Experiments with the Brauer-Manin obstruction

Jörg Jahnel

University of Siegen

University of Sydney, The magma group
March 7, 2013

joint work with
Andreas-Stephan Elsenhans (Sydney)

A Diophantine equation

Example

Consider the Diophantine equation

$$3x^3+2x^2z+xy^2-2xyz-2xyw-xzw+2xw^2-yzw-yw^2-z^3+z^2w=0\,.$$

Observation

There are 18 non-trivial solutions of height ≤ 10:

$(0:0:0:1)$, $(0:0:1:1)$, $(0:1:0:0)$, $(0:2:-3:9)$, $(0:3:-2:4)$, $(0:4:-6:-3)$, $(0:9:-6:-2)$, $(1:-6:5:-8)$, $(1:0:5:4)$, $(2:-4:-3:-1)$, $(2:-2:-3:-1)$, $(2:2:3:5)$, $(3:-6:-5:-2)$, $(3:6:-1:8)$, $(3:6:3:2)$, $(4:-2:9:1)$, $(4:4:6:1)$, $(4:8:0:7)$,

Fact

There are no solutions $(x:y:z:w)\in \mathbf{P}^3(\mathbb{Z})$ *such that the reduction modulo 3 is* $(1:0:0:0)$, $(1:0:1:1)$, *or* $(1:0:1:-1)$.
Thus, weak approximation *is violated.*

A Diophantine equation II

Modulo 3, this equation has exactly ten solutions, six of which occur as reductions of integral solutions. These are $(0 : 0 : 0 : 1)$, $(0 : 0 : 1 : 1)$, $(0 : 1 : 0 : 0)$, $(1 : 0 : -1 : 1)$, $(1 : 1 : 0 : 1)$, and $(1 : -1 : 0 : 1)$.

Further, there are the three solutions given above and $(1 : 1 : 1 : 0)$. The latter does not lift to a 3-adic solution.

Remark

From the geometric point of view,

$$3x^3 + 2x^2z + xy^2 - 2xyz - 2xyw - xzw + 2xw^2 - yzw - yw^2 - z^3 + z^2w = 0$$

defines a smooth cubic surface C over $\mathbb{Q}$. Classical algebraic geometry gives us a lot of information about such surfaces.

In our case, C has bad reduction at 2, 3, and 5. The reduction modulo 3 is of the type of a Cayley cubic, having four isolated singular points. Among these, $(1 : 1 : 0 : 1)$ is the only $\mathbb{F}_3$-rational one. The three others are defined over $\mathbb{F}_{27}$.

Another Diophantine equation

Example

Consider the Diophantine equation

$$z^2 = x(x-1)(x-25)u(u+25)(u+36)\,.$$

Trivial solutions: $x \in \{0, 1, 25\}$ or $u \in \{0, -25, -36\}$.

Observation

There are 64 non-trivial solutions of height <100:

$(-2, -24; \pm 216)$, $(9, -24; \pm 576)$, $(-2, -3; \pm 594)$, $(4, -18; \pm 756)$, $(5, -20; \pm 800)$, $(4, -14; \pm 924)$, $(-5, -20; \pm 1200)$, $(9, -3; \pm 1584)$, $(29, -29; \pm 1624)$, $(10, -40; \pm 1800)$, $(5, -45; \pm 1800)$, $(8, -8; \pm 1904)$, $(-7, -18; \pm 2016)$, $(4, -50; \pm 2100)$, $(22, -11; \pm 2310)$, $(-7, -14; \pm 2464)$, $(-5, -45; \pm 2700)$, $(18, -8; \pm 2856)$, $(-10, -11; \pm 3850)$, $(-15, -40; \pm 4800)$, $(-7, -50; \pm 5600)$, $(-24, -40; \pm 8400)$, $(5, -80; \pm 8800)$, $(-5, -80; \pm 13200)$, $(-32, -44; \pm 20064)$, $(14, -88; \pm 24024)$, $(-55, -11; \pm 30800)$, $(-63, -11; \pm 36960)$, $(-27, -64; \pm 52416)$, $(64, 14; \pm 65520)$, $(64, 27; \pm 117936)$ $(-56, -63; \pm 129276)$,

Another Diophantine equation II

Fact

There are no solutions $(x, u, z) \in \mathbb{Z}^3$ *such that* $x \equiv 2 \pmod 5$ *and* $u \equiv 5 \pmod{25}$.
Thus, weak approximation *is violated.*

Observe that $x = 2$ and $u = 5$ lead to a solution in 5-adic integers. Indeed, $2 \cdot (2-1) \cdot (2-25) \cdot 5 \cdot (5+25) \cdot (5+36) = -11\,316 \cdot 5^2$ is a 5-adic square.

Remark

From the geometric point of view, $z^2 = x(x-1)(x-25)u(u+25)(u+36)$ defines a $K3$ surface S over $\mathbb{Q}$, more precisely a Kummer surface.
It is obtained form the product $E \times E'$ of the elliptic curves

$$E \colon y^2 = x(x-1)(x-25) \text{ and } E' \colon y'^2 = u(u+25)(u+36)$$

by identifying (x, y, u, y') with $(x, -y, u, -y')$.

The Hilbert symbol

Definition

For k a local field and $0 \neq \alpha, \beta \in k$ define $(\alpha, \beta)_k \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ by

$$(\alpha, \beta)_k := \begin{cases} 0 & \text{if } \alpha X^2 + \beta Y^2 - Z^2 \text{ non-trivially represents } 0 \text{ over } k\,, \\ \frac{1}{2} & \text{otherwise}\,. \end{cases}$$

This is called the *Hilbert symbol* of α and β.

Fact

For $0 \neq \alpha, \beta \in \mathbb{Q}$*, there is the* sum formula $\sum\limits_{p \in \{2,3,5,\ldots;\infty\}} (\alpha, \beta)_p = 0$.

The Hilbert symbol II

For $3x^3+2x^2z+xy^2-2xyz-2xyw-xzw+2xw^2-yzw-yw^2-z^3+z^2w=0$, there is a homogeneous form $F_{30} \in \mathbb{Q}[x,y,z,w]$ of degree 30 such that

- for every real or p-adic solutions ($p \neq 3$), one automatically has $(F_{30}(x,y,z,w),-3)_p = 0$.
- There are 3-adic solutions such that $(F_{30}(x,y,z,w),-3)_p = \frac{1}{2}$.

For the equation $z^2 = x(x-1)(x-25)u(u+25)(u+36)$, we may show the following

- For every non-trivial real or p-adic solution ($p \neq 5$), one automatically has $((x-1)(x-25),(u+25)(u+36))_p = 0$.
- There are, however, non-trivial 5-adic solutions such that $((x-1)(x-25),(u+25)(u+36))_5 = \frac{1}{2}$.

Thus, $C(\mathbb{Q}_3)$ and $S(\mathbb{Q}_5)$ split into two sorts of points (*red* and *green* points). We have *colourings* on these p-adic manifolds. Only one sort may be approximated by $\mathbb{Q}$-rational points.

The Brauer group

Definition

Let S be any scheme. Then the (cohomological) *Brauer group* of S is defined by $\mathrm{Br}(S) := H^2_{\text{ét}}(S, \mathbb{G}_m)$.

Remarks

1. This definition is not very explicit. In general, Brauer groups are not easily computable.
2. One has $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$, $\mathrm{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, and
$$\mathrm{Br}(\mathbb{Q}) = \ker(\mathrm{sum}\colon \bigoplus_{p \in \{2,3,5,\ldots\}} \mathrm{Br}(\mathbb{Q}_p) \oplus \bigoplus_{\nu\colon K \to \mathbb{R}} \mathrm{Br}(\mathbb{R}) \to \mathbb{Q}/\mathbb{Z}).$$
3. Let $\alpha \in \mathrm{Br}(S)$ be any Brauer class. Then, for every K-rational point $p \in S(K)$, there is $\alpha|_p \in \mathrm{Br}(\operatorname{Spec} K)$.
Hence, an adelic point *not* fulfilling the condition that the sum zero cannot be approximated by $\mathbb{Q}$-rational points.
This is called the *Brauer-Manin obstruction* to weak approximation.

The Brauer group II

The cohomological Brauer group of a variety S over a field k is equipped with a canonical filtration, defined by the Hochschild-Serre spectral sequence.

1. $\mathrm{Br}_0(S) \subseteq \mathrm{Br}(S)$ is the image of $\mathrm{Br}(k)$ under the natural map. At least when S has a k-rational point, $\mathrm{Br}_0(S) \cong \mathrm{Br}(k)$. $\mathrm{Br}_0(S)$ does not contribute to the Brauer-Manin obstruction.
2. One has

$$\mathrm{Br}_1(S)/\mathrm{Br}_0(S) \cong H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}(S_{k^{\mathrm{sep}}}))\,.$$

   This subquotient is called the algebraic part of the Brauer group. For k a number field, it is responsible for the so-called *algebraic* Brauer-Manin obstruction.
3. Finally, $\mathrm{Br}(S)/\mathrm{Br}_1(S)$ injects into $\mathrm{Br}(S_{k^{\mathrm{sep}}})$. This quotient is called the transcendental part of the Brauer group. For k a number field, the corresponding obstruction is called a *transcendental* Brauer-Manin obstruction.

Smooth cubic surfaces—Algebraic Brauer-Manin obstruction

Let $C \subset \mathbf{P}^3$ be a smooth cubic surface over an algebraically closed field.

- C is isomorphic to $\mathbf{P}^2$, blown up in six points. These are in general position.
- C contains precisely 27 lines.
- The configuration of the 27 lines is highly symmetric. The group of all permutations respecting the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840.
- There are many combinatorial structures determined by the 27 lines. For example, the are 72 *sixers* of mutually skew lines, forming 36 *double-sixes*.

- There is a pentahedron associated with general C (Sylvester).
- There are (at least) two kinds of moduli spaces, coming out of the classical invariant theory.
  - The coarse moduli space of smooth cubic surfaces (Salmon, Clebsch).
  - The fine moduli space of *marked cubic surfaces* (Cayley, Coble).

The Brauer group of smooth cubic surfaces

Lemma

Let C be a smooth cubic surface over an algebraically closed field. Then $\mathrm{Br}(C) = 0$.

Idea of **proof:** One has $\mathrm{Br}(\mathbf{P}^2) = 0$ and a blow-up does not change the Brauer group.

Corollary

Let C be a smooth cubic surface over a field k of characteristic zero.

- *Then the transcendental part* $\mathrm{Br}(C)/\mathrm{Br}_1(C)$ *of the Brauer group vanishes.*
- *The canonical map*

$$\delta\colon H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(C_{\overline{k}})) \longrightarrow \mathrm{Br}(C)/\mathrm{Br}(k)$$

  is an isomorphism.

The Brauer group of smooth cubic surfaces II

Theorem (Manin 1969)

Let C be a smooth cubic surface over a field k. Then

$$H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(C_{\overline{k}})) \cong \mathrm{Hom}((NF \cap F_0)/NF_0, \mathbb{Q}/\mathbb{Z})$$

Here, $F \subset \mathrm{Div}(C)$ is the subgroup generated by the 27 lines on C. $F_0 \subset F$ is the subgroup pf all principal divisors in F. Finally, N is the norm map from the field of definition of the 27 lines to k.

Thus, the $\mathrm{Gal}(\overline{k}/k)$-module structure on $F \cong \mathbb{Z}^{27}$, i.e. the Galois operation on the 27 lines, determines the Brauer group $\mathrm{Br}(C)/\mathrm{Br}(k)$ completely.

Remark

$\mathrm{Gal}(\overline{k}/k)$ permutes the 27 lines in such a way that the intersection matrix is respected. Thus, every smooth cubic surface over k defines a homomorphism $\varrho\colon \mathrm{Gal}(\overline{k}/k) \to W(E_6) \subseteq S_{27}$. The subgroup $\mathrm{im}\,\varrho$ determines the Brauer group.

Systematic computation

There are 350 conjugacy classes of subgroups in $W(E_6)$.

It turns out that $H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(C_{\overline{k}}))$ is isomorphic to

0	for 257 classes,
$\mathbb{Z}/2\mathbb{Z}$	for 65 classes,
$\mathbb{Z}/3\mathbb{Z}$	for 16 classes,
$(\mathbb{Z}/2\mathbb{Z})^2$	for 11 classes,
$(\mathbb{Z}/3\mathbb{Z})^2$	for one class.

Fact (Swinnerton-Dyer 1993, Elsenhans+J. 2009)

Let C be a smooth cubic surface over a field k.

1. *If $H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(C_{\overline{k}})) = \mathbb{Z}/2\mathbb{Z}$ then, on C, there is a Galois-invariant double-six.*
2. *If $H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(C_{\overline{k}})) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then, on C, there are three Galois-invariant double-sixes that are azygetic to each other. Azygeticity means every pair has six lines in common.*

The hexahedral form

We constructed examples over $\mathbb{Q}$ for each of the 350 conjugacy classes.

Cubic surfaces with a Galois invariant double-six are related to the hexahedral form.

Definition (Hexahedral form)

The cubic surface $S^{(a_0,\ldots,a_5)}$ given in $\mathbf{P}^5$ by

$$\begin{aligned} X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 &= 0\,, \\ X_0 + X_1 + X_2 + X_3 + X_4 + X_5 &= 0\,, \\ a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 &= 0\,. \end{aligned}$$

is said to be in *hexahedral form*.

Remarks

- There are the 15 *obvious lines* given by the equations $X_{i_0} + X_{i_1} = X_{i_2} + X_{i_3} = X_{i_4} + X_{i_5} = 0$ for $\{i_0, \ldots, i_5\} = \{0, \ldots, 5\}$.

The hexahedral form II

Remarks (continued)

- The twelve *non-obvious* lines form a *double-six*. I.e., a configuration of the type $\{l_0, \ldots, l_5, l'_0, \ldots, l'_5\}$ with l_i meeting l'_j if and only if $l \neq j$, the l_i being pairwise skew, and the l'_i being pairwise skew.
- The group of all permutations of $\{l_0, \ldots, l_5, l'_0, \ldots, l'_5\}$ respecting the intersection product is isomorphic to $S_6 \times \mathbb{Z}/2\mathbb{Z}$ of order 1440, generated by the permutations of the indices and the flip.
- A permutation of the coordinates $X_0, \ldots, X_5$ operates on the double-six as an element of $S_6 \subset S_6 \times \mathbb{Z}/2\mathbb{Z}$. However, an outer automorphism of S_6 comes in!
- A cubic surface has 45 *tritangent planes* cutting the surface in three lines. There are 15 *obvious* tritangent planes, given by $X_{j_0} + X_{j_1} = 0$ for $0 \leq j_0 < j_1 \leq 5$, and 30 *non-obvious* ones.

  Every obvious tritangent plane contains three obvious lines. A non-obvious plane contains two non-obvious lines and one obvious line.

The hexahedral form III

Definition

Let σ_i denote the i-th elementary symmetric function in $a_0, \dots, a_5$. Then, the form

$$d_4 := \sigma_2^2 - 4\sigma_4 + \sigma_1(2\sigma_3 - \tfrac{3}{2}\sigma_1\sigma_2 + \tfrac{5}{16}\sigma_1^3)$$

is called the *Coble quartic*.

Theorem (Coble 1915)

The field of definition of the 27 lines on $S^{(a_0,\dots,a_5)}$ *is* $\mathbb{Q}(\sqrt{d_4})$.

The trace construction

Algorithm (Trace construction—Computation of the Galois descent)

Given a separable polynomial $f \in \mathbb{Q}[T]$ of degree six, this algorithm computes a cubic surface $S_{(a_0,\ldots,a_5)}$.

1. Compute, according to the definition, the traces $t_i := \operatorname{tr} T^i$ for $i = 0, \ldots, 5$. Use these values to compute $t_6 := \operatorname{tr} T^6$.
2. Determine the kernel of the 2×6-matrix

$$\begin{pmatrix} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{pmatrix}.$$

   Choose linearly independent kernel vectors $(c_i^0, \ldots, c_i^5) \in \mathbb{Q}^6$ for $i = 0, \ldots, 3$.
3. Compute the term $\left[\sum_{j=0}^{5}(c_0^j x_0 + \ldots + c_3^j x_3) T^j\right]^3$ modulo $f(T)$. This is a cubic form in $x_0, \ldots, x_3$ with coefficients in $\mathbb{Q}[T]/(f)$.
4. Finally, apply the trace coefficient-wise and output the resulting cubic form in $x_0, \ldots, x_3$ with 20 rational coefficients.

The trace construction II

Remark

$S_{(a_0,\dots,a_5)}$ is a cubic surface over $\mathbb{Q}$ such that $S_{(a_0,\dots,a_5)} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}}$ is isomorphic to the surface $S^{(a_0,\dots,a_5)}$ in $\mathbf{P}^5$ given by

$$\begin{aligned} X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 &= 0\,, \\ X_0 + X_1 + X_2 + X_3 + X_4 + X_5 &= 0\,, \\ a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 &= 0\,. \end{aligned}$$

Here, $a_0, \dots, a_5 \in \overline{\mathbb{Q}}$ are the zeroes of f.

Proposition (Elsenhans+J. 2009)

An element $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ flips the double-six if and only if it defines the conjugation of $\mathbb{Q}(\sqrt{D})$ for $D := d_4 \cdot \Delta(f)$.

The local evaluation map

Proposition (Elsenhans+J. 2009)

Let $f \in \mathbb{Q}[T]$ be a polynomial of degree six and $C := S_{(a_0,\dots,a_5)}$ be the corresponding cubic surface. Then there is a Brauer class $\alpha \in \mathrm{Br}(C)_2$ such that, for every prime p, the local evaluation map

$$\mathrm{ev}_\alpha \colon C(\mathbb{Q}_p) \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}$$

is given by

$$(x:y:z:w) \mapsto \mathrm{ev}_\alpha(x:y:z:w) = \left(\tfrac{F_{30}(x,y,z,w)}{F_{15}^2(x,y,z,w)}, D\right)_p$$

for every $(x:y:z:w) \in C(\mathbb{Q}_p)$, not contained in any of the 27 lines.

Here, $D := d_4 \cdot \Delta(f)$, F_{15} is a product of linear forms corresponding to the 15 obvious tritangent planes and F_{30} is a product of linear forms corresponding to the 30 non-obvious tritangent planes.

The local evaluation map II

Idea of **proof** (only for generic orbit structure [12, 15]):

- Manin's formula: Need a rational function F such that $\operatorname{div} F = ND$ for a (non-principal) divisor $D \in \operatorname{Div}(C_{\mathbb{Q}(\sqrt{D})})$.
- Put $D_1 := \sum\limits_{l \text{ non-obv. line}} l$ and $D_2 := \sum\limits_{l \text{ obv. line}} l$. Then the intersection matrix is

$$\begin{pmatrix} 48 & 60 \\ 60 & 75 \end{pmatrix}$$

  Thus, $5D_1 - 4D_2$ is a principal divisor.
- On the other hand, $\operatorname{div} F_{30} = 5D_1 + 2D_2$ and $\operatorname{div} F_{15} = 3D_2$. Hence, $5D_1 - 4D_2 = \operatorname{div}(F_{30}/F_{15}^2)$.
- Finally, over $\mathbb{Q}(\sqrt{D})$, the double-six is split, $D_1 = D_1^{(1)} + D_1^{(2)}$. Therefore,

$$5D_1 - 4D_2 = N_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(5D_1^{(1)} - 2D_2)\,. \qquad \square$$

Constancy of the evaluation map

Proposition (Elsenhans+J. 2009)

Let C be a non-singular cubic surface and $\alpha \in \mathrm{Br}(C)$. Then, for a prime number p such that

- *the field extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ splitting the double-six is unramified at p,*
- *the reduction C_p is geometrically irreducible and no $\mathbb{Q}_p$-rational point on C reduces to a singularity of C_p,*

the value of $\mathrm{ev}_\alpha(x)$ is independent of $x \in C(\mathbb{Q}_p)$.

In particular, the evaluation is constant on $C(\mathbb{Q}_p)$, for p any prime of good reduction.

Back to the introductory example

The equation

$$3x^3 + 2x^2z + xy^2 - 2xyz - 2xyw - xzw + 2xw^2 - yzw - yw^2 - z^3 + z^2w = 0$$

was obtained using the starting polynomial

$$F := T(T^5 - 60T^3 - 90T^2 + 675T + 810).$$

One has $\text{disc}(F) = -2^{12}3^{21}5^{8}13^{2}$, while Coble's radicand d_4 is a perfect square. Thus, $D = -3$.

The proposition shows that the local evaluation map is constant for all primes $p \neq 2, 3, 5$, and ∞. Constancy at $2, 5$, and ∞ is true as well.

Transcendental Brauer-Manin obstruction – Particular Kummer surfaces

Proposition (Skorobogatov/Zarhin 2011)

Let $E\colon y^2 = x(x-a)(x-b)$ *and* $E'\colon v^2 = u(u-a')(u-b')$ *be two elliptic curves over a field* k, $\operatorname{char} k = 0$. *Suppose that their 2-torsion points are defined over* k *and that* $E_{\overline{k}}$ *and* $E'_{\overline{k}}$ *are not isogenous to each other.*

Further, let $S := \operatorname{Kum}(E \times E')$ *be the corresponding Kummer surface. Then*

$$\operatorname{Br}(S)_2/\operatorname{Br}(k)_2 = \operatorname{im}(\operatorname{Br}(S)_2 \to \operatorname{Br}(S_{\overline{k}})_2) \cong \ker(\mu\colon \mathbb{F}_2^4 \to (k^*/k^{*2})^4)\,,$$

where μ *is given by the matrix*

$$M_{aba'b'} := \begin{pmatrix} 1 & ab & a'b' & -aa' \\ ab & 1 & aa' & a'(a'-b') \\ a'b' & aa' & 1 & a(a-b) \\ -aa' & a'(a'-b') & a(a-b) & 1 \end{pmatrix}.$$

Transcendental Brauer-Manin obstruction – Particular Kummer surfaces II

Remarks

1. In general, there is the short exact sequence
$$0 \to \mathrm{Pic}(S)/2\,\mathrm{Pic}(S) \to H^2_{\text{ét}}(S, \mu_2) \to \mathrm{Br}(S)_2 \to 0\,.$$
2. $S := \mathrm{Kum}(E \times E')$ over algebraically closed field k. Then $\mathrm{Br}(S)_2 \cong \mathbb{F}_2^4$. More canonically,
$$\mathrm{Br}(S)_2 \cong H^2_{\text{ét}}(E \times E', \mu_2)/(H^2_{\text{ét}}(E, \mu_2) \oplus H^2_{\text{ét}}(E', \mu_2)) \cong \mathrm{Hom}(E[2], E'[2]).$$
3. $S := \mathrm{Kum}(E \times E')$ over an arbitrary field k, $\mathrm{char}\, k = 0$. Then the assumption that the 2-torsion points are defined over k implies that $\mathrm{Gal}(\overline{k}/k)$ operates trivially on $\mathrm{Br}(S_{\overline{k}})_2$. Nevertheless, in general,
$$\mathrm{Br}(S)_2/\,\mathrm{Br}(k)_2 \subsetneqq \mathrm{Br}(S_{\overline{k}})_2^{\mathrm{Gal}(\overline{k}/k)} \cong \mathbb{F}_2^4\,.$$

Algebraic versus transcendental Brauer-Manin obstruction

- Algebraic Brauer-Manin obstruction:

  Explicit computations have been done for many classes of varieties. Most examples were Fano.

  Cubic surfaces:
  The example from above is rather typical.
  The classical counterexamples to the Hasse principle (Mordell and Cassels/Guy) are in fact algebraic Brauer-Manin obstruction (Manin).

  Computations for diagonal quartic surfaces, due to M. Bright.

- Transcendental Brauer-Manin obstruction:

  Much less understood, seemingly more difficult.
  First explicit example: Harari 1993.
  Literature still very small. Often enormous efforts.
  E.g., a whole Ph.D. thesis on one diagonal quartic surface, by Th. Preu.

The local evaluation map

Remark

The result of Skorobogatov/Zarhin gives us a class of varieties, for which the transcendental Brauer group is exceptionally well accessible. The same is true for the local evaluation map.

Fact

Over the function field $k(S)$, each of the 16 vectors in $\mathbb{F}_2^4$ defines a Brauer class. Consider the four quaternion algebras

$$A_{\mu,\nu} := ((x-\mu)(x-b),(u-\nu)(u-b')), \qquad \mu = 0, a,\ \nu = 0, a'.$$

Then e_1 corresponds to $A_{a,a'}$, e_2 to $A_{a,0}$, e_3 to $A_{0,a'}$, and e_4 to $A_{0,0}$.

The local evaluation map II

Lemma

Let k be a local field, $\operatorname{char} k = 0$, $a, b, a', b' \in k$ *be such that*

$$E\colon y^2 = x(x-a)(x-b) \quad \textit{and} \quad E'\colon v^2 = u(u-a')(u-b')$$

are elliptic curves. Consider $S := \operatorname{Kum}(E \times E')$, *given explicitly by*

$$z^2 = x(x-a)(x-b)u(u-a')(u-b').$$

Let $\alpha \in \operatorname{Br}(S)$ *be a Brauer class, represented over* $k(S)$ *by the central simple algebra* $\bigotimes_i A_{\mu_i,\nu_i}$.

Then the local evaluation map $\operatorname{ev}_\alpha\colon S(k) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ *is given by*

$$(x, u; z) \mapsto \operatorname{ev}_\alpha((x, u; z)) = \sum_i ((x-\mu_i)(x-b), (u-\nu_i)(u-b'))_k\,.$$

Constancy near the singular points

Lemma (Elsenhans+J. 2012)

Let $p > 2$ be a prime number and $a, b, a', b' \in \mathbb{Z}_p$ be such that $E\colon y^2 = x(x-a)(x-b)$ and $E'\colon v^2 = u(u-a')(u-b')$ are elliptic curves, not isogenous to each other. Suppose $\gcd(a,b) = \gcd(a',b') = 1$ *and put*

$$l := \max(\nu_p(a), \nu_p(b), \nu_p(a-b), \nu_p(a'), \nu_p(b'), \nu_p(a'-b')) \,.$$

Consider the surface S over $\mathbb{Q}_p$, given by

$$z^2 = x(x-a)(x-b)u(u-a')(u-b') \,.$$

Then, for every $\alpha \in \mathrm{Br}(S)_2$, the evaluation map $S(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ is constant on the subset

$$T := \{(x,u;z) \in S(\mathbb{Q}_p) \mid \nu_p(x) < 0 \text{ or } \nu_p(u) < 0 \text{ or } x \equiv \mu, u \equiv \nu \pmod{p^{l+1}}, \mu = 0, a, b, \ \nu = 0, a', b'\} \,.$$

The case of good reduction

Proposition

Let $E\colon y^2 = x(x-a)(x-b)$ *and* $E'\colon v^2 = u(u-a')(u-b')$ *be two elliptic curves over a local field* k*, not isogenous to each other. Suppose that* $a, b, a', b' \in k$*. Further, let* $S := \mathrm{Kum}(E \times E')$ *be the corresponding Kummer surface.*

Suppose that either $k = \mathbb{R}$ *or* k *is a p-adic field and both* E *and* E' *have good reduction. Then, for every* $\alpha \in \mathrm{Br}(S)_2$*, the evaluation map* $\mathrm{ev}_\alpha\colon S(k) \to \mathbb{Q}/\mathbb{Z}$ *is constant.*

- The case $k = \mathbb{Q}_p$ is a particular case of a very general result, due to J.-L. Colliot-Thélène and A. N. Skorobogatov. It also follows from the lemma above.

The case of good reduction II

- $k = \mathbb{R}$: Without loss of generality, suppose $a > b > 0$ and $a' > b' > 0$. Then

$$M_{aba'b'} = \begin{pmatrix} + & + & + & - \\ + & + & + & + \\ + & + & + & + \\ - & + & + & + \end{pmatrix}$$

has kernel $\langle e_2, e_3 \rangle$. Representatives for e_2 and e_3 are $((x-a)(x-b), u(u-b'))_{\mathbb{R}}$ and $(x(x-b), (u-a')(u-b'))_{\mathbb{R}}$.

e_2: $((x-a)(x-b), u(u-b'))_{\mathbb{R}} = \frac{1}{2}$ would mean $(x-a)(x-b) < 0$ and $u(u-b') < 0$. Hence, $b < x < a$ and $0 < u < b'$. But then $x(x-a)(x-b)u(u-a')(u-b') < 0$. There is no real point on S corresponding to (x, u).

For e_3, the argument is analogous.

An algorithm determining the local evaluation map

Algorithm

Let the parameters $a, b, a', b' \in \mathbb{Z}$, a Brauer class $\alpha \in \mathrm{Br}(S)_2$ as a combination of Hilbert symbols, and a prime number p be given.

1. Calculate $l := \max(\nu_p(a), \nu_p(b), \nu_p(a-b), \nu_p(a'), \nu_p(b'), \nu_p(a'-b'))$.
2. Initialize three lists S_0, S_1, and S_2, the first two being empty, the third containing all triples (x_0, u_0, p) for $x_0, u_0 \in \{0, \dots, p-1\}$. A triple (x_0, u_0, p^e) shall represent the subset

$$\{(x, u; z) \in S(\mathbb{Q}_p) \mid \nu_p(x - x_0) \geq e, \nu_p(u - u_0) \geq e\}.$$

3. Run through S_2. For each element (x_0, u_0, p^e), execute, in this order, the following operations.

- Test whether the corresponding set is non-empty. Otherwise, delete it.
- If $e \geq l+1$, $\nu_p(x - \mu) \geq l+1$ and $\nu_p(u - \nu) \geq l+1$ for some $\mu \in \{0, a, b\}$ and $\nu \in \{0, a', b'\}$ then move (x_0, u_0, p^e) to S_0.

An algorithm determining the local evaluation map II

- Test naively, using the elementary properties of the Hilbert symbol, whether the elements in the corresponding set all have the same evaluation. If this test succeeds then move (x_0, u_0, p^e) to S_0 or S_1, accordingly.
- Otherwise, replace (x_0, u_0, p^e) by the p^2 triples $(x_0 + ip^e, u_0 + jp^e, p^{e+1})$ for $i, j \in \{0, \dots, p-1\}$.

4. If S_2 is empty then output S_0 and S_1 and terminate. Otherwise, go back to step 3.

Remark

This algorithm terminates after finitely many steps only because constancy near the singular points is known.

Back to the introductory example

The introductory example $S\colon z^2 = x(x-1)(x-25)u(u+25)(u+36)$ has the Skorobogatov-Zarhin matrix

$$M = \begin{pmatrix} 1 & 25 & 900 & 25 \\ 25 & 1 & -25 & -275 \\ 900 & -25 & 1 & -24 \\ 25 & -275 & -24 & 1 \end{pmatrix} \widehat{=} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -11 \\ 1 & -1 & 1 & -6 \\ 1 & -11 & -6 & 1 \end{pmatrix},$$

with $\ker M = \langle e_1 \rangle$. Thus, there is a non-trivial Brauer class.

Furthermore, S has bad reduction at $2, 3, 5$, and 11. Running the algorithm for these four primes, one sees that the local evaluation maps at $2, 3$, and 11 are constant, while that at 5 is not.

Some kind of normal form

Observation

Let k be a field, $a, b, a', b' \in k^*$, $a \neq b$, $a' \neq b'$, and S be the Kummer surface $z^2 = x(x-a)(x-b)u(u-a')(u-b')$. There are two types of non-trivial Brauer classes $\alpha \in \operatorname{Br}(S)_2/\operatorname{Br}(k)_2$.

Type 1. α may be expressed by a single Hilbert symbol.
There are nine cases for the kernel vector of $M_{aba'b'}$. A suitable translation of $\mathbf{A}^1 \times \mathbf{A}^1$ transforms the surface into one with kernel vector e_1. Then $ab, a'b', (-aa') \in k^{*2}$.
This implies $(-ba'), (-ab'), (-bb') \in k^{*2}$, too.

Type 2. To express α, two Hilbert symbols are necessary.
There are six cases for the kernel vector of $M_{aba'b'}$. A suitable translation of $\mathbf{A}^1 \times \mathbf{A}^1$ transforms the surface into one with kernel vector $e_2 + e_3$. Then $aa', bb', (a-b)(a'-b') \in k^{*2}$.

A criterion for trivial evaluation

Theorem (Elsenhans+J. 2012)

Let $p > 2$ be a prime number and $0 \neq a, b, a', b' \in \mathbb{Z}_p$ such that $a \neq b$ and $a' \neq b'$. Let S be the Kummer surface, given by $z^2 = x(x-a)(x-b)u(u-a')(u-b')$.

Assume that e_1 is a kernel vector of the matrix $M_{aba'b'}$ and let $\alpha \in \mathrm{Br}(S)_2$ be the corresponding Brauer class.

1. *Suppose $a \equiv b \not\equiv 0 \pmod{p}$ or $a' \equiv b' \not\equiv 0 \pmod{p}$. Then the evaluation map $\mathrm{ev}_\alpha \colon S(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ is constant.*
2. *If $a \not\equiv b \pmod{p}$, $a' \not\equiv b' \pmod{p}$, and not all four numbers are p-adic units then the evaluation map $\mathrm{ev}_\alpha \colon S(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ is non-constant.*

Remark

Consider $a = 1, b = 25,\ a' = -25, b' = -36$.
By 1, we have constancy at $2, 3, 11$. By 2, there is non-constancy at 5.

A sample

Determined all Kummer surfaces of the form

$$z^2 = x(x-a)(x-b)u(u-a')(u-b')$$

allowing coefficients of absolute value ≤ 200 and having a transcendental 2-torsion Brauer class.

More precisely,

- we determined all $(a, b, a', b') \in \mathbb{Z}^4$ such that $\gcd(a, b) = 1$, $\gcd(a', b') = 1$, $a > b > 0$, $a - b, b \leq 200$, as well as $a' < b' < 0$, $a' - b', b' \geq -200$ and the matrix $M_{aba'b'}$ has a non-zero kernel.
- We made sure that (a, b, a', b') was not listed when $(-a', -b', -a, -b)$, $(a, a-b, a', a'-b')$, or $(-a', b'-a', -a, b-a)$ was already in the list. We ignored the quadruples where (a, b) and (a', b') define geometrically isomorphic elliptic curves.

A sample II

This led to

- 3075 surfaces with a kernel vector of type 1, among them 26 have $\mathrm{Br}(S)_2/\mathrm{Br}(\mathbb{Q})_2 = 0$, due to a $\mathbb{Q}$-isogeny.
- 367 surfaces with a kernel vector of type 2
- two surfaces with $\dim \mathrm{Br}(S)_2/\mathrm{Br}(\mathbb{Q})_2 = 2$, $(25, 9, -169, -25)$ and $(25, 16, -169, -25)$.

The generic case is that $\dim \mathrm{Br}(S)_2/\mathrm{Br}(\mathbb{Q})_2 = 0$.

Definition

1. We say that a Brauer class $\alpha \in \mathrm{Br}(S)$ *works* at a prime p if the local evaluation map $\mathrm{ev}_{\alpha,p}$ is non-constant.
2. A prime number p is *BM-relevant* for S if there is a Brauer class working at p.

BM-relevant primes

$(25, 9, -169, -25)$:
One Brauer class works at 2 and 13, another at 5 and 13, and the third at all three.

$(25, 16, -169, -25)$:
One Brauer class works at 3 and 13, another at 5 and 13, and the last at all three.

Remaining surfaces:

# relevant primes	# surfaces
-	6
1	428
2	1577
3	1119
4	276
5	9
6	1

For $(196, 75, -361, -169)$, the Brauer class works at 2, 5, 7, 11, 13, and 19.

$\mathbb{Q}$-rational points

Assume $\alpha \in \mathrm{Br}(S)$ works at l primes $p_1, \ldots, p_l$. There are 2^l vectors consisting only of zeroes and $\frac{1}{2}$'s. By the Brauer-Manin obstruction, half of them are forbidden as values of

$$(\mathrm{ev}_{\alpha,p_1}(x), \ldots, \mathrm{ev}_{\alpha,p_l}(x))$$

for $\mathbb{Q}$-rational points $x \in S(\mathbb{Q})$.

Table: Search bounds to get all vectors by rational points

		bound N insufficient for								
#primes	#surfaces	$N = 50$	100	200	400	800	1600	3200	6400	12800
2	1577	190	56	22	-					
3	1119	555	187	48	1	-				
4	262	262	200	127	67	36	24	13	4	-
5	9	9	9	8	8	8	5	3	1	-

Table: Numbers of vectors in the case $(196, 75, -361, -169)$

bound	50	100	200	400	800	1600	3200	6400	12 800	25 600	50 000
vectors	5	10	14	20	24	26	28	30	31	31	32

Summary

Summary

- We investigated the transcendental Brauer-Manin obstruction for a sample of particular Kummer surfaces.
  Actually, most of the surfaces had $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q}) = 0$, but there was a 2-torsion Brauer class on more than 3000 of the surfaces.
- In our situation, the local evaluation map could be expressed in terms of the Hilbert symbol. This is in close analogy with computations of the algebraic Brauer-Manin obstruction.
- In our sample, the Brauer classes never works at the infinite place. As is known, they do not work at good places, either.
- We tested at which (bad) primes the Brauer classes actually work. There were form zero (in six cases) to six BM-relevant primes.
- We carried out a relatively extensive point search, but no other exceptional phenomena showed up. Our results are perfectly compatible with the idea that there are no further obstructions.

Thank you!!