Line bundles on arithmetic surfaces and intersection theory

Jörg Jahnel*

Abstract

For line bundles on arithmetic varieties we construct height functions using arithmetic intersection theory. In the case of an arithmetic surface, generically of genus g, for line bundles of degree g equivalence is shown to the height on the Jacobian defined by Θ. We recover the classical formula due to Faltings and Hriljac for the Néron-Tate height on the Jacobian in terms of the intersection pairing on the arithmetic surface.

1 Introduction

In this paper we will suggest a construction for height functions for line bundles on arithmetic varieties. Following the philosophy of [BoGS] heights should be objects in arithmetic geometry analogous to degrees in algebraic geometry. So let K be a number field, $\mathcal{O}_K$ its ring of integers and $\mathcal{X}/\mathcal{O}_K$ an arithmetic variety by which we will mean a scheme, projective and flat over $\mathcal{O}_K$. In order to have a good intersection product available we assume $\mathcal{X}$ to be regular. Its generic fiber will be denoted by X/K. Then we have to fix a metrized line bundle $(\mathcal{T}, \|.\|)$ or, equivalently, its first Chern class

$$\overset{\wedge}{c_1}(\mathcal{T}, \|.\|) = (T, g_T) \in \overset{\wedge}{\mathrm{CH}}{}^1(\mathcal{X}).$$

The height of a line bundle $\mathcal{L}$ on $\mathcal{X}$ should be the arithmetic degree of the intersection of $\overset{\wedge}{c_1}(\mathcal{L})$ with $(T, g_T)^{\dim X}$. For this a natural hermitian metric has to be chosen on $\mathcal{L}$. We fix a Kähler metric ω_0 on $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation F_∞, as in [Ar]. Then it is well known that the condition on the Chern form to be harmonic defines $\|.\|$ up to a locally constant factor.

In order to determine this factor we require

$$\overset{\wedge}{\deg}\left(\overset{\wedge}{c_1}\left(\det R\pi_*\mathcal{L}, \|.\|_Q\right)\right) = 0.$$

Here $\pi : \mathcal{X} \longrightarrow \mathrm{Spec}\ \mathcal{O}_K$ is the structural morphism and $\|.\|_Q$ is Quillen's metric ([Qu], [BGS]) at the infinite places of K.

*Mathematisches Institut, Universität Göttingen, Bunsenstraße 3-5, 37073 Göttingen, Germany; email: ßjjahnel@cfgauss.wwwuser.gwdg.de

1.1 Fact. a) *If the Euler characteristic* $\chi(\mathcal{L}_K)$ *does not vanish, such a metric exists.*
b) $\stackrel{\wedge}{c_1}(\mathcal{L}, \|.\|)$ *is uniquely determined up to a summand* $(0, C)$, *where* $C = (C_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}$ *is a system of constants on* $X \times_{\mathrm{SpecK},\sigma} \mathrm{Spec}\,\mathbb{C}$ *with*

$$\sum_{\sigma: K \hookrightarrow \mathbb{C}} C_\sigma = 0 \qquad (\textit{and} \;\; C_\sigma = C_{\bar{\sigma}}).$$

1.2 Fact. *Such* $(0, C) \in \stackrel{\wedge}{\mathrm{CH}^1}(\mathcal{X})$ *are numerically trivial.*

1.3 We note that in order to prove the Fact 1.1 one mainly uses the following
Lemma. *Let* $f : X \longrightarrow Y$ *be a smooth proper map of complex manifolds, where* X *is equipped with a Kähler form* ω *and* Y *is connected, and* E *be a holomorphic vector bundle on* X. *For a hermitian metric* $\|.\|$ *on* E *and a constant factor* $D > 0$ *we have*

$$\|\cdot\|_{Q,(E, D\cdot\|.\|)} = \|\cdot\|_{Q,(E,\|.\|)} \cdot D^{\chi(E_K)}$$

on the line bundle $\det R\pi_* E$ *on* Y.

1.4 Now we can state our fundamental
Definition. *The* height *of the line bundle* $\mathcal{L}$ *is given by*

$$h_{T,\omega_0}(\mathcal{L}) := \stackrel{\wedge}{\deg} \;\; \pi_* \left[\stackrel{\wedge}{c_1}(\mathcal{L}, \|.\|) \cdot (T, g_T)^{\dim X} \right],$$

where $\|.\|$ *is one of the distinguished metrics specified above.*

1.5 An arithmetic surface is an arithmetic variety of absolute dimension 2. In this paper we will analyze Definition 1.4 in that case. Our main result is
Theorem (Equivalence). *Let* $\pi : \mathcal{C} \longrightarrow \mathcal{O}_K$ *be a regular arithmetic surface. Denote by* $C := \mathcal{C} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}\, K$ *its generic fiber. Assume* C *to be connected and to admit a* K*-valued point* $x \in C(K)$ *and let* Θ *be the Theta divisor on the Jacobian* $J = \mathrm{Pic}^g(C)$ *(transferred from* $\mathrm{Pic}^{g-1}(C)$ *by* x*), where* g *denotes the genus of* C. *On*

$$\mathcal{C}_{\mathbb{C}} := \coprod_{\sigma: K \hookrightarrow \mathbb{C}} \left(C \times_{\mathrm{Spec}K,\sigma} \mathrm{Spec}\,\mathbb{C} \right)(\mathbb{C})$$

let ω *be a Kähler form invariant under* F_∞.

Then, for line bundles $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$, *fiber-by-fiber of degree* g *and of degree of absolute value less than* H *on every irreducible component of the special fibers of* $\mathcal{C}$ *(with some constant* $H \in \mathbb{N}$*)*

$$h_{x,\omega}(\mathcal{L}) = h_\Theta(\mathcal{L}_K) + \mathrm{O}(1),$$

where h_Θ *is the height on* J *defined using the ample divisor* Θ.

1.6 Remark. Another connection between heights on the Jacobian of a curve and arithmetic intersection theory was obtained by Faltings [Fa84] and Hriljac [Hr]. Recently it has been generalized to higher dimensions and higher codimension Chow groups by Künnemann [Kü]. They write down an explicit formula for the Néron-Tate height pairing on the Jacobian (higher Picard variety) in terms of arithmetic intersection theory. The main point is that they consider line bundles (cycles) algebraically equivalent to zero. So there is no need to scale a metric (to specify the infinite part of the arithmetic cycles occuring). We recover the Faltings-Hriljac formula for the Néron-Tate height by considering some kind of asymptotic limit over the heights constructed here.

1.7 Theorem (Asymptotic behaviour)**.** *Let $\pi : \mathcal{C} \longrightarrow \mathcal{O}_K$ be a regular arithmetic surface. Equip $\mathcal{C}_{\mathbb{C}}$ with a Kähler form ω invariant under complex conjugation F_∞.*

Let $\overline{\mathcal{E}}$ and $\overline{\mathcal{M}}$ be line bundles on $\mathcal{C}$ equipped with a distinguished hermitian metric und $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}\,(\mathcal{C})$ be a hermitian line bundle. Assume $n \in \mathbb{N}$ is chosen such that $\chi(\mathcal{E}_K \otimes \mathcal{M}_K^{\otimes n})$. Then

$$\begin{aligned}\chi(\mathcal{E}_K \otimes \mathcal{M}_K^{\otimes n}) h_{\overline{\mathcal{L}}}(\mathcal{E} \otimes \mathcal{M}^{\otimes n}) &= \chi(\mathcal{E}_K) h_{\overline{\mathcal{L}}}(\mathcal{E}) \\ &+ n\ [(\deg(\mathcal{M}_K) h_{\overline{\mathcal{L}}}(\mathcal{E}) + \chi(\mathcal{E}_K) D - \deg(\mathcal{L}_K) B] \\ &+ n^2 [\deg(\mathcal{M}_K) D - \deg(\mathcal{L}_K) \frac{A}{2}].\end{aligned}$$

Here χ always denotes the Euler characteristic and A, B and D are abbreviations for the arithmetic intersection numbers

$$\begin{aligned}A &:= \widehat{\deg}\ \pi_*(\hat{c}_1\,(\overline{\mathcal{M}})^2), \\ B &:= \widehat{\deg}\ \pi_*\Big(\hat{c}_1\,(\overline{\mathcal{M}})\Big(\hat{c}_1\,(\overline{\mathcal{E}}) + \frac{1}{2}\,\hat{c}_1\,(\overline{T_f})\Big)\Big) \qquad \text{and} \\ D &:= \widehat{\deg}\ \pi_*(\hat{c}_1\,(\overline{\mathcal{M}})\ \hat{c}_1\,(\overline{\mathcal{L}})).\end{aligned}$$

By $\hat{c}_1\,(\overline{T_f})$ we mean the first arithmetic Chern class of the relative tangent bundle equipped with the metric associated with ω.

1.8 Remark. Consider the special case, where $\deg(\mathcal{E}_{\mathbb{C}}) = g$, $\deg(\mathcal{L}_{\mathbb{C}}) = 1$ and $\deg(\mathcal{M}_{\mathbb{C}}) = 0$. Then $h_{\overline{\mathcal{L}}}(\mathcal{E} \otimes \mathcal{M}^{\otimes n})$ is a quadratic polynomial in n, whose leading coefficient is just $-\frac{1}{2}A = -\frac{1}{2}\ \widehat{\deg}\ \pi_*(\hat{c}_1\,(\overline{\mathcal{M}})^2)$, one half of the term considered by Faltings and Hriljac. In fact we have reproven the coincidence of the Néron-Tate height on the Jacobian with the arithmetic self-intersection number $-\ \widehat{\deg}\ \pi_*(\hat{c}_1\,(\overline{\mathcal{M}})^2)$ since the Néron-Tate height is actually defined as a limit of the type considered.

1.9 Corollary. *Let Θ be the Theta divisor on* $\mathrm{Pic}^0(C)$ *(transferred from* $\mathrm{Pic}^{g-1}(C)$ *by x). Then*

$$h_{\mathrm{NT},\Theta}(\mathcal{M}_K) = -\frac{1}{2}\ \widehat{\deg}\ \pi_*(\hat{c}_1\,(\overline{\mathcal{M}})^2).$$

1.10 Corollary (Faltings, Hriljac)**.**

$$h_{\mathrm{NT},\Theta+\Theta^-}(\mathcal{M}_K) = -\ \widehat{\deg}\ \pi_*(\hat{c}_1\,(\overline{\mathcal{M}})^2).$$

Proof. The endomorphism $[-1]$ does not change the right hand side. □

1.11 Remark. The proof of the equivalence theorem is relatively long but mainly consists of using elementary algebraic geometry and analysis. There is another (even more involved?) proof for that theorem in [J1]. To the contrary the asymptotic behaviour is an almost direct application of the arithmatic Riemann-Roch theorem.

We note once more that Definition 1.4 works in every dimension. We deal with the case of dimension greater than two in another paper [J2].

2 Proof of the Equivalence Theorem

2.1 Divisors versus points of the Jacobian

2.1.1 The remainder of this paper is devoted to the proof of Theorem 1.5. So let C/K be a regular proper algebraic curve of genus g with $C(K) \neq \emptyset$. We consider a regular projective model $\mathcal{C}/\mathcal{O}_K$. Denote by $J = \mathrm{Pic}^g_{C/K}$ the Jacobian of C. When $x \in C(K)$ is chosen we have a canonical isomorphism $\mathrm{Pic}^{g-1}_{C/K} \longrightarrow \mathrm{Pic}^g_{C/K} = J$ and thus the divisor Θ on J. Θ induces a closed embedding $i^{'} : J \hookrightarrow \mathbf{P}^N_K$ and a "naive" height for K-valued points of J:

$$h_\Theta(D) := \log \left(\prod_{\nu \in M_K} \max \{ \|i(D)_0\|_\nu, \ \ldots \ , \|i(D)_N\|_\nu \} \right).$$

Accordingly $j^*(\Theta)$ induces a morphism $i : C^g \xrightarrow{j} J \xrightarrow{i^{'}} \mathbf{P}^N_K$ and a height function $h_{j^*(\Theta)}$ for K-valued points of C^g. Here j denotes the natural map sending a divisor to its associated line bundle. A general construction for heights defined by a divisor, the "height machine", is given in [CS, Chapter VI, Theorem 3.3].

The underlying height h for K-valued points of $\mathbf{P}^N_K$ is a height in the sense of Arakelov theory [BoGS] as follows: We choose the regular projective model $\mathbf{P}^N_{\mathcal{O}_K} \supseteq \mathbf{P}^N_K$. Every K-valued point y of $\mathbf{P}^N_K$ can be extended uniquely to an $\mathcal{O}_K$-valued point $\underline{y}$ of $\mathbf{P}^N_{\mathcal{O}_K}$. Let $\overline{\mathcal{O}(1)}$ be the hermitian line bundle on $\mathbf{P}^N_{\mathcal{O}_K}$, where the hermitian metrics at the infinite places are given by

$$\|x_i\| := \left(\left|\frac{x_0}{x_i}\right|^2 + \ldots + 1 + \ldots + \left|\frac{x_N}{x_i}\right|^2 \right)^{-\frac{1}{2}}.$$

Then $h = h_{\overline{\mathcal{O}(1)}}$ is the height defined by $\overline{\mathcal{O}(1)}$ in the sense of [BoGS, Definition 3.1.; formula (3.1.6)].

2.1.2 Remark. We need a better understanding of $\mathcal{O}(j^*(\Theta))$. By Riemann's Theorem [GH, Chapter 2, §7] one has $\Theta = \frac{1}{(g-1)!} j_*((x) \times C^{g-1})$, where $j : C^g \xrightarrow{p} C^{(g)} \xrightarrow{c} J$ factors into a morphism finite flat of degree $g!$ and a birational morphism. So $j^*(\Theta)$ is an effective divisor containing the summands $\pi_k^*(x)$, where $\pi_k : C^g \longrightarrow C$ denotes k-th projection.

$$\mathcal{O}\Big(j^*(\Theta)\Big) = \bigotimes_{k=1}^{g} \pi_k^*\Big(\mathcal{O}(x)\Big) \otimes \mathcal{O}\Big(p^*(R)\Big)$$

Intuitively, the divisor R on $C^{(g)}$ corresponds to the divisors on C moving in a linear system. This can be made precise, but we will not need that here.

2.1.3 Remark. It is a difficulty that there are no regular projective models available for J and C^g, such that arithmetic intersection theory does not work immediately. So we follow [BoGS, Remark after Proposition 3.2.1.] and consider a projective (not necessarily regular) model of C^g, namely $\mathcal{C}^g := \mathcal{C} \times_{\mathrm{Spec}\mathcal{O}_K} \ldots \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$. Hereon let $\overline{\mathcal{U}}$ be a line bundle extending $\bigotimes_{k=1}^{g} \pi_k^* \mathcal{O}(x)$ equipped with a hermitian metric.

Consider more generally a projective (singular) arithmetic variety $\mathcal{X}/\mathcal{O}_K$ and a hermitian line bundle $\overline{\mathcal{U}}$ on $\mathcal{X}$. Then there is a morphism $\iota : \mathcal{X} \longrightarrow P$ into a projective variety P smooth over Spec $\mathcal{O}_K$ and a line bundle $\mathcal{U}_{\mathcal{P}}$ on P such that $\iota^*(\mathcal{U}_{\mathcal{P}}) = \mathcal{U}$ (see [Fu, Lemma 3.2.], cf. [BoGS, Remark 2.3.1.ii)]). We can even choose ι in such a way that the hermitian metric on $\overline{\mathcal{U}}$ is a pullback of one on $\mathcal{U}_P$ (e.g. as a closed embedding).

$$\iota^*\Big(\overline{\mathcal{U}_P}\Big) = \overline{\mathcal{U}}$$

Then for an $\mathcal{O}_K$-valued point $\underline{y}$ of $\mathcal{X}$ one defines

$$\begin{aligned} h_{\overline{\mathcal{U}}}\big(\underline{y}\big) &:= h_{\overline{\mathcal{U}}_P}\Big(\iota_*(\underline{y})\Big) \\ &= \widehat{\deg}\,\Big(\,\hat{c}_1\,(\overline{\mathcal{U}_P})\Big|\iota_*(y)\Big), \end{aligned}$$

where $(.|.)$ denotes the pairing $\widehat{\mathrm{CH}}^1\,(P) \times \mathrm{Z}_1(P) \longrightarrow \widehat{\mathrm{CH}}^1\,(\mathrm{Spec}\ \mathcal{O}_K)_{\mathbb{Q}}$ from [BoGS, 2.3.]. In [BoGS, Remark after Proposition 3.2.1.] independence of this definition of the ι chosen is shown. In particular it becomes clear at this point that the pairing $\Big(\,\hat{c}_1\,(.)\Big|.\Big)$ can be extended to arbitrary (singular) projective arithmetic varieties over $\mathcal{O}_K$ and satisfies the projection formula

$$\Big(\,\hat{c}_1\,(L)\Big|f_*(Z)\Big) = \Big(\,\hat{c}_1\,(f^*(L))\Big|Z\Big).$$

2.1.4 Lemma. *Let $\mathcal{X}/\mathcal{O}_K$ be a (singular) projective arithmetic variety and X/K its generic fiber which is assumed to be regular. Further, let D be a divisor on X and $\overline{\mathcal{U}}$ be a hermitian line bundle extending $\mathcal{O}(D)$. Then $h_D = h_{\overline{\mathcal{U}}} + \mathrm{O}(1)$ for K-valued points of X, i.e. the naive height defined by the divisor D and the height in the sense of Arakelow theory are equivalent.*

Proof. This is proven between the lines of [BoGS]. □

2.1.5 The height defined by an extension $\mathcal{U}$ of $\bigotimes_{k=1}^{g} \pi_k^*(\mathcal{O}(x))$ is understood by the following

Proposition. *On $\mathcal{C}/\mathcal{O}_K$ let $\overline{\mathcal{S}}$ be the line bundle $\mathcal{O}(\overline{x})$, where $\overline{x}$ denotes the closure of x in $\mathcal{C}$, equipped with a hermitian metric. For L-valued points $P = (P_1, \ldots, P_g)$ of C^g we consider the divisor $\underline{P} := (P_1) + \ldots + (P_g)$ on C. Then*

$$h_{\overline{\mathcal{S}}}(\underline{P}) = h_{\overline{\mathcal{U}}}(P) + O([L:K]),$$

i.e. there is a universal constant A such that the difference of the two heights is bounded by $A[L:K]$ for all number fields L.

Proof. By [BoGS, Proposition 3.2.2.ii)] we may assume $\mathcal{U} = \mathcal{O}\Big(\sum_{k=1}^{g} \pi_k^*(\overline{x})\Big)$, where $\overline{x}$ is the closure of x in $\mathcal{C}$. The extensions of P and $\underline{P}$ over $\mathcal{C}$, respectively $\mathcal{C}^g$ will be denoted by $(\overline{P_1}) + \ldots + (\overline{P_g})$, respectively $(\overline{P_1}, \ldots, \overline{P_g})$. Then

$$\begin{aligned}
h_{\overline{\mathcal{S}}}\Big((\overline{P_1}) + \ldots + (\overline{P_g})\Big) &= \widehat{\deg}\left(\hat{c}_1\, (\overline{\mathcal{S}})\Big|(\overline{P_1}) + \ldots + (\overline{P_g})\right) \\
&= \sum_{k=1}^{g} \widehat{\deg}\left(\hat{c}_1\, (\overline{\mathcal{S}})\Big|(\overline{P_k})\right) \\
&= \sum_{k=1}^{g} \widehat{\deg}\left(\hat{c}_1\, \left(\pi_k^*(\overline{\mathcal{S}})\right)\Big|(\overline{P_1},\ \ldots\ ,\overline{P_g})\right) \\
&= \widehat{\deg}\left(\hat{c}_1\left(\bigotimes_{k=1}^{g} \pi_k^*\left(\overline{\mathcal{S}}\right)\right)\ \Big|\ \left(\overline{P_1},\ \ldots\ ,\overline{P_g}\right)\right).
\end{aligned}$$

But by construction $\bigotimes_{k=1}^{g} \pi_k^*(\overline{\mathcal{S}})$ is the line bundle $\mathcal{U}$, equipped with a hermitian metric (and by definition the formula $\left(\hat{c}_1\, (\bigotimes_k \overline{\mathcal{L}_k})\Big|Z\right) = \sum_k \left(\hat{c}_1\, (\overline{\mathcal{L}_k})\Big|Z\right)$ holds in singular case, too). So we have

$$h_{\overline{\mathcal{S}}}\Big((\overline{P_1}) + \ldots + (\overline{P_g})\Big) = h_{\overline{\mathcal{U}}'}\left((\overline{P_1},\ \ldots\ ,\overline{P_g})\right),$$

where $\overline{\mathcal{U}}'$ differs from $\overline{\mathcal{U}}$ only by the hermitian metric. The claim follows from [BoGS, Proposition 3.2.2.i)]. □

2.1.6 Corollary. *Let $P \in C^g(L)$ and $\underline{P}$ be the associated divisor on C. Then*

$$h_\Theta(\mathcal{O}(\underline{P})) = h_{\overline{\mathcal{S}}}(\underline{P}) + h_R(p_*P) + \mathrm{O}([L:K]),$$

where h_R denotes the height for L-valued points of $C^{(g)}$ defined by R.

2.2 An observation concerning the tautological line bundle

In this section we start analyzing the fundamental definition 1.3. First we will consider only varieties over number fields and forget about integral models.

2.2.1 Definition. *Let Δ be the diagonal in $C \times C$. Then*

$$\underline{\underline{\mathcal{E}}} := \bigotimes_{k=1}^{g} \pi_{k,g+1}^*\Big(\mathcal{O}(\Delta)\Big)$$

will be called the tautological line bundle *on $C^g \times C$. Note that the restriction of $\underline{\underline{\mathcal{E}}}$ to $\{(P_1, \ldots, P_g)\} \times C$ equals $\mathcal{O}(P_1 + \ldots + P_g)$. By construction $\underline{\underline{\mathcal{E}}}$ is the pullback of some line bundle $\mathcal{E}$, said to be* the tautological *one on $C^{(g)} \times C$.*

$$\underline{\underline{\mathcal{E}}} = (p \times id)^*(\mathcal{E})$$

2.2.2 Proposition. *We have* $\det R\pi_*\mathcal{E} = \mathcal{O}_{C^{(g)}}(-R)$.

2.2.3 This will be a direct consequence of the following
Lemma. *Let* $\underline{\mathcal{E}} := \mathcal{E} \otimes \pi_C^*(\mathcal{O}(-x))$ *be a tautological line bundle fiber-by-fiber of degree* $g-1$. *Then*

$$\det R\pi_*\underline{\mathcal{E}} = \mathcal{O}_{C^{(g)}}\Big(-c^*(\Theta)\Big).$$

Proof. The canonical map $c : C^{(g)} \longrightarrow J$ is given by $\mathcal{E}$ using Picard functoriality. So for a tautological line bundle $\mathcal{M}$, fiber-by-fiber of degree g on $J \times C$, one has

$$\mathcal{E} = (c \times id)^*\mathcal{M} \otimes \pi^*\mathcal{H},$$

where $\mathcal{H}$ is a line bundle on $C^{(g)}$. Putting $\mathcal{M}_0 := \mathcal{M} \otimes \pi_C^*\mathcal{O}(-x)$, where $\pi_C : J \times C \longrightarrow C$ denotes here the canonical projection from $J \times C$, we get

$$\underline{\mathcal{E}} = (c \times id)^*\mathcal{M}_0 \otimes \pi^*\mathcal{H}.$$

It follows

$$\begin{aligned} \det R\pi_*\underline{\mathcal{E}} &\cong \det R\pi_*\Big[(c \times id)^*\mathcal{M}_0 \otimes \pi^*\mathcal{H}\Big] \\ &= \det R\pi_*\Big[(c \times id)^*\mathcal{M}_0\Big] \\ &= c^* \det R\pi_*\mathcal{M}_0, \end{aligned}$$

where we first used the projection formula, which is particularly simple here, since line bundles, fiber-by-fiber of degree $g-1$, have relative Euler characteristic 0, and afterwords noted that the determinant of cohomology commutes with arbitrary base change [KM]. But by [MB, Proposition 2.4.2] or [Fa, p. 396] we know $\det R\pi_*\mathcal{M}_0 = \mathcal{O}_J(-\Theta)$. The assertion follows. $\square$

2.2.4 Proof of the Proposition. The short exact sequence

$$0 \longrightarrow \underline{\mathcal{E}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_{C^{(g)} \times \{x\}} \longrightarrow 0$$

gives

$$\begin{aligned} \det R\pi_*\mathcal{E} &= \det R\pi_*\underline{\mathcal{E}} \otimes \mathcal{O}\left(\frac{1}{g!}p_*\left(\sum_{k=1}^{g} \pi_k^*(x)\right)\right) \\ &= \mathcal{O}\left(-c^*(\Theta)\right) \otimes \mathcal{O}\left(\frac{1}{g!}p_*\left(\sum_{k=1}^{g} \pi_k^*(x)\right)\right) \\ &= \mathcal{O}(-R). \end{aligned}$$

$\square$

2.2.5 Corollary. $\det R\pi_*(\mathcal{E} \otimes \pi^*\mathcal{O}(R)) = \mathcal{O}_{C^{(g)}}$.
Proof. This is the projection formula for the determinant of cohomology. $\square$

2.2.6 Remark. Every degree g line bundle on C is represented by a K-valued point of the symmetric power $C^{(g)}$.

Indeed, by Riemann-Roch such a line bundle admits a section s. $\mathrm{div}(s)$ is defined over K, i.e. it is a $\mathrm{Gal}(\overline{K}/K)$-invariant formal sum of g $\overline{K}$-valued points. But this defines just a $\mathrm{Gal}(\overline{K}/K)$-invariant $\overline{K}$-valued point of the symmetric power $C^{(g)}$ and those descend to K-valued points.

We note that this phenomenon is no more true when working with the power C^g. So we prefer the symmetric power in what follows in order to be not forced to deal with field extensions all the time.

2.3 An integral model for the symmetric power

2.3.1 Remark. We want to lift the results of the previous subsection to the level of models over $\mathcal{O}_K$. So we need a model $\mathcal{I}$ for $C^{(g)}$ satisfying the following two conditions simultaneously.

i) Every K-valued point of $C^{(g)}$ extends to an $\mathcal{O}_K$-valued point of $\mathcal{I}$.

ii) The tautological line bundle $\mathcal{E}$ on $C^{(g)} \times C$ can be extended to a line bundle $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$.

We note that i) could be realized when $\mathcal{I}$ was proper while ii) is possible when $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$ would be regular, but in general there are no smooth proper models for $C^{(g)}$.

2.3.2 Proposition. *There exists a scheme $\mathcal{I}$ over $\mathcal{O}_K$ satisfying the following conditions.*

i) $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}\ K \cong C^{(g)}$,

ii) Every K-valued point of $C^{(g)}$ extends to an $\mathcal{O}_K$-valued point of $\mathcal{I}$,

iii) $\mathcal{I}$ is smooth over $\mathcal{O}_K$.

Proof. First consider the scheme-theoretic symmetric power $\mathcal{I}' := \mathcal{C}^g/S_g$. We note particularly that this object really exists. In fact we know that every finite set of points in $\mathcal{C}^g$ lies in an affine subscheme as this is clearly true for the projective space $\mathbf{P}^n_{\mathbb{Z}}$. So $\mathcal{C}^g$ can be covered by powers of affine subsets of $\mathcal{C}$ and those are invariant under the action of the symmetric group.

It is clear that $\mathcal{I}'$ is a model of $C^{(g)}$ over $\mathcal{O}_K$. As the image of $\mathcal{C}^g$ under the projection map it is clearly proper, so K-valued points can be extended to $\mathcal{O}_K$-valued ones.

Let $\mathcal{I}$ be the smooth locus of $\mathcal{I}'$. Everything have to show now is that the extension of a K-valued point of $C^{(g)}$ automatically lands in the smooth locus. But that is a standard argument. (Compare [CS, Ch. VIII by M. Artin, Proposition (1.15)].) $\square$

2.3.3 $\mathcal{I}$ is smooth over $\mathcal{O}_K$, consequently $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$ is smooth over $\mathcal{C}$ and therefore regular. The tautological line bundle $\mathcal{E}$ on $C^{(g)} \times C$, fiber-by-fiber of degree g, can be extended over $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$. For this let $\mathcal{E} = \mathcal{O}(D)$ with some Weil divisor D on $C^{(g)} \times C$. Its closure $\overline{D}$ in $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$ is obviously flat over $\mathcal{O}_K$ and therefore it has codimension 1. We choose the extension $\mathcal{O}(\overline{D})$ and denote it by $\mathcal{F}'$.

$\mathcal{F}'$ is a perfect complex of $\mathcal{O}_{\mathcal{I}\times_{\mathrm{Spec}\,\mathcal{O}_K}\mathcal{C}}$-modules. For the existence of the Knudsen-Mumford determinant $\det R\pi_*\mathcal{F}'$ we need that

$$\pi : \mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C} \longrightarrow \mathcal{I}$$

has finite Tor-dimension. But this is clear by [SGA 6, Exposé III, Proposition 3.6] as π is a projective morphism into a regular scheme.

$\mathcal{F}'$ has, relative to π, Euler characteristic 1. Therefore $\mathcal{F}'$ can be changed by an inverse image of a line bundle on $\mathcal{I}$, in such a way, that we are allowed to assume

$$\det R\pi_*\mathcal{F} \cong \mathcal{O}_{\mathcal{I}}.$$

Obviously, $\mathcal{F}$ is an extension of $\mathcal{E} \otimes \mathcal{O}(R)$ to the integral model $\mathcal{I} \times \mathcal{C}$.

2.4 Analytic Part

For simplicity assume the Kähler form ω on $\mathcal{C}_{\mathbb{C}} = \coprod_{\sigma:K\hookrightarrow\mathbb{C}} \left(\mathcal{C} \times_{\mathrm{Spec}\mathcal{O}_K,\sigma} \mathrm{Spec}\ \mathbb{C}\right)(\mathbb{C})$ is normalized by

$$\int_{(\mathcal{C}\times_{\mathrm{Spec}\mathcal{O}_K,\sigma}\mathrm{Spec}\mathbb{C})(\mathbb{C})} \omega = 1$$

for every σ.

2.4.1 Fact. *On $\mathcal{F}_{\mathbb{C}}$ there exists a hermitian metric $\underline{h}$ such that for every point $y \in C^{(g)}(\mathbb{C})$ the curvature form satisfies*

$$c_1(\mathcal{F}_{\mathbb{C},y}, \underline{h}_y) = g\omega$$

on $(\{y\} \times C)(\mathbb{C}) \cong C(\mathbb{C})$.

Proof. The statement is local in $C^{(g)}(\mathbb{C})$ by partition of unity. By the Theorem on cohomology and base change $R^0\pi_*\mathcal{F}_{\mathbb{C}}(g-1)$ is locally free and commutes with arbitrary base change. Hence there exists, locally on $C^{(g)}(\mathbb{C})$, a rational section s of $\mathcal{F}$ that is neither undefined nor identically zero in any fiber.

First we choose an arbitrary hermitian metric $\|.\|$ on $\mathcal{F}_{\mathbb{C}}$. Then

$$\omega' := -d_C d_C^c \log \|s\|^2 \tag{2}$$

defines a smooth $(1,1)$-form on $(C^{(g)} \times C)(\mathbb{C})\backslash \mathrm{div}(s)$. Since construction (2) is independent of s as soon as it makes sense at a point, we obtain ω' as a smooth $(1,1)$-form on $(C^{(g)} \times C)(\mathbb{C})$ closed under d_C and cohomologous to $g\omega$ on $\{y\} \times C(\mathbb{C})$ for any $y \in C^{(g)}(\mathbb{C})$.

The setup $\|.\|_{\underline{h}} = f \cdot \|.\|$ gives the equation

$$\omega' - g\omega = d_C d_C^c \log |f|^2. \tag{3}$$

But dd^c is an elliptic differential operator on the Riemann surface $C(\mathbb{C})$, so by Hodge theory it permits a Green's operator G compact with respect to every Sobolew norm $\|.\|_\alpha$. Consequently, there exists a solution f of (3) being smooth on $(C^{(g)} \times C)(\mathbb{C})$. $\square$

2.4.2 We note, that $\det R\pi_*\mathcal{F} \cong \mathcal{O}_\mathcal{I}$ and the isomorphism is uniquely determined up to units of $\mathcal{O}_K$. Namely, one has $\mathrm{Aut}_{\mathcal{O}_\mathcal{I}}(\mathcal{O}_\mathcal{I}) = \Gamma(\mathcal{I}, \mathcal{O}^*_\mathcal{I})$ and already $\Gamma(C^{(g)}, \mathcal{O}^*_{C^{(g)}})$ consists of constants only. In particular, there is a unitary section, uniquely determined up to units of $\mathcal{O}_K$,

$$\mathbf{1} \in \Gamma(\mathcal{I}, \det R\pi_*\mathcal{F}).$$

2.4.3 Corollary. *Let $R \in \mathbb{R}$. Then, on $\mathcal{F}_\mathbb{C}$ there exists exactly one hermitian metric h, such that for every point $y \in C^{(g)}(\mathbb{C})$*

i) the curvature form $c_1(\mathcal{F}_{\mathbb{C},y}, h_y) = g\omega$ and

ii) for the Quillen metric one has $h_{Q,h}(\mathbf{1}) = R$, where $\mathbf{1} \in \Gamma(\{y\}, \det R\pi_*\mathcal{F}_{\mathbb{C},y})$.

Proof. Let $\underline{h}$ be the hermitian metric from the preceeding fact. We may replace $\underline{h}$ by $f \cdot \underline{h}$ with $f \in C^\infty(C^{(g)}(\mathbb{C}))$ without any effort on the curvature forms, since they are invariant under scalation. As $\mathcal{F}$ has relative Euler characteristic 1,

$$h := \frac{R}{h_{Q,\underline{h}}(\mathbf{1})} \cdot \underline{h}$$

exactly satisfies the conditions required. $\square$

2.4.4 We have to consider $\mathcal{F}_\mathbb{C}$ on $C^{(g)}(\mathbb{C}) \times (\coprod_{\sigma: K \hookrightarrow \mathbb{C}} C(\mathbb{C}))$. The metric h on $\mathcal{F}_\mathbb{C}$ has to be invariant under F_∞, its curvature form is required to be $g\omega$ and we want to realise

$$\prod_{\sigma: K \hookrightarrow \mathbb{C}} h_{Q,h}(\mathbf{1}) = 1 \tag{4}$$

simultaneously for all $y \in C^{(g)}(\mathbb{C})$.

The first is possible since ω is invariant under F_∞ and the corollary above already gives conditions uniquely determining h. (4) can be obtained by scalation with a constant factor over all $C^{(g)}(\mathbb{C}) \times (\coprod_{\sigma: K \hookrightarrow \mathbb{C}} C(\mathbb{C}))$.

Altogether, for every line bundle of degree g on C we have found a distinguished hermitian metric and seen that it depends, in some sense, continuously on the moduli space J. One obtains

2.4.5 Proposition. *Let K be a number field and $(\mathcal{C}/\mathcal{O}_K, \omega)$ a regular connected Arakelov surface. Then, on the (non proper) Arakelov variety $(\mathcal{I} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathcal{C}, \pi_C^*\omega)$ there is a hermitian line bundle $\overline{\mathcal{F}}$ with the following properties.*

a)

$$\mathcal{F}|_{C^{(g)} \times C} = \mathcal{E} \otimes \pi^*\mathcal{O}(R)$$

is the modified tautological line bundle found in Corollary 2.2.5.

b) *The hermitian metric h on $\mathcal{F}|_{\coprod_{\sigma: K \hookrightarrow \mathbb{C}}(C^{(g)} \times C)(\mathbb{C})}$ is invariant under F_∞ and has curvature form $g\omega$.*

c) *For any $y \in C^{(g)}(K)$ one has $\overline{\{y\}} \subseteq \mathcal{I}$ and*

$$\widehat{\deg}\left(\widehat{c}_1\left(\det R\pi_*(\mathcal{F}|_{\overline{\{y\}} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathcal{C}}, h_{\mathcal{F},y}), \|\cdot\|_{Q,h}\right)\right) = 0.$$

2.5 Decomposition into two summands

2.5.1 Remark. Decompose $\overline{\mathcal{F}}$ into a tensor product $\overline{\mathcal{K}} \otimes \pi^*\overline{\mathcal{R}}$ of hermitian line bundles, where $\mathcal{K}$ is an extension of $\mathcal{E}$ to the model $\mathcal{I} \times_{\mathrm{Spec}\mathcal{O}_K} \mathcal{C}$ and $\mathcal{R}$ extends $\mathcal{O}(R)$ to $\mathcal{I}$. Then for any $\mathcal{O}_K$-valued point $y : \mathrm{Spec}\ \mathcal{O}_K \longrightarrow \mathcal{I}$ one has

$$\begin{aligned} h_{x,\omega}\left(\mathcal{F}|_{y\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}}\right) &= \widehat{\deg}\ \pi_*\left[\hat{c}_1\left(\overline{\mathcal{K}}|_{y\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}}\right)\cdot(x,g_x)\right] \\ &+ \widehat{\deg}\ \pi_*\left[\hat{c}_1\left(\pi^*\overline{\mathcal{R}}|_{y\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}}\right)\cdot(x,g_x)\right]. \end{aligned}$$

2.5.2 Let us investigate the first summand. We have $\mathcal{K}|_{C^{(g)}\times C} = \mathcal{E}$ and this line bundle has a canonical section s, which can be extended over the finite places. Using this section we obtain the arithmetic cycle $(\mathrm{div}\ (s), -\log\|s\|^2)$ representing

$$\hat{c}_1\ (\overline{\mathcal{K}}) \in \widehat{CH}^1\ (\mathcal{I}\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}).$$

The scheme part $\mathrm{div}(s)$ of this cycle is an extension of the tautological divisor representing $c_1(\mathcal{E}) \in \mathrm{CH}^1(C^{(g)}\times C)$ (whose restriction to $\{(x_1,\ldots,x_g)\}\times C$ is $(x_1)+\ldots+(x_g)$). So $\mathrm{div}(s)$ is the closure of that divisor, possibly plus a finite sum of divisors over the finite places. Consequently, if y corresponds to the K-valued point in $C^{(g)}$ representing to the divisor D on C, then

$$c_1\left(\mathcal{K}|_{y\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}}\right) = \left(\overline{D}\right) + (\text{correction terms}),$$

where $\overline{D}$ denotes the closure of D over $\mathcal{C}$ and the correction terms are vertical divisors which (over all the y) occur only over a finite amount of finite places. Their intersection numbers with (x, g_x) are obviously bounded.

The infinite part f of $\hat{c}_1\ (\overline{\mathcal{K}}) = (\mathcal{D}, f)$ is a function on $(C^{(g)}\times C)\backslash\mathrm{div}\ (s)$ whose pullback to $C^g\times C$ satisfies all the assumptions of Lemma 2.6.4 below. We obtain

$$\begin{aligned} \widehat{\deg}\ \pi_*\left[\hat{c}_1\left(\overline{\mathcal{K}}|_{y\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}}\right)\cdot(x,g_x)\right] &= \widehat{\deg}\left(\hat{c}_1\left(\overline{\mathcal{O}(x)}\right)\middle|\left(\mathcal{D}|_{y\times_{\mathrm{Spec}\ \mathcal{O}_K}\mathcal{C}}\right)\right) \\ &+ \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\int_{C(\mathbb{C})} f_D\omega_x \\ &= h_{\overline{\mathcal{S}}}(D) + \mathrm{O}(1). \end{aligned}$$

2.5.3 The second summand is simpler. One has

$$\begin{aligned} \hat{c}_1\left(\pi^*\overline{\mathcal{R}}|_{y\times_{\mathrm{Spec}\mathcal{O}_K}\mathcal{C}}\right)\cdot(x,g_x) &= \pi^*\ \hat{c}_1\ (\mathcal{R}|_y)\cdot(x,g_x) \\ &= \hat{c}_1\ (\mathcal{R}|_y) + a\Big(g_x\omega_R(y)\Big), \end{aligned}$$

when one identifies $\mathcal{I}\times_{\mathrm{Spec}\ \mathcal{O}_K}\overline{\{x\}}$ with $\mathcal{I}$. The integral $\int_{C(\mathbb{C})} g_x\omega_R(.)$ depends smoothly on the parameter, in particular it is bounded. On the other hand, the push-forward of $\hat{c}_1\ (\mathcal{R}|_y)$ to $\mathrm{Spec}\ \mathcal{O}_K$ is by definition $= \left(\hat{c}_1\ (\overline{\mathcal{R}})\ \middle|\ y\right)$. Now Lemma 2.1.4 gives

$$\widehat{\deg}\left(\hat{c}_1\ (\overline{\mathcal{R}})\ \middle|\ y\right) = h_R(D),$$

where D is the divisor corresponding to the restriction of y to $C^{(g)}$.

2.5.4 We obtain

Proposition. *Assume $\mathcal{L} = \mathcal{O}(\overline{D})$, where $\overline{D}$ is the closure of some divisor on C. Then Theorem 1.5 is true.*

Proof. By the computations above and Corollary 2.1.6 this is now proven for line bundles coming by restriction from $\mathcal{F}$. This way one can realize the line bundles $\mathcal{O}(D)$ on the generic fiber C for arbitrary divisors D (defined over K) of degree g over C. Consider the degrees

$$\deg \mathcal{F}|_{y \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathcal{C}_{\mathfrak{p},i}}$$

for $\mathcal{O}_K$-valued points y of $\mathcal{I}$, where $\mathcal{C}_{\mathfrak{p},i}$ denote the irreducible components of the special fiber $\mathcal{C}_{\mathfrak{p}}$. They are even defined for $\overline{\mathcal{O}/\mathfrak{p}}$-valued points, where the bar denotes algebraic closure here, and are locally constant over the special fiber $\mathcal{I}_{\mathfrak{p}}$. In particular they are bounded since $\mathcal{I}$ is of finite type by construction. The Proposition follows from Lemma 2.6.3 below. $\square$

2.6 Some technical Lemmata

2.6.1 Lemma. (Fibers do not change the height.)
If $\mathcal{L}$ is a line bundle on $\mathcal{C}/\mathcal{O}_K$ with $\chi(\mathcal{L}) \neq 0$, then

$$h_{x,\omega}(\mathcal{L} \otimes \mathcal{O}(\mathfrak{p})) = h_{x,\omega}(\mathcal{L})$$

for every prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$.

Proof. One has $\mathcal{O}(\mathfrak{p}) = \pi^*(\mathfrak{p}^{-1})$, hence by projection formula

$$\det R\pi_*(\mathcal{L} \otimes \mathcal{O}(\mathfrak{p})) \cong \det R\pi_*\mathcal{L} \otimes \mathcal{O}(\mathfrak{p})^{-\chi(\mathcal{L})}.$$

Let $\|.\|$ be one of the distinguished metrics on the line bundle $\mathcal{L}_{\mathbb{C}}$ on $\coprod_{\sigma: K \hookrightarrow \mathbb{C}} C(\mathbb{C})$. We put $\|.\|_{\mathfrak{p}} = C \cdot \|.\|$ for a distinguished hermitian metric on $(\mathcal{L} \otimes \mathcal{O}(\mathfrak{p}))_{\mathbb{C}} = \mathcal{L}_{\mathbb{C}}$. It follows $h_{Q,\det R\pi_*(\mathcal{L}\otimes\mathcal{O}(\mathfrak{p}))} = C^{\chi(\mathcal{L})} \cdot h_{Q,\det R\pi_*\mathcal{L}}$ and

$$\begin{aligned}\widehat{\deg}\left(\det R\pi_*(\mathcal{L} \otimes \mathcal{O}(\mathfrak{p})), h_{Q,\det R\pi_*(\mathcal{L}\otimes\mathcal{O}(\mathfrak{p}))}\right) &= \widehat{\deg}\left(\det R\pi_*\mathcal{L}, h_{Q,\det R\pi_*\mathcal{L}}\right) \\ &+ \chi(\mathcal{L})\Big[[K:\mathbb{Q}]\log C - \log(\sharp\mathcal{O}/\mathfrak{p})\Big].\end{aligned}$$

Thus a distinguished hermitian metric on $(\mathcal{L} \otimes \mathcal{O}(\mathfrak{p}))_{\mathbb{C}}$ can be given by $\|.\|_{\mathfrak{p}} = (\sharp\mathcal{O}/\mathfrak{p})^{\frac{1}{[K:\mathbb{Q}]}} \cdot \|.\|$ and it follows

$$\widehat{c}_1(\mathcal{L} \otimes \mathcal{O}(\mathfrak{p}), \|.\|_{\mathfrak{p}}) = \widehat{c}_1(\mathcal{L}, \|.\|) + \pi^*\left(\mathfrak{p}; -\tfrac{2}{[K:\mathbb{Q}]}\log(\sharp\mathcal{O}/\mathfrak{p}), \ldots, -\tfrac{2}{[K:\mathbb{Q}]}\log(\sharp\mathcal{O}/\mathfrak{p})\right).$$

But the arithmetic cycle

$$\left(\mathfrak{p}; -\frac{2}{[K:\mathbb{Q}]}\log(\sharp\mathcal{O}/\mathfrak{p}), \ldots, -\frac{2}{[K:\mathbb{Q}]}\log(\sharp\mathcal{O}/\mathfrak{p})\right) \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathcal{O}_K)$$

vanishes after multiplication with the class number $\sharp\mathrm{Pic}\,(\mathrm{Spec}\,\mathcal{O}_K)$, hence it is torsion and therefore numerically trivial. $\square$

2.6.2 Lemma. *Let F be some vertical divisor on $\mathcal{C}/\mathcal{O}_K$. Then, for line bundles $\mathcal{L}/\mathcal{C}$, fiber-by-fiber of degree g, and of bounded degrees on the irreducible components of the special fibers*

$$h_{x,\omega}(\mathcal{L}(F)) = h_{x,\omega}(\mathcal{L}) + \mathrm{O}(1).$$

Proof. By Lemma 2.6.1 we may assume that $E := -F$ is effective. Using induction we are reduced to the case E is an irreducible curve. We have a short exact sequence

$$0 \longrightarrow \mathcal{L}(F) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_E \longrightarrow 0$$

inducing the isomorphism

$$\det R\pi_*\mathcal{L}(F) \cong \det R\pi_*\mathcal{L} \otimes (\det R\pi_*\mathcal{L}_E)^\vee.$$

But $\det R\pi_*\mathcal{L}_E$ depends only on the Euler characteristic of $\mathcal{L}_E$ and for the degree of that bundle there are only finitely many possibilities. So up to numerical equivalence there are only finitely possibilities for

$$\hat{c}_1\left(\mathcal{L}(F), \|.\|_{\mathcal{L}(F)}\right) - \hat{c}_1\left(\mathcal{L}, \|.\|_{\mathcal{L}}\right),$$

where $\|.\|_{\mathcal{L}}$ and $\|.\|_{\mathcal{L}(F)}$ denote distinguished hermitian metrics. □

2.6.3 Lemma. *Consider line bundles $\mathcal{L}$, generically of degree g on $\mathcal{C}$, equipped with a section $s \in \Gamma(C, \mathcal{L}_C)$ over the generic fiber, and assume the degrees $\deg \mathcal{L}|_{\mathcal{C}_{\mathfrak{p},i}}$ of the restrictions of $\mathcal{L}$ to the irreducible components of the special fibers to be fixed. Then*

$$h_{x,\omega}(\mathcal{L}) = h_{x,\omega}\left(\mathcal{O}\left(\overline{\mathrm{div}(s)}\right)\right) + \mathrm{O}(1).$$

Proof. We have $\mathcal{L} = \mathcal{L}'(E)$, where $\mathcal{L}' = \mathcal{O}\left(\overline{\mathrm{div}(s)}\right)$ is a line bundle induced by a horizontal divisor and E is a vertical divisor. By Lemma 2.6.1 we may assume E to be concentrated in the reducible fibers of $\mathcal{C}$. So, using induction, let E be in one such fiber $\mathcal{C}_{\mathfrak{p}}$. Then for the degrees $\deg \mathcal{O}(E)|_{\mathcal{C}_{\mathfrak{p},i}}$ there are only finitely many possibilities. But by [Fa, Theorem 4.a)] the intersection form on $\mathcal{C}_{\mathfrak{p}}$ is negative semi-definite where only multiples of the fiber have square 0. Hence, for E there are only finitely many possibilities up to addition of the whole fiber, which does not change the height. Lemma 2.6.2 gives the claim. □

2.6.4 Lemma. *Let X be a compact Riemann surface and $g \in \mathbb{N}$ be a natural number. Denote by Δ the diagonal in $X \times X$, by δ_M the δ-distribution defined by M and by $\pi_i : X^g \times X \longrightarrow X$ (resp. $\pi_{i,g+1} : X^g \times X \longrightarrow X \times X$) the canonical projection on the i-th component (resp. to the product of the i-th and $(g+1)$-th component.) Further let*

$$f : (X^g \times X) \setminus \bigcup_{i=1}^{g} \pi_{i,g+1}^{-1}(\Delta) \longrightarrow \mathbb{C}$$

be a smooth function such that the restriction of

$$-d_X d_X^c f + \delta_\Delta \circ \pi_{1,g+1} + \ldots + \delta_\Delta \circ \pi_{g,g+1} = \rho,$$

to $\{(x_1, \ldots, x_g)\} \times X$ *is a smooth* $(1,1)$*-form smoothly varying with* $(x_1, \ldots, x_g)$. *Let* ω *be a smooth* $(1,1)$*-form on* X. *Then*

$$\int_X f(x_1, \ldots, x_g, \cdot)\omega$$

depends smoothly on $(x_1, \ldots, x_g) \in X^g$.

Proof. Without restriction we may assume $\int_X \omega = 1$. Then, for any $x \in X$ there exists a function $h \in C^\infty(X \backslash \{x\})$, having a logarithmic singularity in x, such that $\omega = -dd^c h + \delta_x$. It follows

$$\begin{aligned}
\int_X f(x_1, \ldots, x_g, \cdot)\omega &= -\int_X f(x_1, \ldots, x_g, \cdot) dd^c h + f(x_1, \ldots, x_g, x) \\
&= -\int_X \Big(d_X d_X^c f(x_1, \ldots, x_g, \cdot)\Big) h + f(x_1, \ldots, x_g, x) \\
&= \int_X \rho(x_1, \ldots, x_g, \cdot) h \\
&- h(x_1) - \ldots - h(x_g) + f(x_1, \ldots, x_g, x) \\
&= \int_X \rho(x_1, \ldots, x_g, \cdot) h \\
&- \Big[h(x_1) - G(x, x_1)\Big] - \ldots - \Big[h(x_g) - G(x, x_g)\Big] \\
&- \Big[G(x, x_1) + \ldots + G(x, x_g) - f(x_1, \ldots, x_g, x)\Big],
\end{aligned}$$

where G is the Green's function of X. Because h has only a logarithmic singularity it is allowed to differentiate under the integral sign. So the integral is smooth. The other summands are solutions of equations of the form $dd^c F = \sigma$ with a smooth $(1,1)$-form σ on X satisfying $\int_X \sigma = 0$ (in $x_1, \ldots, x_g$, respectively x). Since dd^c is elliptic, these solutions exist as smooth functions and are unique up to constants. In particular, also the last summand must depend smoothly on $(x_1, \ldots, x_g)$, even when some of the x_i equal x. Note that the symmetry of the Green's function is used here essentially. □

3 Asymptotic Behaviour

3.1 This is a direct computation using the arithmetic Riemann-Roch Theorem.

Proposition. *Let* $\pi : \mathcal{C} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ *be an arithmetic surface and assume* $\mathcal{C}_{\mathbb{C}}$ *to be equipped with a Kähler form* ω *invariant under complex conjugation* F_∞. *Let* $\overline{\mathcal{E}}$ *be a line bundle on* $\mathcal{C}$ *equipped with a distinguished hermitian metric and* $\mathcal{M} \in \widehat{\operatorname{Pic}}\,(\mathcal{C})$ *be a hermitian line bundle. Then*

$$\widehat{\deg}\,\Big(\hat{c}_1\Big(\det R\pi_*(\overline{\mathcal{E}} \otimes \overline{\mathcal{M}}^{\otimes n}), \|.\|_Q\Big)\Big) = \frac{1}{2}An^2 + Bn,$$

where A and B denote the arithmetic intersection numbers

$$
\begin{aligned}
A &:= \widehat{\deg}\,\pi_*\Big(\,\hat{c}_1\,(\overline{\mathcal{M}})^2\Big) \quad \text{and}\\
B &:= \widehat{\deg}\,\pi_*\Big(\,\hat{c}_1\,(\overline{\mathcal{M}})\Big(\,\hat{c}_1\,(\overline{\mathcal{E}})+\frac{1}{2}\,\hat{c}_1\,(\overline{T_f})\Big)\Big)
\end{aligned}
$$

and $\hat{c}_1\,(\overline{T_F})$ is the arithmetic Chern class of the relative tangent bundle equipped with the hermitian metric associated with ω.

Proof. The arithmetic Riemann-Roch Theorem [GS92, Fa92] states

$$
\begin{aligned}
&\widehat{\deg}\,\left(\hat{c}_1\left(\det R\pi_*(\overline{\mathcal{E}}\otimes\overline{\mathcal{M}}^{\otimes n}),\|.\|_Q\right)\right)\\
=\;&\widehat{\deg}\,\left(\pi_*\left(\widehat{\mathrm{ch}}\,(\overline{\mathcal{E}})\;\widehat{\mathrm{ch}}\,(\overline{\mathcal{M}}^{\otimes n})\;\widehat{\mathrm{Td}}\,(\overline{T_f})\right)\right)^{(1)}\\
+\;&\frac{1}{2}\left[\int_{X(\mathbb{C})}\Big(\,\mathrm{ch}(\mathcal{E}_{\mathbb{C}})\;\mathrm{ch}(\mathcal{M}_{\mathbb{C}}^{\otimes n})\;\mathrm{Td}(T_{f,\mathbb{C}})\;\mathrm{R}(T_{f,\mathbb{C}})\Big)^{(1,1)}\right].
\end{aligned}
$$

Here $\widehat{\mathrm{ch}}$ and $\widehat{\mathrm{Td}}$ denote the arithmetic Chern character, respectively Todd genus, while ch, Td and R are the Chern character form, the Todd form and the R-genus in the sense of Chern-Weil theory, respectively. The superscripts $^{(1)}$ and $^{(1,1)}$ indicate restriction to the codimension one Chow group, respectively to forms of type $(1,1)$. It is obvious that the right hand side is a quadratic polynomial in n, whose absolute term is 0, since $\overline{\mathcal{E}}$ carries a distinguished hermitian metric. As we work with line bundles on an arithmetic surface there are the identities

$$\widehat{\mathrm{ch}}\,(\overline{\mathcal{M}}^{\otimes n}) = 1+\,\hat{c}_1\,(\overline{\mathcal{M}})n+\frac{1}{2}\,\hat{c}_1\,(\overline{\mathcal{M}})^2n^2$$

and

$$\mathrm{ch}(\mathcal{M}_{\mathbb{C}}^{\otimes n}) = 1+c_1(\mathcal{M}_{\mathbb{C}})n.$$

Thus for the coefficient of n^2 we get

$$\frac{1}{2}\,\widehat{\deg}\,\pi_*\Big(\,\hat{c}_1\,(\overline{\mathcal{M}})^2\Big)$$

and the coefficient of n is

$$\widehat{\deg}\,\pi_*\Big(\,\hat{c}_1\,(\overline{\mathcal{M}})\Big(\,\hat{c}_1\,(\overline{\mathcal{E}})+\frac{1}{2}\,\hat{c}_1\,(\overline{T_f})\Big)\Big).$$

The term with the integral does not play any role in both calculations, since $\mathrm{R}(T_{f,\mathbb{C}})$ starts with a $(1,1)$-form. □

3.2 Corollary. *Theorem 1.7 is true.*

Proof. The height of $\mathcal{E}\otimes\mathcal{M}^{\otimes n}$ is given by the intersection number of its first arithmetic Chern class with $\hat{c}_1\,(\overline{\mathcal{L}})$, where the hermitian metric on $\mathcal{E}\otimes\mathcal{M}^{\otimes n}$ has to be corrected appropriately. The Proposition above gives

$$\hat{c}_1\left(\det R\pi_*(\overline{\mathcal{E}}\otimes\overline{\mathcal{M}}^{\otimes n}),\|.\|_Q\right)-\hat{c}_1\left(\det R\pi_*(\mathcal{E}\otimes\mathcal{M}^{\otimes n}),\|.\|_{Q,\mathrm{dist}}\right) = a(An^2+2Bn),$$

where $\|.\|_{Q,\mathrm{dist}}$ denotes the Quillen metric induced by some distinguished hermitian metric on $\mathcal{E}\otimes\mathcal{M}^{\otimes n}$, i.e. one making the arithmetic degree vanish. Lemma 1.3 implies that there exists a distinguished hermitian metric satisfying

$$\hat{c}_1\left(\overline{\mathcal{E}}\otimes\overline{\mathcal{M}}^{\otimes n}\right)-\hat{c}_1\left(\mathcal{E}\otimes\mathcal{M}^{\otimes n},\|.\|_{\mathrm{dist}}\right)=a\left(\frac{An^2+2Bn}{\chi(\mathcal{E}_K\otimes\mathcal{M}_K^{\otimes n})}\right).$$

Therefore we obtain for the height of $\mathcal{E}\otimes\mathcal{M}^{\otimes n}$ the following formula.

$$h_{\overline{\mathcal{L}}}(\mathcal{E}\otimes\mathcal{M}^{\otimes n})=h_{\overline{\mathcal{L}}}(\mathcal{E})+\widehat{\deg}\left(\hat{c}_1(\overline{\mathcal{L}})\,\hat{c}_1(\overline{\mathcal{M}})\right)n-\frac{An^2+2Bn}{2\chi(\mathcal{E}_K\otimes\mathcal{M}_K^{\otimes n})}\deg\mathcal{L}_K$$

Using Riemann-Roch for curves it follows that

$$\begin{aligned}
\chi(\mathcal{E}_K\otimes\mathcal{M}_K^{\otimes n})h_{\overline{\mathcal{L}}}(\mathcal{E}\otimes\mathcal{M}^{\otimes n}) &= \chi(\mathcal{E}_K\otimes\mathcal{M}_K^{\otimes n})\Big(h_{\overline{\mathcal{L}}}(\mathcal{E})+nD\Big)\\
&- \Big(\frac{1}{2}An^2+Bn\Big)\deg\mathcal{L}_K\\
&= \Big(\chi(\mathcal{E}_K)+\deg(\mathcal{M}_K)\ n\Big)\Big(h_{\overline{\mathcal{L}}}(\mathcal{E})+nD\Big)\\
&- \Big(\frac{1}{2}An^2+Bn\Big)\deg\ \mathcal{L}_K\\
&= \chi(\mathcal{E}_K)h_{\overline{\mathcal{L}}}(\mathcal{E})\\
&+ n\ [(\deg\ (\mathcal{M}_K)h_{\overline{\mathcal{L}}}(\mathcal{E})+\chi(\mathcal{E}_K)D-\deg(\mathcal{L}_K)B]\\
&+ n^2[\deg(\mathcal{M}_K)D-\deg(\mathcal{L}_K)\frac{A}{2}].
\end{aligned}$$

□

Acknowledgement. When doing this work, the author had fruitful discussions with U. Bunke, who explained him much of the analytic part of Arakelow theory. The author also wishes to thank M. Kneser for his friendly encouragement.

References

[Ar] Аракелов, С. Ю.: Теория пересечений дивизоров на арифметической поверхности, *Изв. Акад. наук СССР*, сер. мат., т.38(1974)1179-1192 (English translation: Arakelov, S. Yu.: Intersection theory of divisors on an arithmetic surface, *Math. USSR Izvestiya* 8(1974)1167-1180)

[SGA6] Berthelot, P., Grothendieck, A., Illusie, L. et al.: *Théorie des intersections et Théorème de Riemann-Roch*, Lecture Notes Math. 225, Springer, Berlin 1971

[BGS] Bismut, J.-M., Gillet, H., Soulé, C.: Analytic torsion and holomorphic determinant bundles I, II, III, *Comm. Math. Phys.* 115(1988)49-78, 79-126, 301-351

[BoGS] Bost, J.-B., Gillet, H., Soulé, C.: Heights of projective varieties and positive Green forms, *J. Amer. Math. Soc.* 7(1994)903-1027

[CS] Cornell, G., Silverman, J.-H. (eds.): *Arithmetic geometry*, Springer, New York 1986

[Fa84] Faltings, G.: Calculus on arithmetic surfaces, *Annals of Math.* 119(1984) 387-424

[Fa92] Faltings, G.: *Lectures on the arithmetic Riemann-Roch theorem*, Princeton University Press, Princeton 1992

[Fu] Fulton, W.: Rational equivalence on singular varieties, *Publ. Math. IHES* 45(1975)147-167

[GS87] Gillet, H., Soulé, C.: Intersection theory using Adams operations, *Inventiones Math.* 90(1987)243-277

[GS90] Gillet, H., Soulé, C.: Arithmetic intersection theory, *Publ. Math. IHES* 72(1990)93-174

[GS92] Gillet, H., Soulé, C.: An arithmetic Riemann-Roch theorem. *Inventiones Math.* 110 (1992)473-543

[Hr] Hriljac, P.: Heights and Arakelov's intersection theory, *Amer. J. Math.* 107(1985)23-38

[J1] Jahnel, J.: Heights for line bundles on arithmetic surfaces, *mathematica Gottingensis* 16/1995

[J2] Jahnel, J.: Heights for line bundles on arithmetic varieties, *mathematica Gottingensis* 34/1995

[KM] Knudsen, F., Mumford, D.: The projectivity of the moduli space of stable curves I, *Math. Scand.* 39(1976)19-55

[Kü] Künnemann, K.: Higher Picard varieties and the height pairing, *Preprint Münster*, Münster 1995

[MB] Moret-Bailly, L.: Métriques permises, in: Szpiro: Séminaire sur les pinceaux arithmétiques: La conjecture de Mordell, *Asterisque* 127(1985)

[Qu] Квиллен, Д.: Детерминанты операторов Коши-Римана на римановых поверхностях, *Функцион. анализ и его прил.*, т. 19(1985)37-41
(English translation: Quillen, D.: Determinants of Cauchy-Riemann operators over a Riemann surface, *Funct. Anal. Appl.* 19(1985)31-34)

[SABK] Soulé, C., Abramovich, D., Burnol, J.-F., Kramer, J.: *Lectures on Arakelov geometry*, Cambridge University Press, Cambridge 1992