primes for it.

No explicit example is known at the moment.

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Remark

More precisely, Charles' result states that there is no good prime only when the Hodge structure in $H^2(X_{\mathbb{C}}, \mathbb{Q})$ complementary to the Picard group, has real multiplication.

Thus, for a "random" $K3$ surface, good primes exist.

Kummer surfaces

Fact (Kummer, Quartics with 16 singularities)

For parameters a, b, c, *put*

$$k := a^2 + b^2 + c^2 - 1 - 2abc\,,$$
$$\phi := x^2 + y^2 + z^2 + w^2 + 2a(yz + xw) + 2b(xz + yw) + 2c(xy + zw)\,.$$

Then

$$16kxyzw - \phi^2 = 0$$

defines a quartic surface. A generic member of this family has exactly 16 singular points.

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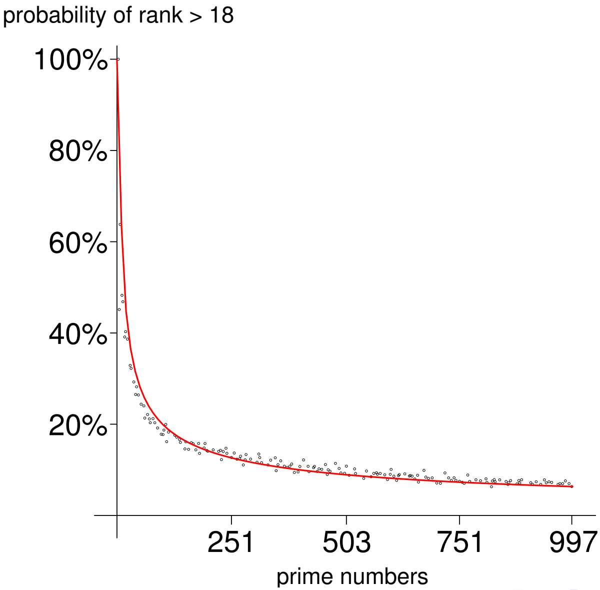
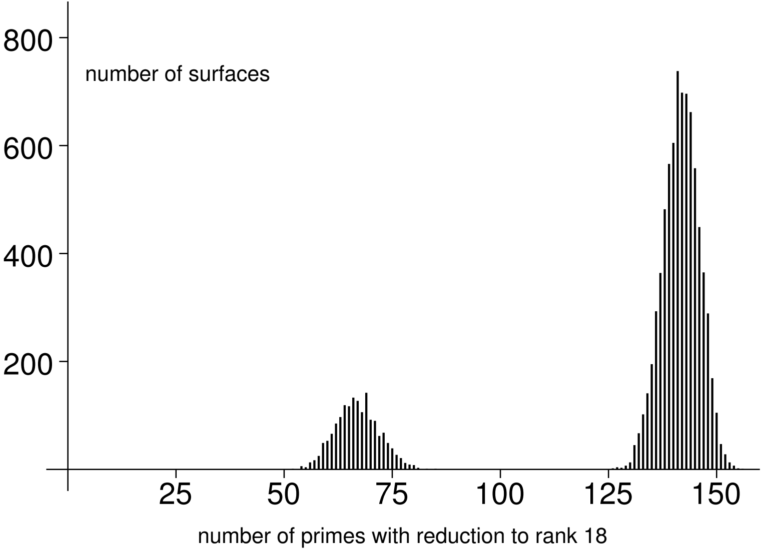
Our sample

- $a, b, c \in \{-30, \ldots, 30\}$. This leads to 9452 essentially different singular quartics.
- We used all the 168 primes < 1000.
- We determined the Picard rank in all cases.

Probability for a prime not to be good



How many primes with reduction to rank 18?



The density $\frac{1}{2}$ case

The plot suggests that, for some surfaces, the density of the good primes is close to $\frac{1}{2}$, while, for others, it is close to 1.

Explanation

- All examples with density $\leq \frac{1}{2}$ have Picard rank 18 over $\overline{\mathbb{Q}}$.
- In many cases, the corresponding abelian surfaces split into two elliptic curves. Usually, this splitting is defined over a quadratic extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$.

  Thus, the resulting elliptic curves are conjugate to each other over $\mathbb{Q}(\sqrt{d})$. Modulo an inert prime, the reductions are isogenous via Frob. We find Picard rank ≥ 20 after reduction modulo such a prime.

Summary

Goal

Compute the geometric Picard groups of $K3$ surfaces. Use R. van Luijk's method.

This requires point counting over relatively large finite fields.

Improvements

- Use the Artin-Tate formula to exclude some characteristic polynomials.
- Verify the rank bound 2 without the most expensive counting step.
- Use the Galois module structure of the Picard group together with the discriminants to reduce the rank bound by more than one.
- Use the fact that $\mathrm{Pic}(V_{\overline{\mathbb{F}}_p}) / \mathrm{Pic}(V_{\overline{\mathbb{Q}}})$ is torsion-free.

Summary II – The statistical test

Statistical test

We tested our improvements of van Luijk's method on $K3$ surfaces given by quartics having 16 singular points.

Observations

- In all cases, the method of van Luijk works when sufficiently large primes are used.
- Good primes seem to have density one in the odd rank case.
- Good primes seem to have density at least $\frac{1}{2}$ in the even rank case.
- We needed primes up to 103 to determine the Picard ranks in our examples.

Point counting took several weeks of CPU time.