On the Hasse principle for lines on del Pezzo surfaces

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joint work with
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Cubic surfaces

Almost two years ago, Daniel Loughran asked me the following question.

Question

Is there a non-singular cubic surface S over $\mathbb{Q}$ having a line over each of the fields $\mathbb{Q}_p$ for $p = 2, 3, 5, 7, 11, \ldots$, but no line defined over $\mathbb{Q}$?

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Classical Algebraic Geometry teaches that a non-singular cubic surface always contains exactly 27 lines. But, although S is defined over $\mathbb{Q}$, this count concerns the lines that are defined over the algebraically closed field $\overline{\mathbb{Q}}$.

Remark

A non-singular cubic surface defined over $\mathbb{R}$ always contains real lines.

In fact, there is a classification due to Schläfli into five types. These have 3, 3, 7, 15, and 27 real lines, accordingly.

Hasse-type principles

Definition (classical Hasse principle)

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Definition (Hasse principle for linear subspaces)

A class of varieties embedded into a fixed projective space $\mathbf{P}^n$ over k satisfies the *Hasse principle for linear subspaces of dimension* r if for each variety in the class, the existence of a linear subspace of dimension r over every completion of k implies the existence of a linear subspace of dimension r over k.

The Hasse principle for zero-dimensional schemes

Remarks

1. For $r = 0$ we recover the classical Hasse principle.
2. There is a Hilbert scheme parametrising the linear subspaces of fixed dimension. We are asking whether this scheme satisfies the classical Hasse principle.

E.g., for cubic surfaces, the Hilbert schemes of lines is a zero-dimensional, reduced scheme of degree 27.

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Zero-dimensional scheme may well violate the Hasse principle.

Example

Let $X \subset \mathbf{A}^1_{\mathbb{Q}}$ be defined by $(x^2 - 2)(x^2 - 17)(x^2 - 34) = 0$. Then X fails the Hasse principle.

Question

Can the Hasse principle for lines fail for cubic surfaces?

Del Pezzo surfaces

Definition

Let k be a field. A *del Pezzo surface* over k is a smooth projective surface S over k with ample anticanonical divisor $(-K_S)$. The *degree* of S is the self-intersection number $d = (-K_S)^2$.

This is the natural class of surfaces cubic surfaces belong to, namely they are the del Pezzo surfaces of degree 3.

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If k is algebraically closed then any del Pezzo surface is either isomorphic to

- $\mathbf{P}^2$ $(d = 9)$,
- $\mathbf{P}^1 \times \mathbf{P}^1$ $(d = 8)$,
- or the blow-up of $\mathbf{P}^2$ in $9 - d$ points in general position $(d \leq 8)$.

Del Pezzo surfaces II

Facts (Lines on del Pezzo surfaces)

Let S be a del Pezzo surface of degree d over k. Then the Hilbert scheme $L(S)$ of lines on S is a reduced projective scheme over k, which satisfies the following:

1. $L(S) \times_k \overline{k} \cong \mathbf{P}^2$, *if* $d = 9$.
2. $L(S) \times_k \overline{k} = \mathbf{P}^1 \sqcup \mathbf{P}^1$, *if* $S \times_k \overline{k} \cong \mathbf{P}^1 \times \mathbf{P}^1$.
3. $L(S)$ *is a finite reduced zero-dimensional scheme of degree*

d	8	7	6	5	4	3	2	1
$\deg L(S)$	1	3	6	10	16	27	56	240

otherwise.

Our results concerning del Pezzo surfaces

Theorem (J.+Loughran 2014)

Let k be a number field and let $1 \leq d \leq 9$. Then the class of del Pezzo surfaces of degree d fails the Hasse principle for lines if and only if

$$d = 8, 5, 3, 2 \text{ or } 1\,.$$

Our proof is constructive. For example, an explicit counter-example in degree 3 over $\mathbb{Q}$ is given by

$$-5x^2w - 5xy^2 - 2xyw + 5xz^2 - 9xzw - 5xw^2 + 9y^3 - 11y^2z + 29y^2w$$
$$+43yz^2 - 52yzw - 4yw^2 - 13z^3 + 14z^2w - 96zw^2 + 45w^3 = 0\,.$$

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Corollary

Let k be a number field. Then there exist smooth cubic surfaces over k which fail the Hasse principle for conics.

Our results concerning del Pezzo surfaces II

Any del Pezzo surface of degree d may be embedded anticanonically into

$$X_d := \begin{cases} \mathbf{P}(1,1,2,3), & \text{if } d = 1\,, \\ \mathbf{P}(1,1,1,2), & \text{if } d = 2\,, \\ \mathbf{P}^d & \text{if } d \geq 3\,. \end{cases}$$

Notation

For k a base field, let $\mathscr{H}_d$ denote the Hilbert scheme over k which parametrises those subschemes of X_d, which are anticanonically embedded del Pezzo surfaces of degree d.

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Lemma

Let $1 \leq d \leq 9$.

1. *Then $\mathscr{H}_d$ is smooth.*
2. *When $d \neq 8$, it is geometrically connected. When $d = 8$, it consists of two connected components, each of which is geometrically connected.*

Our results concerning del Pezzo surfaces III

Definition (Serre 1997)

Given a variety X over a number field k, a subset $\Omega \subset X(k)$ is said to be *thin* if it is a finite union of subsets which are either

- contained in a proper closed subvariety of X, or
- in some $\pi(Y(k))$ where $\pi\colon Y \to X$ is a generically finite dominant morphism of degree exceeding 1, with Y irreducible.

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Example

The set of all rational squares,

$$\{q^2 \mid q \in \mathbb{Q}\} \subset \mathbf{A}^1_{\mathbb{Q}}(\mathbb{Q}),$$

is Zariski dense but a thin subset.

Our results concerning del Pezzo surfaces IV

Theorem (Thinness of the counterexamples, J.+Loughran 2014)

Let k be a number field and let $1 \leq d \leq 9$.

There exists a thin subset $\Omega_d \subset \mathscr{H}_d(k)$ such that the Hasse principle for lines holds for those del Pezzo surfaces corresponding to the points of $\mathscr{H}_d(k) \backslash \Omega_d$.

Our results concerning del Pezzo surfaces IV

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Theorem (Zariski density of the counterexamples, J.+Loughran 2014)

Let k be a number field and let $1 \leq d \leq 9$.

1. *The collection of del Pezzo surfaces of degree 8 in $\mathscr{H}_8^1(k)$ which fail the Hasse principle for lines is Zariski dense in $\mathscr{H}_8^1$.*
2. *If $d = 5, 3, 2$ or 1 then the collection of del Pezzo surfaces of degree d in $\mathscr{H}_d(k)$ which fail the Hasse principle for lines is Zariski dense inside $\mathscr{H}_d$.*

Degrees 9 and 8

Elementary cases

- $d = 9$: Here, $L(S)$ is a Brauer-Severi surface (a twist of $\mathbf{P}^2$), which satisfies the Hasse principle, according to Chatelet. Hence, S fulfills the Hasse principle for lines.

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- $d = 8$: Two cases.
  - $S_{\overline{k}}$ is $\mathbf{P}^2$ blown up in one point:

    Then $S_{\overline{k}}$ contains exactly one line, which must be fixed by $\mathrm{Gal}(\overline{k}/k)$. The Hasse principle for lines is therefore fulfilled.

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    Then $S_{\overline{k}}$ contains exactly one line, which must be fixed by $\mathrm{Gal}(\overline{k}/k)$. The Hasse principle for lines is therefore fulfilled.
  - $S_{\overline{k}}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$: Then $L(S)_{\overline{k}}$ is isomorphic to $\mathbf{P}^1 \sqcup \mathbf{P}^1$.
    Suppose that the components are interchanged under $\mathrm{Gal}(k(\sqrt{d})/k)$, for some non-square $d \in k^*$. Then, for all inert primes ν, $L(S)$ has no k_ν-rational point. The Hasse principle for lines is fulfilled trivially.

    Remaining case: $S \cong C_1 \times C_2$ for conics C_1, C_2. Then $L(S) \cong C_1 \sqcup C_2$.
    $C_1(k_\nu) = \emptyset$ iff $\nu \in S_1$ and $C_2(k_\nu) = \emptyset$ iff $\nu \in S_2$.
    If S_1 and S_2 are non-empty and disjoint then $C_1 \times C_2$ is a counterexample to the Hasse principle for lines.

A group-theoretic condition

If $d \leq 7$ then $S_{\overline{k}}$ contains only finitely many lines. $\mathrm{Gal}(\overline{k}/k)$ permutes them.

E.g., if $d = 3$ then we have a permutation representation

$$\iota_S\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow S_{27}\,.$$

Do we know more about this permutation representation?

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Do we know more about this permutation representation?

We do!

1. Each element $\sigma \in \mathrm{Gal}(\overline{k}/k)$ must respect the intersection matrix.
2. The permutations respecting the intersection matrix form a subgroup $W \subset S_{27}$ isomorphic to the Weyl group $W(E_6)$ of order $51,840$.

A group-theoretic condition II

In general, the permutations respecting the intersection matrix form a group that is isomorphic to the group below.

d	7	6	5	4	3	2	1
$W(E_{9-d})$	$\mathbb{Z}/2\mathbb{Z}$	D_6	$W(A_4)$	$W(D_5)$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\#W(E_{9-d})$	2	12	120	1920	51,840	2,903,040	696,729,600
#lines	3	6	10	16	27	56	240

Let us use $W(E_{9-d})$ as a general notation. We understand $W(E_{9-d})$ as a permutation group, acting on the lines.

Thus, we have

Lemma

Let k be a number field, $1 \leq d \leq 7$, and S a del Pezzo surface of degree d over k. Then the image of the permutation representation ι_S is a subgroup of $W(E_{9-d})$.

A group-theoretic condition III

Proposition (Reduction to a group-theoretic question—ignoring ramification)

Let k be a number field, $1 \leq d \leq 7$, and S a del Pezzo surface of degree d over k. Put $G := \operatorname{im} \iota_S$ and let K be the field of definition of the lines on $S_{\overline{k}}$. Then

1. *S contains a line over k if and only if G has a fixed point.*
2. *S contains a line over k_ν for every prime ν that is either archimedean or unramified in K if and only if each element of G has a fixed point.*

Idea of proof: 1. is clear.

2. One has $\operatorname{Gal}(K/\mathbb{Q}) \cong G$.

The local Galois group $\operatorname{Gal}(K_w/k_\nu)$ is isomorphic to the decomposition group $D_w \subseteq G$. For ν unramified, D_w is cyclic, generated by the Frobenius Frob_ν. Moreover, according to Chebotarev's density theorem, Frob_ν runs through all conjugacy classes of elements of G. □

A group-theoretic condition IV

Hence, to have a counterexample to the Hasse principle for lines, we need a subgroup $G \subseteq W(E_{9-d})$ satisfying the following

Group-theoretic condition

$G \subseteq W(E_{9-d})$ is fixed-point free, but every element of G has a fixed point.

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Observation

1. Cyclic subgroups never satisfy this condition.
2. Neither do transitive ones.

Idea of proof: 1. is clear.

2. (C. Jordan) Let $G \subseteq S_n$, $n \geq 2$, be transitive. Then exactly $\#G/n$ elements fix 1, exactly $\#G/n$ elements fix 2, exactly $\#G/n$ elements fix 3, etc.

Altogether, $\#G/n + \#G/n + \#G/n + \ldots = \#G$. Hence, if every element of G had a fixed point then no element of G could have more than one. But the neutral element has n fixed points.

Degrees 7 and 6

High degrees

- $d = 7$: The three lines form the path graph on three vertices. The automorphism group of the graph fixes a vertex, so there is always a line defined over k. The Hasse principle for lines trivially holds.

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- $d = 6$: Here, the six lines form the cycle graph on six vertices, which one may identify with a regular hexagon. The automorphism group is the dihedral group D_6, acting in the usual way.

  In this case, every non-cyclic subgroup contains a non-trivial rotation. Hence, no subgroup of D_6 fulfills the group-theoretic condition above. The Hasse principle for lines holds.

Low degrees—A computer experiment

For $1 \leq d \leq 5$, we used a computer. An experiment in `magma` quickly finds all conjugacy classes of subgroups of $W(E_{9-d})$ fulfilling the group-theoretic condition.

d	5	4	3	2	1
#conjugacy classes	19	197	350	8074	62092
#satisfying condition	2	0	3	60	8742

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In particular, this proves the Hasse principle for lines for del Pezzo surfaces of degree 4.

Low degrees—The subgroups

$d = 5$:

Here, $W(A_4) \cong S_5$. The operation on the 10 lines is the operation of S_5 on pairs. The two subgroups are

G	orbit type
V_4	$[2, 2, 2, 4]$
A_4	$[4, 6]$

The size four orbits consist of skew lines. Thus, surfaces with these group operations on the lines may be obtained by blowing up $\mathbf{P}^2$ in a closed point of degree 4 in general position.

Remark

In fact, all degree 5 del Pezzo surfaces with these group operations are obtained in this way.

Indeed, blowing down the four skew lines leads to a del Pezzo surface of degree 9. But, according to a theorem of Enriques and Swinnerton-Dyer, every degree 5 del Pezzo surface contains a rational point.

Low degrees—The subgroups II

$d = 3$:

The three subgroups are

isomorphy type of G	orbit type
D_5	$[2, 5, 5, 5, 10]$
$\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$	$[2, 5, 10, 10]$
S_5	$[2, 5, 10, 10]$

Construction

1. Choose two closed points P and Q in $\mathbf{P}^2$ of degrees 2 and 5, respectively, such that $P \sqcup Q$ is in general position.
2. The del Pezzo surface $S = \mathrm{Bl}_{P \sqcup Q}\mathbf{P}^2$ of degree 2 contains exactly two rational lines. These are the strict transforms $\widetilde{L}$ and $\widetilde{C}$ of the line L through the two quadratic points and the conic C passing through the five quintic points over $\overline{k}$.
3. Contracting $\widetilde{L}$, we obtain a cubic surface S which contains no lines, since $\widetilde{L}$ and $\widetilde{C}$ intersect.

Low degrees—The subgroups III

The lines on S correspond to the following curves:

1. The 2 singular cubic curves which pass through all 7 points and which have a double point at exactly one of the quadratic points.
2. The 5 exceptional curves above the quintic points.
3. The 10 conics passing through the two quadratic points and three of the five quintic points.
4. The 10 lines passing through one of the quadratic points and one of the quintic points.

$d = 2$: For example, $G := V_4 \times T$ or $G := A_4 \times T$, for $T \subseteq S_3$ transitive, operating on the seven blow-up points in the obvious way.

$d = 1$: For example, $G := V_4 \times T$ or $G := A_4 \times T$, for $T \subseteq S_4$ transitive, operating on the eight blow-up points in the obvious way.

Thus, although the inverse Galois problem for del Pezzo surfaces is, in general, unsolved, we do not run into difficulties with this. In each of the degrees $5, 3, 2, 1$, some of the subgroups sought for are extremely easy to realize.

Ramified primes

Lemma (Sonn 2008)

Let k be a number field and let G be a solvable group. Then there exists a Galois extension K/k with Galois group G, all of whose decomposition groups are cyclic.

Let G be one of the groups above. Take K as in the lemma. Then, making K the field of definition of the lines on S, one achieves that S has lines over k_ν for *all* primes ν. And there is no line over k.

This completes the proof for the existence of del Pezzo surfaces of degrees 5, 3, 2, and 1 over any number field, violating the Hasse principle for lines. □

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This completes the proof for the existence of del Pezzo surfaces of degrees 5, 3, 2, and 1 over any number field, violating the Hasse principle for lines. □

Example ($d = 5$, $G = V_4$)

Take rational primes p, q such that $p \equiv 1 \pmod 8$, $q \equiv 1 \pmod p$, and $k(\sqrt{p}, \sqrt{q})/k$ is of degree 4. Then all decomposition groups are cyclic.

Moreover, $\mathbf{P}^2$ blown up in the closed point which is the union of $(\pm\sqrt{p} : \pm\sqrt{q} : 1)$, is a del Pezzo surface of degree 5 violating the Hasse principle for lines.

Explicit examples

Concrete examples are often easier to achieve by looking at the splitting behaviour at the ramified primes rather than the decomposition groups.

Example ($d = 5$, $G = A_4$)

For $k := \mathbb{Q}$, let

$$f(x) := x^4 - x^3 - 7x^2 + 2x + 9$$

and L be the field defined by f. Then $\mathbf{P}^2$ blown up in a generic closed point with residue field L is a degree 5 del Pezzo surface being a counterexample to the Hasse principle for lines.

In fact, $\operatorname{disc} L = 163^2$ and its Galois closure has Galois group A_4. Moreover, there is the factorisation $(163) = \mathfrak{p}_1\mathfrak{p}_2^3$ into prime ideals.

As $e(\mathfrak{p}_1|163) = f(\mathfrak{p}_1|163) = 1$, there is a homomorphism $L \to \mathbb{Q}_{163}$. Thus, the surface contains a line defined over $\mathbb{Q}_{163}$.

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As $e(\mathfrak{p}_1|163) = f(\mathfrak{p}_1|163) = 1$, there is a homomorphism $L \to \mathbb{Q}_{163}$. Thus, the surface contains a line defined over $\mathbb{Q}_{163}$.

The polynomial f [as well as all explicit polynomials used in this project] was taken from the database `http://galoisdb.math.upb.de/`, maintained by J. Klüners and G. Malle.

Explicit examples II

Example ($d = 3$, $G = D_5$)

For $k := \mathbb{Q}$, let

$$f(x) := x^5 - 2x^4 + 2x^3 - x^2 + 1$$

and L be the field defined by f. Then $\operatorname{disc} L = 47^2$ and its Galois closure has Galois group D_5. The unique quadratic subfield is $\mathbb{Q}(\sqrt{-47})$.

Moreover, in L there is the factorisation $(47) = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2$ into prime ideals of inertia degree 1.

$\mathrm{Bl}_{P \sqcup Q}\mathbf{P}^2$, for closed points P and Q of residue fields L and $\mathbb{Q}(\sqrt{-47})$ such that $P \sqcup Q$ is in general position, is a del Pezzo surface of degree 2 containing two $\mathbb{Q}$-rational lines. Blowing down one of them yields a cubic surface that is a counterexample to the Hasse principle for lines.

Explicit equation:

$$-5x^2w - 5xy^2 - 2xyw + 5xz^2 - 9xzw - 5xw^2 + 9y^3 - 11y^2z + 29y^2w$$
$$+ 43yz^2 - 52yzw - 4yw^2 - 13z^3 + 14z^2w - 96zw^2 + 45w^3 = 0\,.$$

Explicit examples III

Example ($d = 3$, $G = \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$)

For $k := \mathbb{Q}$, let

$$f(x) := x^5 - 101$$

and L be the field defined by f. Then $\operatorname{disc} L = 5^3 {\cdot} 101^4$ and its Galois closure has Galois group $\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. The unique quadratic subfield is $\mathbb{Q}(\sqrt{5})$.

Moreover, 101 splits in $\mathbb{Q}(\sqrt{5})$ and in L there is the factorisation $(5) = \mathfrak{p}_1 \mathfrak{p}_2^4$ into prime ideals of inertia degree 1.

$\mathrm{Bl}_{P \sqcup Q} \mathbf{P}^2$, for closed points P and Q of residue fields L and $\mathbb{Q}(\sqrt{5})$ such that $P \sqcup Q$ is in general position, is a del Pezzo surface of degree 2 containing two $\mathbb{Q}$-rational lines. Blowing down one of them yields a cubic surface that is a counterexample to the Hasse principle for lines.

Explicit equation:

$$5x^2y - 10x^2z - 5x^2w + 6xy^2 - xyz - 2xyw - 6xz^2 + xzw + 5xw^2 + y^3$$
$$+ 9y^2z + 8y^2w - yz^2 - yzw + 17yw^2 - 10z^3 - 9z^2w + 13zw^2 + 19w^3 = 0\,.$$

Explicit examples IV

Example ($d = 3$, $G = S_5$)

For $k := \mathbb{Q}$, let

$$f(x) := x^5 - x^4 - 5x^3 + 5x^2 + 2x - 1$$

and L be the field defined by f. Then $\operatorname{disc} L = 101833$, which is prime, and the Galois closure of L has Galois group S_5. The unique quadratic subfield is $\mathbb{Q}(\sqrt{101833})$.

Moreover, in L there is the factorisation $(101833) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^2$ into prime ideals of inertia degree 1.

$\mathrm{Bl}_{P \sqcup Q}\mathbf{P}^2$, for closed points P and Q of residue fields L and $\mathbb{Q}(\sqrt{101833})$ such that $P \sqcup Q$ is in general position, is a del Pezzo surface of degree 2 containing two $\mathbb{Q}$-rational lines. Blowing down one of them yields a cubic surface that is a counterexample to the Hasse principle for lines.

Explicit equation:

$$-4x^3+4x^2y+9x^2z-x^2w+2xy^2-4xyz+6xyw+2xz^2+xzw+10xw^2$$
$$-2y^3-7y^2z+6y^2w-15yz^2+23yzw+13yw^2+z^3-11z^2w+zw^2+w^3=0.$$

How to obtain the explicit equations

Algorithm

1. Determine three linearly independent cubic forms c_0, c_1, c_2 on $\mathbf{P}^2$ that vanish at the seven blow-up points. Choose c_0 to be the product of a linear form vanishing on the size-two orbit and a quadratic form vanishing on the size-five orbit.
2. Also determine a sextic form s which admits a double point at each of the seven blow-up points, and is not a linear combination of products of the cubic forms found.
3. Calculate the unique relation of the type

$$s^2 + f_2(c_0, c_1, c_2)s + f_4(c_0, c_1, c_2) = 0$$

between these forms. This leads to a del Pezzo surface of degree 2 with the explicit equation

$$w^2 + f_2(x_0, x_1, x_2)w + f_4(x_0, x_1, x_2) = 0 \subset \mathbf{P}(1, 1, 1, 2)\,.$$

[Calculations up to here: Only linear algebra.]

How to obtain the explicit equations II

4. Moreover, the pre-image of the "$x_0 = 0$" splits. Thus, a linear transformation yields an equation of the form
$$w^2 + f_2'(x_0, x_1, x_2)w + x_0 \cdot f_3(x_0, x_1, x_2) = 0\,.$$
5. Explicitly blow down one of the lines over "$x_0 = 0$". The result is
$$x_0 \cdot w^2 + f_2'(x_0, x_1, x_2)w + f_3(x_0, x_1, x_2) = 0$$
[C. F. Geiser 1869].
6. To obtain small coeficients, one may apply Kollar's reduction and minimisation algorithm (Kollar 1997+Elsenhans).

Degrees 2 and 1

Here, we constructed examples only for some of the admissible Galois groups.

Rational surfaces:

Blow up a degree 5 counterexample to the Hasse principle for lines in a generic closed point of degree 3 [4]. Then the result is a del Pezzo surface of degree 2 [1] violating the Hasse principle for lines.

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Non-rational surfaces: Unlike degrees 5 and 3, the lists here also contain groups corresponding to surfaces that are non-rational over the base field.

Example

Let S be a conic bundle surface of the shape

$$f(t)x^2 + g(t)y^2 + h(t)z^2 = 0 \subset \mathbf{A}^1 \times \mathbf{P}^2 \,,$$

where

$$f(t) = a(t-13)(2-t)\,, \quad g(t) = b(t+14)(3-t)\,, \quad h(t) = (t+2)(t-11)\,,$$

and $a, b \in k^*$.

Degrees 2 and 1 II

By work of Browning, Matthiesen, and Skorobogatov from 2014, the closure of this in $\mathbf{P}^1 \times \mathbf{P}^2$ is a del Pezzo surface S of degree 2.

The projection to $\mathbf{P}^1$ realises S as a conic bundle over $\mathbf{P}^1$, with exactly six singular fibres splitting over $\mathbb{Q}(\sqrt{a})$, $\mathbb{Q}(\sqrt{b})$, and $\mathbb{Q}(\sqrt{ab})$. Choosing a and b as above [degree 5], for each prime ν at least of them splits into two lines.

The orbit type is $[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 4, 4]$. By work of Iskovskikh (1970), such conic bundle surfaces cannot be rational over the base field.

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Remark

There is a similar, but more complicated, example in degree 1.

Thinness of the counterexamples

$d = 5, 3, 2, 1$:

Let $\ell_d \colon \mathscr{L}_d \to \mathscr{H}_d$ be the universal family of lines over the Hilbert scheme $\mathscr{H}_d$. The generic fiber of ℓ_d is a thick point of degree 10, 27, 56 or 240.

By Hilbert irreducibility, outside of a thin subset $\Omega_d \subset \mathscr{H}_d$, the special fibres are irreducible. But then the Galois operation on the lines is transitive. The Hasse principle for lines therefore holds.

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$d = 8$:

Consider the Stein factorisation

$$\begin{array}{ccc} \mathscr{L}_8^1 & \xrightarrow{\ell_8^1} & \mathscr{H}_8^1 \\ & \searrow & \uparrow \\ & & \mathscr{P} \, . \end{array}$$

$\mathscr{P}$ is irreducible as $\mathscr{L}_8^1$ is. $\mathscr{P} \to \mathscr{H}_8^1$ is finite étale of degree 2. Outside of a thin subset $\Omega_8 \subset \mathscr{H}_8^1$, the special fibres are irreducible. But then $L(S)$ does not split over k. The Hasse principle for lines holds.

Zariski density of the counterexamples

Idea of proof: $d = 5, 3, 2, 1$:

Suppose not. Then the counterexamples would be contained in a closed subvariety $X \subset \mathscr{H}_d$. Put $\delta := \dim \mathscr{H}_d$. By the Lang-Weil estimates, $X(\mathbb{F}_l) \leq C \cdot l^{\delta-1}$, for a constant C. Thus, it suffices to find a sequence of primes tending to ∞ such that the number of different reductions of the counterexamples grows faster than $C \cdot l^{\delta-1}$.

Consider the examples constructed by blow-up [and down] points defined over a field extension K/k. Let $\mathfrak{p}_i$ the sequence of primes that are completely split. Without destroying general position, one may require arbitrary locations in $\mathbf{P}^2(\mathbb{F}_{\#k/\mathfrak{p}_i})$ for the reductions of the blowup points. This alone leads to a growth of type $c \cdot l^\delta$. Contradiction!

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$d = 8$:

Let $S \subset \mathbf{P}^8$ be a counterexample to the Hasse principle for lines. Identify S with an element of $\mathscr{H}_8^1(k)$. As $\mathscr{H}_8^1$ is a PGL_9-homogeneous space, Zariski density follows. □

A complementary result

Theorem (J.+Loughran 2014)

Over every number field k, there exists a cubic surface S violating the Hasse principle for lines such that the Galois group operating on the 27 lines is isomorphic to S_5.

Idea of proof: Decomposition groups are always solvable. The only solvable subgroups of S_5 not fixing a line are those of order 20 [isomorphic to $\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$]. We need

Lemma

Let k be a number field. Then there exists an S_5-extension K/k such that no decomposition group is isomorphic to $\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$.

[This concerns only the ramified primes.] The result follows from work of K. Kedlaya from 2012. He shows existence of infinitely many S_5-extensions $F/\mathbb{Q}$ of square-free discriminant. Then $F/\mathbb{Q}(\sqrt{\operatorname{disc} F})$ is unramified. Hence, the decomposition group must have a normal subgroup of order 2, which $\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ does not have. Take Fk/k. □

Intersections of two quadrics

Theorem (J.+Loughran 2014)

Let $n \geq 0$ and let k be a number field. Let X be a smooth $2n$-dimensional complete intersection of two quadrics over k. Then X satisfies the Hasse principle for n-planes.

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This explains more conceptually why the Hasse principle for lines is fulfilled for del Pezzo surfaces of degree 4.

Idea of proof: Let X be given by $Q_1(x) = Q_2(x) = 0$ in $\mathbf{P}^{n+2}$. Then one associates the discriminant $D :=$ "$\det(\lambda Q_1 + \mu Q_2) = 0$" $\subset \mathbf{P}^1$ parametrising the singular quadrics in the pencil. One has $D \cong \operatorname{Spec} K$ for an étale k-algebra K of degree $2n+3$. It turns out (e.g., Reid 1972) that $L(S)$ is a torsor under the group scheme $R_{K/k}(\mu_2)/\mu_2$.

For any commutative group scheme G, let

$$Ш(k, G) := \ker \left(H^1(k, G) \longrightarrow \prod_\nu H^1(k_\nu, G)\right)$$

be the associated *Tate-Shafarevich group*. Then, what we need is

Intersections of two quadrics II

Proposition

Let k be a number field and let K be an étale k-algebra of odd degree. Then

$$Ш(k, R_{K/k}(\mu_2)/\mu_2) = 0\,.$$

Idea of proof: As K is of odd degree, $R_{K/k}(\mu_2)/\mu_2$ is isomorphic to the norm-1 subgroup of $R_{K/k}(\mu_2)$, the sequence

$$0 \to R^1_{K/k}(\mu_2) \to R_{K/k}(\mu_2) \to R_{K/k}(\mu_2)/\mu_2 \to 0$$

being split. Thus, it suffices to show the vanishing of $Ш(k, R_{K/k}(\mu_2))$, which, by Shapiro's lemma coincides with $\prod_i Ш(K_i, \mu_2)$, for $K = \prod_i K_i$. However, $H^1(K_i, \mu_2)$ classifies quadratic extensions of K_i, hence this follows from the fact that any quadratic extension is non-split at at least one prime. □

Thank you!!