On the Brauer–Manin obstruction for cubic surfaces*

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Abstract

We describe a method to compute the Brauer-Manin obstruction for smooth cubic surfaces over $\mathbb{Q}$ such that $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ is of order two or four. This covers the vast majority of the cases when this group is non-zero. Our approach is to associate a Brauer class with every Galois invariant double-six. We show that all order two Brauer classes may be obtained in this way. We also recover Sir Peter Swinnerton-Dyer's result that $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ may take only five values.

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1 Introduction

1.1. — For cubic surfaces, weak approximation and even the Hasse principle are not always fulfilled. The first example of a cubic surface violating the Hasse principle was constructed by Sir Peter Swinnerton-Dyer [SD1]. A series of examples generalizing that of Swinnerton-Dyer is due to L. J. Mordell [Mo]. An example of a different sort was given by J. W. S. Cassels and M. J. T. Guy [CG].

A way to explain these examples in a unified manner was provided by Yu. I. Manin in his book [Ma]. This is what today is called the Brauer-Manin obstruction. Manin's idea is that a non-trivial Brauer class may be responsible for the failure of weak approximation. We will recall the Brauer-Manin obstruction in some detail in section 2.

An important point is that only the factor group $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ of the Grothendieck-Brauer group of the cubic surface S is relevant. That is isomorphic to the Galois cohomology group $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$. A theorem of Sir Peter Swinnerton-Dyer [SD2]

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states that, for this group, there are only five possibilities. It may be isomorphic to 0, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. We observed that, today, Swinnerton-Dyer's theorem from 1993 may easily be established by a script in `GAP`.

The effect of the Brauer-Manin obstruction has been studied by several authors. For example, for diagonal cubic surfaces, the computations were carried out by J.-L. Colliot-Thélène and his coworkers in [CKS]. In this case, $\operatorname{Br}(S)/\operatorname{Br}(\mathbb{Q}) = \mathbb{Z}/3\mathbb{Z}$. The same applies to the examples of Mordell or Cassels-Guy which were explained by the Brauer-Manin obstruction in [Ma].

1.2. — It seems that, for the cases that $H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, no computations have been done up to now. The goal of the present paper is to fill this gap.

Our starting point is a somewhat surprising observation. It turns out that $H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \operatorname{Pic}(S_{\overline{\mathbb{Q}}}))$ is of order two or four only in cases when, on S, there is a Galois invariant double-six. This reduces the possibilities for the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the 27 lines. In general, the automorphism group of the configuration of the 27 lines is the Weyl group $W(E_6)$ [Ma, Theorem 23.9]. Among the 350 conjugacy classes of subgroups in $W(E_6)$, exactly 158 stabilize a double-six.

In a previous paper [EJ2], we described a method to construct smooth cubic surfaces with a Galois invariant double-six. Our method is based on the hexahedral form of L. Cremona and Th. Reye and an explicit Galois descent. It is able to produce examples for each of the 158 conjugacy classes.

Among them, however, there are 56 which even stabilize a sixer. Those may be constructed by blowing up six points in $\mathbf{P}^2$ and, thus, certainly fulfill weak approximation. There are 26 further conjugacy classes which lead to $H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) = 0$.

1.3. — In this article, we compute the Brauer-Manin obstruction for each of the 76 cases such that $H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. We start with two "model cases" for the Brauer groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. These are the maximal subgroup $U_1 \subset W(E_6)$ stabilizing a double-six and the maximal subgroup $U_3 \subset W(E_6)$ stabilizing a triple of azygetic double sixes [Ko].

In both cases, we compute the Brauer group explicitly. This means, we produce representatives which we describe as Azumaya algebras. We then show that every subgroup $H \subset W(E_6)$ which leads to a Brauer group of order four is actually contained in U_3. Recall that every subgroup $H \subset W(E_6)$ leading to a Brauer group of order two is contained in U_1. Finally, we prove the main result that the restriction map is bijective in each of the cases.

1.4. — The article is concluded by examples showing the effect of the Brauer-Manin obstruction. It turns out that, unlike the situation described in [CKS] where a Brauer class of order three typically excludes two thirds of the adelic points, various fractions are possible.

2 The Brauer-Manin obstruction – Generalities

2.1. — For cubic surfaces, all known counterexamples to the Hasse principle or weak approximation are explained by the following observation.

2.2. Definition. — Let X be a projective variety over $\mathbb{Q}$ and $\mathrm{Br}(X)$ its Grothendieck-Brauer group. Then, we will call

$$\mathrm{ev}_\nu\colon\ \mathrm{Br}(X)\times X(\mathbb{Q}_\nu)\longrightarrow \mathbb{Q}/\mathbb{Z}\,,\quad (\alpha,\xi)\mapsto \mathrm{inv}_\nu(\alpha|_\xi)$$

the *local evaluation map*. Here, $\mathrm{inv}_\nu\colon\ \mathrm{Br}(\mathbb{Q}_\nu)\to\mathbb{Q}/\mathbb{Z}$ (and $\mathrm{inv}_\infty\colon\ \mathrm{Br}(\mathbb{R})\to\frac{1}{2}\mathbb{Z}/\mathbb{Z}$) denote the canonical isomorphisms.

2.3. Observation (Manin)**. —** *Let $\pi\colon\ X\to\mathrm{Spec}(\mathbb{Q})$ be a projective variety over $\mathbb{Q}$. Choose an element $\alpha\in\mathrm{Br}(X)$. Then, every $\mathbb{Q}$-rational point $x\in X(\mathbb{Q})$ gives rise to an adelic point $(x_\nu)_\nu\in X(\mathbf{A}_\mathbb{Q})$ satisfying the condition*

$$\sum_{\nu\in\mathrm{Val}(\mathbb{Q})}\mathrm{ev}_\nu(\alpha,x_\nu)=0\,.$$

2.4. Remarks. — i) It is obvious that altering $\alpha\in\mathrm{Br}(X)$ by some Brauer class $\pi^*\rho$ for $\rho\in\mathrm{Br}(\mathbb{Q})$ does not change the obstruction defined by α. Consequently, it is only the factor group $\mathrm{Br}(X)/\pi^*\mathrm{Br}(\mathbb{Q})$ which is relevant for the Brauer-Manin obstruction.

ii) The local evaluation map $\mathrm{ev}_\nu\colon\ \mathrm{Br}(X)\times X(\mathbb{Q}_\nu)\to\mathbb{Q}/\mathbb{Z}$ is continuous in the second variable.

iii) Further, for every projective variety X over $\mathbb{Q}$ and every $\alpha\in\mathrm{Br}(X)$, there exists a finite set $S\subset\mathrm{Val}(\mathbb{Q})$ such that $\mathrm{ev}(\alpha,\xi)=0$ for every $\nu\notin S$ and $\xi\in X(\mathbb{Q}_\nu)$.

These facts imply that the Brauer-Manin obstruction, if present, is an obstruction to the principle of weak approximation.

2.5. Lemma. — *Let $\pi\colon\ S\to\mathrm{Spec}\,\mathbb{Q}$ be a non-singular cubic surface. Then, there is a canonical isomorphism*

$$\delta\colon\ H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathrm{Pic}(S_{\overline{\mathbb{Q}}}))\longrightarrow\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$$

making the diagram

$$\begin{tikzcd}
H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \arrow[r, "\delta"] \arrow[d, "d"'] & \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) \arrow[dd, "\mathrm{res}"] \\
H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\overline{\mathbb{Q}}(S)^*/\overline{\mathbb{Q}}^*) \arrow[d, equal] & \\
H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\overline{\mathbb{Q}}(S)^*)/\pi^*\mathrm{Br}(\mathbb{Q}) \arrow[r, "\mathrm{inf}"] & \mathrm{Br}(\mathbb{Q}(S))/\pi^*\mathrm{Br}(\mathbb{Q})
\end{tikzcd}$$

commute. Here, d is induced by the short exact sequence

$$0 \to \overline{\mathbb{Q}}(S)^*/\overline{\mathbb{Q}}^* \to \mathrm{Div}(S_{\overline{\mathbb{Q}}}) \to \mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \to 0$$

and the other morphisms are the canonical ones.

Proof. The equality at the lower left corner comes from the fact [Ta, section 11.4] that $H^3(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \overline{\mathbb{Q}}^*) = 0$. The main assertion is [Ma, Lemma 43.1.1]. □

2.6. Remark. — The group $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is always finite. Hence, by Remark 2.4.iii), we know that only finitely many primes are relevant for the Brauer-Manin obstruction.

3 One double-six

3.1. — Recall that, on a non-singular cubic surface, there are exactly 27 lines. Their configuration was studied intensively by the geometers of the 19th century. They showed that the 27 lines contain 72 *sixers* of lines which are mutually skew. For each sixer, there is a complementary one formed by the lines meeting exactly five lines of the sixer. A pair of complementary sixers is called a *double-six*.

The automorphism group $W(E_6)$ acts transitively on double-sixes. In the blown-up model, one double-six may easily be written down explicitly. For this, we refer to [Ha, Remark V.4.9.1] from which we also adopt the notation for the lines.

3.2. Lemma. — *Let S be a non-singular cubic surface over* $\mathbb{Q}$. *Suppose that, under the operation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *the 27 lines on S decompose into orbits one of which is of size 15. Then, the complementary twelve lines form a double-six.*

Proof. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates via a certain subgroup $G \subseteq W(E_6)$. Our assumption implies that $5 | \#G$. I.e., G contains the 5-Sylow subgroup of $W(E_6)$.

The operation of this is easily described in the blown-up model. The cyclic group $\langle(12345)\rangle \subset S_6$ acts on the indices. The two lines E_0 and G_0 are stationary while the others form five orbits of size five each. These are $E := \{E_1, \ldots, E_5\}$, $G := \{G_1, \ldots, G_5\}$, $F_0 := \{F_{01}, \ldots, F_{05}\}$, $F_1 := \{F_{12}, F_{23}, F_{34}, F_{45}, F_{15}\}$, and, finally, $F_2 := \{F_{13}, F_{24}, F_{35}, F_{14}, F_{25}\}$.

The intersection matrix of the five latter orbits turns out to be

$$\begin{pmatrix} -5 & 20 & 5 & 10 & 10 \\ 20 & -5 & 5 & 10 & 10 \\ 5 & 5 & -5 & 15 & 15 \\ 10 & 10 & 15 & 5 & 5 \\ 10 & 10 & 15 & 5 & 5 \end{pmatrix}.$$

We assert that a size 15 orbit must be formed by F_0, F_1, and F_2.

Indeed, three orbits of size five may be put together to form an orbit only if, for the corresponding divisors, $D(D+D'+D'') = D'(D+D'+D'') = D''(D+D'+D'')$. This excludes all combinations, except for $E \cup G \cup F_1$, ($E \cup G \cup F_2$, and $F_0 \cup F_1 \cup F_2$). The set $E \cup G \cup F_1$

contains, however, only two fivers of skew lines, namely E and G. As the lines in F_1 are missing, this is a contradiction. $\square$

3.3. Remark. — U_1, the largest subgroup of $W(E_6)$ stabilizing a double-six is isomorphic to $S_6 \times \mathbb{Z}/2\mathbb{Z}$ of order 1440.

3.4. Notation. — Let S be a non-singular cubic surface. Assume that twelve of the 27 lines on S form a double-six which is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. Choose such a double-six. Then, there are two kinds of tritangent planes. We have 15 tritangent planes which meet S only within the 15 complementary lines. The other 30 tritangent planes contain one of the 15 lines and two from the double-six.

We write F_{30} for a product over the linear forms defining the 30 tritangent planes and F_{15} for a product over the linear forms defining the 15 others. Note that $F_{30}/F_{15}^2 \in \mathbb{Q}(S)$.

3.5. Theorem. — *Let* $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ *be a non-singular cubic surface such that the 27 lines have orbit structure* $[12,15]$ *under the operation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

i) *Then,* $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$.

ii) *For* $0 \neq c \in \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$, *a representative* $\underline{c}$ *of* $\mathrm{res}(c) \in \mathrm{Br}(\mathbb{Q}(S))/\pi^*\mathrm{Br}(\mathbb{Q})$ *is given as follows.*

Consider the quadratic number field $\mathbb{Q}(\sqrt{D})$ *splitting the double-six into two sixers. Then, apply to the class*

$$(F_{30}/F_{15}^2) \in \widehat{H}^0(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}), \mathbb{Q}(\sqrt{D})(S)^*) = \mathbb{Q}(S)^*/N\mathbb{Q}(\sqrt{D})(S)^*$$

the periodicity isomorphism to H^2 *and the inflation map.*

Proof. *First step.* Inflation.

We have the isomorphism $\delta\colon H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \to \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$ and will work with the group on the left.

An element of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ may either flip the two sixers forming the twelve lines or not. Therefore, there is an index two subgroup stabilizing the sixers. This group corresponds to the quadratic number field $\mathbb{Q}(\sqrt{D})$. By Fact 3.6 below, we know $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D})), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = 0$. The inflation-restriction-sequence yields that

$$\inf\colon H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}) \longrightarrow H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$$

is an isomorphism.

Second step. Divisors.

The orbit structure of the 27 lines under the operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$ is $[6,6,15]$. Indeed, when going over to an index two subgroup an orbit of odd size must not split. Denote by E, G, and F the divisors formed by summing over the first, second, and third orbit, respectively. E, G, and F clearly define elements in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$. Write P for the subgroup generated by these three divisors.

P is of finite index in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$. Indeed, every element of $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ is an integral linear combination of the divisors given by the 27 lines. Therefore, every element in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$ is a $\mathbb{Q}$-linear combination of E, G, and F and the denominators are at most six or 15.

We claim that the index of P is a divisor of 15. In fact, we have the relation $5E+5G-4F \sim 0$. Further, the discriminant of the lattice spanned by E and F is

$$\begin{vmatrix} -6 & 30 \\ 30 & 75 \end{vmatrix} = -1350 = (-6)\cdot 15^2 .$$

Consequently, P is of odd index in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$. This implies that the natural homomorphism

$$H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) \longrightarrow H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))})$$

is a bijection.

Third step. The fundamental class.
As $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$ is a cyclic group of order two, we have

$$H^2(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} .$$

Write u for the non-zero element. Then, the periodicity isomorphism is given by

$$\cup u\colon \widehat{H}^{-1}(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) \longrightarrow H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) .$$

Observe that, for a cyclic group of order two, this isomorphism is canonical as there is no ambiguity in the choice of u.

Fourth step. Computing $\widehat{H}^{-1}$.
We have $P = S/S_0$ for $S := \mathbb{Z}E \oplus \mathbb{Z}G \oplus \mathbb{Z}F$ and S_0 the group of the principal divisors contained in S. The relation $\widehat{H}^{-1}(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),S) = 0$ follows immediately from the definition. Hence, the short exact sequence

$$0 \to S_0 \to S \to P \to 0$$

yields

$$\begin{aligned}
&\widehat{H}^{-1}(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) = \\
&\qquad = \ker(\widehat{H}^0(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),S_0) \to \widehat{H}^0(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),S)) \\
&\qquad = \ker(S_0^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})}/NS_0 \to S^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})}/NS) \\
&\qquad = (S_0^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})} \cap NS)/NS_0 .
\end{aligned}$$

Here, the norm map acts by the rule $N\colon aE+bG+cF \mapsto (a+b)E+(a+b)G+2cF$. Hence, $NS = \langle E+G, 2F\rangle$. Principal divisors are characterized by the property that all intersection numbers are zero. A direct calculation shows

$$S_0^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})} \cap NS = \langle 5E+5G-4F\rangle .$$

The generator is the norm of any divisor of the form $aE + (5-a)G - 2F$. None of these is principal. Indeed, the intersection number with E is equal to $-6a+30(5-a)-60 = -36a+90$ and this terms does not vanish for $a \in \mathbb{Z}$. Assertion i) is proven.

Fifth step. The representative.

We actually constructed a non-zero element $c' \in H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$. The calculations given above show that

$$d(c') = (F_{30}/F_{15}^2) \cup u\,.$$

Indeed, it is easy to see that $\mathrm{div}(F_{30}/F_{15}^2) = 5E+5G-4F$. The assertion now follows from the commutative diagram given in Lemma 2.5. $\square$

3.6. Fact. — *Let S be a non-singular cubic surface over a field K obtained by blowing up $\mathbf{P}^2_K$ in six $\overline{K}$-rational points which form a Galois invariant set.*

Then, $H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{Pic}(S_{\overline{K}})) = 0$.

Proof. This is, of course, a particular case of the very general [Ma, Theorem 29.1]. Let us give an elementary proof here.

According to Shapiro's lemma, we may replace $\mathrm{Gal}(\overline{K}/K)$ by a finite quotient G. We have $\mathrm{Pic}(S_{\overline{K}}) = \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_6$ for H the hyperplane section of $\mathbf{P}^2$ and $E_1, \ldots, E_6$ the exceptional divisors. Therefore, as a G-module,

$$\mathrm{Pic}(S_{\overline{K}}) = \mathbb{Z} \oplus \mathbb{Z}[G/H_1] \oplus \ldots \oplus \mathbb{Z}[G/H_l]$$

for l the number of Galois orbits and certain subgroups $H_1, \ldots, H_l$. Clearly, we have $H^1(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z}) = 0$.

On the other hand, for any subgroup H, the G-module $\mathbb{Z}[G/H]$ is equipped with a non-degenerate pairing. Hence, $\mathbb{Z}[G/H] \cong \mathrm{Hom}(\mathbb{Z}[G/H], \mathbb{Z})$ and

$$\begin{aligned} H^1(G, \mathbb{Z}[G/H]) &\cong H^1(G, \mathrm{Hom}(\mathbb{Z}[G/H], \mathbb{Z})) \\ &\cong \widehat{H}^0(G, \mathrm{Hom}(\mathbb{Z}[G/H], \mathbb{Q}/\mathbb{Z})) \\ &\cong \mathrm{Hom}\big(\widehat{H}^{-1}(G, \mathbb{Z}[G/H]), \mathbb{Q}/\mathbb{Z}\big) \end{aligned}$$

by the duality theorem for cohomology of finite groups [CE, Chap. XII, Corollary 6.5]. Finally, $\widehat{H}^{-1}(G, \mathbb{Z}[G/H])$ vanishes as is seen immediately from the definition. $\square$

4 Triples of azygetic double-sixes

4.1. Definition. — Let $\pi\colon S \to \mathrm{Spec}\,\mathbb{Q}$ be a smooth cubic surface and $\mathscr{D}$ be a Galois invariant double-six. This induces a group homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to U_1$, given by the operation on the 27 lines.

i) Then, the image of the non-zero element under the natural homomorphism

$$\mathbb{Z}/2\mathbb{Z} \cong H^1(U_1, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \longrightarrow H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \stackrel{\delta}{\longrightarrow} \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$$

is called *the Brauer class associated with* the double-six $\mathscr{D}$. We denote it by $\mathrm{cl}(\mathscr{D})$.

ii) This defines a map

$$\mathrm{cl}\colon \Phi_S^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \longrightarrow \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$$

from the set of all Galois invariant double-sixes which we will call the *class map*.

4.2. — Two double-sixes may have either four or six lines in common. In the former case, the two are called syzygetic, in the latter azygetic. A pair of azygetic double-sixes is built as follows.

$$\begin{pmatrix} E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \\ G_0 & G_1 & G_2 & G_3 & G_4 & G_5 \end{pmatrix}, \quad \begin{pmatrix} E_0 & E_1 & E_2 & F_{45} & F_{35} & F_{34} \\ F_{12} & F_{02} & F_{01} & G_3 & G_4 & G_5 \end{pmatrix}.$$

The twelve lines which appear only once form a third double-six

$$\begin{pmatrix} F_{12} & F_{02} & F_{01} & E_3 & E_4 & E_5 \\ G_0 & G_1 & G_2 & F_{45} & F_{35} & F_{34} \end{pmatrix}$$

azygetic to the other two.

4.3. — The largest group U_3 stabilizing a triple of azygetic double-sixes is isomorphic to $(S_3 \times S_3) \ltimes \mathbb{Z}/2\mathbb{Z}$ of order 72. The induced orbit structure is $[6,6,6,9]$. The orbits themselves are, in the notation above, $\{E_0,E_1,E_2,G_3,G_4,G_5\}$, $\{G_0,G_1,G_2,E_3,E_4,E_5\}$, $\{F_{01},F_{02},F_{12},F_{34},F_{35},F_{45}\}$, and $\{F_{03},F_{04},F_{05},F_{13},F_{14},F_{15},F_{23},F_{24},F_{25}\}$.

The quadratic extension $\mathbb{Q}(\sqrt{D})$ splitting one of the three double-sixes into two sixers automatically splits the others, too. The operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$ yields the orbits $\{E_0,E_1,E_2\}$, $\{E_3,E_4,E_5\}$, $\{G_0,G_1,G_2\}$, $\{G_3,G_4,G_5\}$, $\{F_{01},F_{02},F_{12}\}$, $\{F_{34},F_{35},F_{45}\}$, and $\{F_{03},F_{04},F_{05},F_{13},F_{14},F_{15},F_{23},F_{24},F_{25}\}$.

4.4. Theorem. — *Let* $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ *be a non-singular cubic surface. Assume that* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *stabilizes a triple* $\{\mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3\}$ *of azygetic double-sixes and that the 27 lines have orbit structure* $[6,6,6,9]$.

i) *Then,* $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

ii) *The three non-zero elements are* $\mathrm{cl}(\mathscr{D}_1)$, $\mathrm{cl}(\mathscr{D}_2)$, *and* $\mathrm{cl}(\mathscr{D}_3)$.

Proof. *First step.* Inflation.

We will follow the same strategy as in the proof of Theorem 3.5. In particular, we will work with the group $H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))})$.

Second step. Divisors.

The orbit structure of the 27 lines under the operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$ is $[3,3,3,3,3,3,9]$. For the lines and double-sixes, we use the notation introduced in 4.2. Further, denote by

$$E^{(1)}, E^{(2)}, G^{(1)}, G^{(2)}, F^{(1)}, F^{(2)}, F^{(3)}$$

the divisors formed by summing over the orbits. These clearly define elements in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$. We write P for the subgroup generated by these seven divisors.

Every element of $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ is an integral linear combination of the divisors given by the 27 lines. Therefore, every element in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$ is a $\mathbb{Q}$-linear combination of $E^{(1)}$, $E^{(2)}$, $G^{(1)}$, $G^{(2)}$, $F^{(1)}$, $F^{(2)}$, and $F^{(3)}$ and the denominators are divisors of nine. Consequently, P is of odd index in $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))}$. This implies that the natural homomorphism

$$H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) \longrightarrow H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))})$$

is a bijection.

Third step. The fundamental class.

Again, we write u for the non-zero element in $H^2(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Then, the periodicity isomorphism is given by

$$\cup u\colon\ \widehat{H}^{-1}(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) \longrightarrow H^1(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P).$$

Fourth step. Computing $\widehat{H}^{-1}$.

We have $P = S/S_0$ for $S := \mathbb{Z}E^{(1)} \oplus \mathbb{Z}E^{(2)} \oplus \mathbb{Z}G^{(1)} \oplus \mathbb{Z}G^{(2)} \oplus \mathbb{Z}F^{(1)} \oplus \mathbb{Z}F^{(2)} \oplus \mathbb{Z}F^{(3)}$ and S_0 the group of the principal divisors contained in S. To simplify formulas, we will use the notation $D^{(1)} := E^{(1)}+G^{(2)}$, $D^{(2)} := E^{(2)}+G^{(1)}$, and $D^{(3)} := F^{(1)}+F^{(2)}$.

The relation $\widehat{H}^{-1}(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),S) = 0$ follows immediately from the definition. Hence, the short exact sequence $0 \to S_0 \to S \to P \to 0$ yields, as above,

$$\widehat{H}^{-1}(\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}),P) = (S_0^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})} \cap NS)/NS_0\,.$$

Here, the norm map acts by the rule

$$\begin{aligned} N\colon\ & a_1E^{(1)} + a_2E^{(2)} + b_1G^{(1)} + b_2G^{(2)} + c_1F^{(1)} + c_2F^{(2)} + c_3F^{(3)} \\ & \mapsto (a_1+b_2)D^{(1)} + (a_2+b_1)D^{(2)} + (c_1+c_2)D^{(3)} + 2c_3F^{(3)}\,. \end{aligned}$$

Hence, $NS = \langle D^{(1)},D^{(2)},D^{(3)},2F^{(3)}\rangle$. Principal divisors are characterized by the property that all intersection numbers are zero. A direct calculation shows

$$\begin{aligned} & S_0^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})} \cap NS \\ = \ & \{d_1D^{(1)} + d_2D^{(2)} + d_3D^{(3)} + 2eF^{(3)} \mid d_1+d_2+d_3+3e = 0\} \\ = \ & \langle D^{(1)}-D^{(2)}, D^{(1)}-D^{(3)}, D^{(1)}+D^{(2)}+D^{(3)}-2F^{(3)}\rangle. \end{aligned}$$

It is easy to see that $E^{(1)}+G^{(1)}+F^{(1)}-F^{(3)}$ is a principal divisor. Hence,

$$D^{(1)}+D^{(2)}+D^{(3)}-2F^{(3)} \in NS_0\,.$$

Further, NS_0 contains all principal divisors which are divisible by 2. As it turns out that these two sorts of elements generate the whole of NS_0, assertion i) follows.

Fifth step. The representatives.

We actually constructed non-zero elements $c_1,c_2,c_3 \in H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$, represented by $D^{(1)}-D^{(2)}, D^{(1)}-D^{(3)}, D^{(2)}-D^{(3)} \in S_0^{\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})} \cap NS$. The first representative is equivalent to

$$3(D^{(1)}-D^{(2)}) + 2(D^{(1)}+D^{(2)}+D^{(3)}-2F^{(3)}) + 6(D^{(2)}-D^{(3)}) = \mathrm{div}(F_{30}/F_{15}^2)\,.$$

Hence, $d(c_1) = (F_{30}/F_{15}^2) \cup u$. The corresponding Brauer class is $\mathrm{cl}\big(\big(\tfrac{E_0 \cdots E_5}{G_0 \cdots G_5}\big)\big)$. For the two other classes, the situation is analogous. $\square$

4.5. Remark. — In the $[6,6,6,9]$-case, the 45 tritangent planes decompose into five orbits.

- $[E_i, G_j, F_{ij}]$ for $i,j \in \{0,1,2\}$ or $i,j \in \{3,4,5\}$, $i \neq j$. (twelve planes)
- $[E_i, G_j, F_{ij}]$ for $i \in \{0,1,2\}$ and $j \in \{3,4,5\}$. (nine planes)
- $[E_i, G_j, F_{ij}]$ for $i \in \{3,4,5\}$ and $j \in \{0,1,2\}$. (nine planes)
- $[F_{i_0 i_1}, F_{i_2 i_3}, F_{i_4 i_5}]$ for $\{i_0, i_1, i_2, i_3, i_4, i_5\} = \{0,1,2,3,4,5\}$,
  $i_0, i_1 \in \{0,1,2\}$, and $i_2, i_3 \in \{3,4,5\}$. (nine planes)
- $[F_{i_0 i_1}, F_{i_2 i_3}, F_{i_4 i_5}]$ for $\{i_0, i_2, i_4\} = \{0,1,2\}$ and $\{i_1, i_3, i_5\} = \{3,4,5\}$. (six planes)

The three forms of type F_{30} are obtained by multiplying the linear forms defining the orbit of size twelve together with those for two of the orbits of size nine. Actually, the size twelve orbit is irrelevant. Up to a scalar factor, it is the square of a sextic form.

4.6. Remark. — Triples of azygetic double-sixes have been studied by the classical algebraic geometers. See, for example, [Ko, §6]. A result from the 19th century states that there are exactly 120 triples of azygetic double-sixes on a smooth cubic surface. Actually, the automorphism group $W(E_6)$ acts transitively on them.

5 The general case of a Galois group stabilizing a double-six

5.1. — To explicitly compute $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ as an abstract abelian group, one may use Manin's formula [Ma, Proposition 31.3]. This means the following.

$\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ is generated by the 27 lines. The group of all permutations of the 27 lines respecting the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840. The group G operates on the 27 lines via a finite quotient G/H which is isomorphic to a subgroup of $W(E_6)$. Then,

$$H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathrm{Hom}(ND \cap D_0/ND_0, \mathbb{Q}/\mathbb{Z}).$$

Here, D is the free abelian group generated by the 27 lines and D_0 is the subgroup of all principal divisors. $N\colon D \to D$ denotes the norm map as a G/H-module.

5.2. — Using Manin's formula, we computed $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ for each of the 350 conjugacy classes of subgroups of $W(E_6)$. The computations in `GAP` took approximately 28 seconds of CPU time. Thereby, we recovered the following result of Sir Peter Swinnerton-Dyer [SD2]. (See also P. K. Corn [Co].)

5.3. Theorem (Swinnerton-Dyer). — *Let S be a non-singular cubic surface over $\mathbb{Q}$. Then, $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ may take only five values,* 0, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, *and* $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. □

5.4. Remark. — One has $H^1(G, \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = 0$ in 257 of the 350 cases.

5.5. — More importantly, we make the following observation.

Proposition. *Let S be a non-singular cubic surface over* $\mathbb{Q}$.

i) *If* $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = \mathbb{Z}/2\mathbb{Z}$ *then, on S, there is a Galois invariant double-six.*

ii) *If* $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *then, on S, there is a triple of azygetic double-sixes stabilized by* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proof. This is seen by a case-by-case study using `GAP`. □

5.6. Remarks. — i) On the other hand, if there is a Galois invariant double-six on S then $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$ is either 0, or $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

ii) Proposition 5.5 immediately provokes the question whether the cohomology classes are always "the same" as in the $[12,15]$- and $[6,6,6,9]$-cases. I.e., of the type $\mathrm{cl}(\mathscr{D})$ for certain Galois invariant double-sixes. Somewhat surprisingly, this is indeed the case.

5.7. Lemma. — *Let* $\mathscr{S}$ *be a non-singular cubic surface over an algebraically closed field, H a group of automorphisms of the configuration of the 27 lines, and* $H' \subseteq H$ *any subgroup. Each of the criteria below is sufficient for*

$$\mathrm{res}\colon H^1(H, \mathrm{Pic}(\mathscr{S})) \to H^1(H', \mathrm{Pic}(\mathscr{S}))$$

being an injection.

i) *H and* H' *generate the same orbit structure.*

ii) H' *is of odd index in H and* $H^1(H, \mathrm{Pic}(\mathscr{S}))$ *is a* 2*-group.*

iii) H' *is a normal subgroup in H and* $\mathrm{rk}\,\mathrm{Pic}(\mathscr{S})^H = \mathrm{rk}\,\mathrm{Pic}(\mathscr{S})^{H'}$.

Proof. i) This follows immediately from the formula of Manin [Ma, Proposition 31.3].

ii) Here, $\mathrm{cores} \circ \mathrm{res}\colon H^1(H, \mathrm{Pic}(\mathscr{S})) \to H^1(H, \mathrm{Pic}(\mathscr{S}))$ is the multiplication by an odd number, hence the identity.

iii) The assumption ensures that H/H' operates trivially on $\mathrm{Pic}(\mathscr{S})^{H'}$. Hence, $H^1(H/H', \mathrm{Pic}(\mathscr{S})^{H'}) = 0$. The inflation-restriction sequence

$$0 \to H^1\big(H/H', \mathrm{Pic}(\mathscr{S})^{H'}\big) \to H^1(H, \mathrm{Pic}(\mathscr{S})) \to H^1(H', \mathrm{Pic}(\mathscr{S}))$$

yields the assertion. □

5.8. Proposition. — *Let* $\mathscr{S}$ *be a non-singular cubic surface over an algebraically closed field,* U_1 *the group of automorphisms of the configuration of the 27 lines stabilizing a double-six and* U_3 *the group stabilizing a triple of azygetic double-sixes.*

a) *Let* $H \subseteq U_1$ *be such that* $H^1(H, \mathrm{Pic}(\mathscr{S})) = \mathbb{Z}/2\mathbb{Z}$. *Then, the restriction*

$$\mathrm{res}\colon H^1(U_1, \mathrm{Pic}(\mathscr{S})) \longrightarrow H^1(H, \mathrm{Pic}(\mathscr{S}))$$

is a bijection.

b) *Let $H \subseteq U_3$ be a subgroup such that $H^1(H, \mathrm{Pic}(\mathscr{S})) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then, the restriction*

$$\mathrm{res}\colon H^1(U_3, \mathrm{Pic}(\mathscr{S})) \longrightarrow H^1(H, \mathrm{Pic}(\mathscr{S}))$$

is a bijection.

Proof. The proof has a computer part. We use the machine to verify that the criteria formulated in Lemma 5.7 suffice to establish the result in all cases.

b) Here, the subgroup $(A_3 \times A_3) \ltimes \mathbb{Z}/2\mathbb{Z}$ of order 18, as well as the two intermediate groups of order 36 produce the same orbit structure $[6,6,6,9]$. It turns out that every subgroup H which leads to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a subgroup of odd (1, 3, or 9) index in one of those.

a) By Lemma 5.7.ii), we may test this on the 2-Sylow subgroups of H and U_1. $U_1^{(2)}$ is a group of order 32 such that the Picard rank is equal to two. It turns out that, for 2-groups H' such that $H^1(H', \mathrm{Pic}(\mathscr{S})) = \mathbb{Z}/2\mathbb{Z}$, the Picard rank may be only two or three.

There is a maximal 2-group such that the Picard rank is three. This is a group of order 16 generating the orbit structure $[1,1,1,4,4,4,4,4,4]$. To prove the assertion for this group, one first observes that it is of index three in a group of order 48 with orbit structure $[3,12,12]$. This group, in turn, is of index two in the maximal group with that orbit structure. That one, being of order 96, is the maximal group stabilizing a double-six and a tritangent plane containing three complementary lines. It is of index 15 in U_1. □

5.9. Corollary. — *Let $H' \subseteq H \subseteq U_1$ be arbitrary. Then, for the restriction map* $\mathrm{res}\colon H^1(H, \mathrm{Pic}(\mathscr{S})) \longrightarrow H^1(H', \mathrm{Pic}(\mathscr{S}))$, *there are the following limitations.*

i) *If $H^1(H, \mathrm{Pic}(\mathscr{S})) = 0$ then $H^1(H', \mathrm{Pic}(\mathscr{S})) = 0$.*

ii) *If $H^1(H, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^1(H', \mathrm{Pic}(\mathscr{S})) \neq 0$ then* res *is an injection.*

iii) *If $H^1(H, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then $H^1(H', \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or 0. In the former case,* res *is a bijection. The latter is possible only when H' stabilizes a sixer.*

Proof. We know from Remark 5.6.i) that both groups may be only 0, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

i) If $H^1(H', \mathrm{Pic}(\mathscr{S}))$ were isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then the restriction from U_1, respectively U_3, to H' would be the zero map.

ii) is immediate from the computations above.

iii) If $H^1(H', \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/2\mathbb{Z}$ then, by composition, we could produce the zero map on $\mathbb{Z}/2\mathbb{Z}$. The final assertion is an experimental observation. □

5.10. Remark (Pairs of syzygetic double-sixes)**.** — U_2, the largest group stabilizing two syzygetic double-sixes is of order 96. In view of Proposition 5.5.ii), this ensures that $H^1(U_2, \mathrm{Pic}(\mathscr{S})) \cong \mathbb{Z}/2\mathbb{Z}$ or 0. Actually, it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Corollary 5.9.ii) implies that the Brauer classes associated with the two double-sixes coincide. Both are equal to the non-zero element.

Actually, the group U_2 leads to an orbit structure $[1,4,6,8,8]$. It is easy to compute $H^1(U_2, \mathrm{Pic}(\mathscr{S}))$ directly using the same methods as in the proof of Theorem 4.4.

5.11. Theorem. — *Let $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ be an arbitrary smooth cubic surface. Then, consider the class map*

$$\mathrm{cl}\colon \Phi_S^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \to \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}),$$

introduced in Definition 4.1.ii), from the set of all Galois invariant double-sixes.

a) cl *has the properties below.*

i) *If $\mathscr{D}_1, \mathscr{D}_2$ are syzygetic double-sixes then* $\mathrm{cl}(\mathscr{D}_1) = \mathrm{cl}(\mathscr{D}_2)$.

ii) *If $\mathscr{D}_1, \mathscr{D}_2$ are azygetic then* $\mathrm{cl}(\mathscr{D}_1) + \mathrm{cl}(\mathscr{D}_2) + \mathrm{cl}(\mathscr{D}_3) = 0$ *for $\mathscr{D}_3$ the third double-six of the corresponding triple.*

b) *If* $\mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q}) \neq 0$ *then*

i) $\mathrm{cl}(\mathscr{D}) \neq 0$ *for every Galois invariant double-six. Further,* $\mathrm{cl}\big(\Phi_S^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}\big)$ *contains exactly the elements of order two.*

ii) *Two double-sixes $\mathscr{D}_1, \mathscr{D}_2$ are syzygetic if and only if* $\mathrm{cl}(\mathscr{D}_1) = \mathrm{cl}(\mathscr{D}_2)$ *and azygetic if and only if* $\mathrm{cl}(\mathscr{D}_1) \neq \mathrm{cl}(\mathscr{D}_2)$. □

6 Explicit Brauer-Manin obstruction

6.1. — Let S be a non-singular cubic surface with a Galois invariant double-six $\mathscr{D}$. This determines a class $c := \mathrm{cl}(\mathscr{D}) \in \mathrm{Br}(S)/\pi^*\mathrm{Br}(\mathbb{Q})$. Choose a representative $\underline{c} \in \mathrm{Br}(S)$ and the corresponding rational function $F_{30}/F_{15}^2 \in \mathbb{Q}(S)$. Finally, let $\mathbb{Q}(\sqrt{D})$ be the quadratic extension splitting the double-six into two sixers.

6.2. Fact. — *The quaternion algebra over $\mathbb{Q}(S)$ corresponding to $\underline{c}$ is*

$$Q := \mathbb{Q}(S)\{X,Y\}/(XY+YX, X^2-D, Y^2-F_{30}/F_{15}^2).$$

6.3. Remark. — It is well known that a class in $\mathrm{Br}(S)$ is uniquely determined by its restriction to $\mathrm{Br}(\mathbb{Q}(S))$. The corresponding quaternion algebra over the whole of S may be described as follows.

Let $x \in S$. We know that $\mathrm{div}(F_{30}/F_{15}^2)$ is the norm of a divisor on $S_{\mathbb{Q}(\sqrt{D})}$. That one is necessarily locally principal. I.e., we have a rational function $f_x = a_x + b_x\sqrt{D}$ such that $\mathrm{div}(Nf_x) = \mathrm{div}(F_{30}/F_{15}^2)$ on a Zariski neighbourhood of x. Over the maximal such neighbourhood U_x, we define a quaternion algebra by

$$Q_x := \mathscr{O}_{U_x}\{X,Y_x\}/(XY_x+Y_xX, X^2-D, Y_x^2-\tfrac{F_{30}}{F_{15}^2 Nf_x}).$$

In particular, in a neighbourhood U_η of the generic point, we have the quaternion algebra $Q_\eta := \mathscr{O}_{U_\eta}\{X,Y\}/(XY+YX, X^2-D, Y^2-F_{30}/F_{15}^2)$.

Over $U_\eta \cap U_x$, there is the isomorphism $\iota_{\eta,x}\colon Q_\eta|_{U_\eta\cap U_x} \to Q_x|_{U_\eta\cap U_x}$, given by

$$X \mapsto X, \quad Y \mapsto (a_x + b_x X)Y_x.$$

For two points $x, y \in S$, the isomorphism $\iota_{\eta,y}\circ\iota_{\eta,x}^{-1}\colon Q_x|_{U_\eta\cap U_x\cap U_y} \to Q_y|_{U_\eta\cap U_x\cap U_y}$ extends to $U_x \cap U_y$.

Hence, the quaternion algebras Q_x may be glued together along these isomorphisms. This yields a quaternion algebra $\mathscr{Q}$ over S.

6.4. Corollary. — *Let $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ be a non-singular cubic surface with a Galois invariant double-six $\mathscr{D}$. Further, let p be a prime number and $\underline{c} \in \operatorname{Br}(S)$ a representative of the class* $\operatorname{cl}(\mathscr{D}) \in \operatorname{Br}(S)/\pi^*\operatorname{Br}(\mathbb{Q})$.

Then, the local evaluation map $\operatorname{ev}_p(\underline{c}, .)\colon S(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ *is given as follows.*

i) *Let $x \in S(\mathbb{Q}_p)$. Choose a rational function f_x such that* $\operatorname{div}(Nf_x) = \operatorname{div}(F_{30}/F_{15}^2)$. *Then,*

$$\operatorname{ev}_p(\underline{c},x) = \begin{cases} 0 & \text{if } \frac{F_{30}}{F_{15}^2 Nf_x}(x) \in \mathbb{Q}_p^* \text{ is in the image of } N\colon \mathbb{Q}_p(\sqrt{D}) \longrightarrow \mathbb{Q}_p\,, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Here, $F_{30}/F_{15}^2 \in \mathbb{Q}(S)$ is the rational function corresponding to the representative $\underline{c}$. $\mathbb{Q}(\sqrt{D})$ is the quadratic field splitting the double-six into two sixers.

ii) *If x is not contained in any of the 27 lines then $f_x \equiv 1$ is allowed.*

Proof. Assertion i) immediately follows from the above. For ii), recall that $\operatorname{div}(F_{30}/F_{15}^2)$ is a linear combination of the 27 lines. □

6.5. — As already noticed in Remark 2.6, the local evaluation map carries information only at finitely many primes. To exclude a particular prime, the following elementary criteria are highly practical.

6.6. Lemma (The local H^1-criterion)**.** — *Let S be a non-singular cubic surface over $\mathbb{Q}$. Suppose that, for the decomposition group $G_p \cong \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ at a prime number p,*

$$H^1(G_p, \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) = 0\,.$$

Then, for every $\alpha \in \operatorname{Br}(S)$, the value of $\operatorname{ev}_p(\alpha,x)$ is independent of $x \in S(\mathbb{Q}_p)$.

Proof. The local evaluation map ev_p factors via $\operatorname{Br}(S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_p)$. By [Ma, Lemma 43.1.1], we have that

$$\operatorname{Br}(S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_p)/\operatorname{Br}(\mathbb{Q}_p) \cong H^1(\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \operatorname{Pic}(S_{\overline{\mathbb{Q}}_p}))\,.$$

Together with the assumption, this yields $\operatorname{Br}(S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_p) = \operatorname{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$. The assertion follows. □

6.7. Remark. — Recall from Remark 5.4 that we have $H^1(G, \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) = 0$ for 257 of the 350 possible conjugacy classes of subgroups.

6.8. Corollary. — *Let $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ be a non-singular cubic surface with a Galois invariant double-six $\mathscr{D}$. Further, let $\underline{c} \in \operatorname{Br}(S)$ be a representative of the class* $\operatorname{cl}(\mathscr{D}) \in \operatorname{Br}(S)/\pi^*\operatorname{Br}(\mathbb{Q})$.

If a prime number p splits in the quadratic number field $\mathbb{Q}(\sqrt{D})$ splitting the two sixers then the value of $\operatorname{ev}_p(\underline{c},x)$ is independent of $x \in S(\mathbb{Q}_p)$.

Proof. This criterion is, of course, an immediate consequence of Corollary 6.4. In view of Fact 3.6, it is also a particular case of the local H^1-criterion. □

6.9. Proposition. — *Let $\pi\colon S \to \operatorname{Spec}\mathbb{Q}$ be a non-singular cubic surface with a Galois invariant double-six $\mathscr{D}$. Further, let $\underline{c} \in \operatorname{Br}(S)$ be a representative of the class* $\operatorname{cl}(\mathscr{D}) \in \operatorname{Br}(S)/\pi^*\operatorname{Br}(\mathbb{Q})$.

Then, for a prime number p such that

• *the field extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ splitting the double-six is unramified at p,*

• *the reduction S_p is geometrically irreducible and no $\mathbb{Q}_p$-rational point on S reduces to a singularity of S_p,*

the value of $\mathrm{ev}_p(\underline{c},x)$ *is independent of* $x \in S(\mathbb{Q}_p)$*.*

Proof. If p splits in the quadratic extension $\mathbb{Q}(\sqrt{D})$ then the assertion is true, trivially. Thus, we may assume that p remains prime in $\mathbb{Q}(\sqrt{D})$. The requirement that $z \in \mathbb{Q}_p^*$ is a norm from $\mathbb{Q}_p(\sqrt{D})$ then simply means that $\nu_p(z)$ is even.

Further, we may restrict our considerations to points $x \in S(\mathbb{Q}_p)$ which are not contained in any of the 27 lines on S. Indeed, the local evaluation map is p-adically continuous and the complement of the 27 lines is dense in $S(\mathbb{Q}_p)$ according to Hensel's lemma. In particular, we may work with F_{30}/F_{15}^2 itself.

By assumption, we have a model $\mathscr{S}$ of S over $\mathbb{Z}_p$ such that the special fiber of $\mathscr{S} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathscr{O}_{\mathbb{Q}_p(\sqrt{D})}$ is irreducible. We delete its singularities to obtain a model $\underline{\mathscr{S}}$, smooth over $\mathscr{O}_{\mathbb{Q}_p(\sqrt{D})}$. According to the last assumption, every $x \in S(\mathbb{Q}_p)$ determines a unique extension $\underline{x} \in \underline{\mathscr{S}}(\mathscr{O}_{\mathbb{Q}_p(\sqrt{D})})$.

It will suffice to construct a Zariski neighbourhood of $\underline{x}$ such that $\mathrm{ev}_p(\underline{c},\,.\,)$ is constant. We have, on the geometric generic fiber,

$$\operatorname{div}(F_{30}/F_{15}^2) = 5E + 5G - 4F\,.$$

Here, the divisors $E := E_1 + \ldots + E_6$, $G := G_1 + \ldots + G_6$, and $F := F_{12} + \ldots + F_{56}$ are $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$-invariant, and, therefore, defined over $S \times_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec}\mathbb{Q}_p(\sqrt{D})$. $\underline{\mathscr{S}}$ is a regular model of that variety. Hence, every divisor on $\underline{\mathscr{S}}$ is locally principal. This yields, in a Zariski neighbourhood $\mathscr{U}_{\underline{x}}$,

$$F_{30}/F_{15}^2 = Cp^k e^5 g^5/f^4$$

for e, g, and f rational functions corresponding to the divisors E, G, and F, respectively, $k \in \mathbb{Z}$, and a certain $C \in \Gamma(\mathscr{U}_{\underline{x}}, \mathscr{O}^*_{\mathscr{U}_{\underline{x}}})$. Note that we get by with one power of p since the special fiber is irreducible.

The scheme $\underline{\mathscr{S}}$ is acted upon by the conjugation $\sigma \in \operatorname{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$. Restricting to an open subscheme, if necessary, we may assume that $\mathscr{U}_{\underline{x}}$ is invariant under σ. The two sixers are interchanged by σ. Consequently,

$$\sigma(e) = cp^l g, \quad \sigma(f) = c'p^{l'} f$$

for $l, l' \in \mathbb{Z}$ and regular functions c, c', invertible on $\mathscr{U}_{\underline{x}}$. This yields

$$F_{30}/F_{15}^2 = Cc'^2 C^{-5} p^{k+2l'-5l} N(e^5/f^2)\,.$$

For $x_0 \in S(\mathbb{Q}_p)$ specializing to $\mathscr{U}_{\underline{x}}$, we therefore see

$$\nu_p\big((F_{30}/F_{15}^2)(x_0)\big) \equiv k + l \pmod 2\,.$$

In particular, the local evaluation map $\mathrm{ev}_p(\underline{c},\,.\,)\colon S(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ is constant on the set of all points specializing to $\mathscr{U}_{\underline{x}}$. As the point $x \in S(\mathbb{Q}_p)$ defining the open subset $\mathscr{U}_{\underline{x}}$ is arbitrary and the special fiber $\mathscr{S}_p$ is irreducible, this implies the assertion. □

6.10. Remarks. —– a) Assuming resolution of singularities in unequal characteristic, there is a proper model $\mathscr{S}$ of S being a regular scheme. Then, for p a prime unramified in $\mathbb{Q}(\sqrt{D})$, the evaluation $\mathrm{ev}_p(\underline{c},x)$ depends only on the component of $\mathscr{S} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_{p^2}$, the point x specializes to. If p is ramified and $p \neq 2$ then we have at least that $\mathrm{ev}_p(\underline{c},x)$ is determined by the reduction of x modulo p.

b) As one might expect from the proof given, Proposition 6.9 is true in more generality. The reader might consult [B, Theorem 1].

7 Explicit Galois descent

7.1. —– Recall that in [EJ2], we described a method to construct non-singular cubic surfaces over $\mathbb{Q}$ with a Galois invariant double-six. The idea was to start with cubic surfaces in hexahedral form. For these, we developed an explicit version of Galois descent.

7.2. —– More concretely, given a starting polynomial $f \in \mathbb{Q}[T]$ of degree six without multiple zeroes, we construct a cubic surface $S_{(a_0,\ldots,a_5)}$ over $\mathbb{Q}$ such that

$$S_{(a_0,\ldots,a_5)} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}}$$

is isomorphic to the surface $S^{(a_0,\ldots,a_5)}$ in $\mathbf{P}^5$ given by

$$\begin{aligned}
X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 &= 0\,, \\
X_0 + X_1 + X_2 + X_3 + X_4 + X_5 &= 0\,, \\
a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 &= 0\,.
\end{aligned}$$

Here, $a_0,\ldots,a_5 \in \overline{\mathbb{Q}}$ are the zeroes of f.

The operation of an element $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S_{(a_0,\ldots,a_5)} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}}$ goes over into the automorphism $\pi_\sigma \circ t_\sigma \colon S^{(a_0,\ldots,a_5)} \to S^{(a_0,\ldots,a_5)}$. Here, π_σ permutes the coordinates according to the rule $a_{\pi_\sigma(i)} = \sigma(a_i)$ while t_σ is the naive operation of σ on $S_{(a_0,\ldots,a_5)}$ as a morphism of schemes twisted by σ.

7.3. Remarks. —– i) More details on the theory are given in [EJ2, Theorem 6.6].

ii) On $S_{(a_0,\ldots,a_5)}$, there are the 15 obvious lines given by

$$X_{i_0} + X_{i_1} = X_{i_2} + X_{i_3} = X_{i_4} + X_{i_5} = 0$$

for $\{i_0,i_1,i_2,i_3,i_4,i_5\} = \{0,1,2,3,4,5\}$. They clearly form a Galois invariant set. The complement is a double-six. Correspondingly, there are the 15 obvious tritangent planes given by $X_i + X_j = 0$ for $i \neq j$.

There are formulas for the 30 non-obvious tritangent planes, too [EJ2, Proposition 7.1.ii)]. What is important is that they are defined over $\mathbb{Q}(a_0,\ldots,a_5,\sqrt{d_4})$ for

$$d_4(a_0,\ldots,a_5) := \sigma_2^2 - 4\sigma_4 + \sigma_1(2\sigma_3 - \tfrac{3}{2}\sigma_1\sigma_2 + \tfrac{5}{16}\sigma_1^3)$$

the Coble quartic. Here, σ_i is the i-th elementary symmetric function in $a_0,\ldots,a_5$.

Further, an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ flips the double-six if and only if it defines the conjugation of $\mathbb{Q}(\sqrt{D})$ for $D := d_4{\cdot}\Delta$, the second factor denoting the discriminant of $a_0,\ldots,a_5$ [EJ2, Proposition 7.4].

iii) The smooth manifold $S(\mathbb{R})$ has two connected components if and only if exactly four of the $a_0,\ldots,a_5$ are real and $d_4(a_0,\ldots,a_5) > 0$. Otherwise, $S(\mathbb{R})$ is connected [EJ2, Corollary 8.4].

iv) The descent variety $S_{(a_0,\ldots,a_5)}$ may easily be computed completely explicitly. In fact, [EJ2, Algorithm 6.7] yields a quaternary cubic form with 20 rational coefficients.

7.4. — Using the criteria provided in section 6, we have the following strategy to compute the Brauer-Manin obstruction on $S_{(a_0,\ldots,a_5)}$.

Strategy (to explicitly compute the Brauer-Manin obstruction on $S_{(a_0,\ldots,a_5)}$).

i) Compute $D := d_4(a_0,\ldots,a_5){\cdot}\Delta(a_0,\ldots,a_5)$. Determine the list L_1 of all primes at which $\mathbb{Q}(\sqrt{D})$ is ramified.

ii) By a Gröbner basis calculation, determine all the primes outside L_1 at which $S_{(a_0,\ldots,a_5)}$ has bad reduction. Write them into a list L_2.

iii) From L_2, delete all primes which split in $\mathbb{Q}(\sqrt{D})$. Further, erase all those primes from L_2 for which the singular points on the reduction modulo p are not defined over $\mathbb{F}_p$ or do not lift to $S_{(a_0,\ldots,a_5)} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}/p^k\mathbb{Z}$ for k large.

iv) Put $L := L_1 \cup L_2$. If $D < 0$ and $S_{(a_0,\ldots,a_5)}(\mathbb{R})$ is not connected then the infinite place has to be added to this list of critical primes.

v) Delete all the primes from L for which the local H^1-criterion works successfully.

vi) Scale the form F_{30} by a constant factor such that the local evaluation maps are zero for all the primes outside L.

vii) For the primes p that remained in L, the form F_{30} has to be evaluated. For that, cover $S(\mathbb{Q}_p)$ by finitely many open subsets which are sufficiently small to ensure that the first p-adic digit of F_{30} does not change. If $p = 2$ then the first three digits have to be taken into account.

In the case that we have a Galois invariant triple of azygetic double-sixes, the last step has to be executed three times, once for each of the corresponding forms of type F_{30}.

8 Application: Manin's conjecture

8.1. — Recall that a conjecture, due to Yu. I. Manin, asserts that the number of $\mathbb{Q}$-rational points of anticanonical height $\leq B$ on a Fano variety S is asymptotically equal to $\tau B \log^{\mathrm{rk\,Pic}(S)-1} B$, for $B \to \infty$. Further, the coefficient $\tau \in \mathbb{R}$ is conjectured to be the Tamagawa-type number $\tau(S)$ introduced by E. Peyre in [Pe]. In the particular case of a cubic surface, the anticanonical height is the same as the naive height.

8.2. — E. Peyre's Tamagawa-type number is defined in [PT, Definition 2.4] as

$$\tau(S) := \alpha(S){\cdot}\beta(S) \cdot \lim_{s \to 1}(s-1)^t L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) \cdot \tau_H\big(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}\big)$$

for $t = \operatorname{rk}\operatorname{Pic}(S)$.

Here, the factor $\beta(S)$ is simply defined as $\beta(S) := \#H^1\left(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})\right)$. $\alpha(S)$ is given as follows [Pe, Définition 2.4]. Let $\Lambda_{\text{eff}}(S) \subset \operatorname{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ be the cone generated by the effective divisors. Identify $\operatorname{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}^t$ via a mapping induced by an isomorphism $\operatorname{Pic}(S) \xrightarrow{\cong} \mathbb{Z}^t$. Consider the dual cone $\Lambda_{\text{eff}}^{\vee}(S) \subset (\mathbb{R}^t)^{\vee}$. Then, $\alpha(S) := t \cdot \operatorname{vol}\{x \in \Lambda_{\text{eff}}^{\vee} \mid \langle x, -K\rangle \leq 1\}$.

$L(\,\cdot\,, \chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})})$ denotes the Artin L-function of the $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\operatorname{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ which contains the trivial representation t times as a direct summand. Therefore, $L(s, \chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})}) = \zeta(s)^t \cdot L(s, \chi_P)$ and

$$\lim_{s \to 1} (s-1)^t L(s, \chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})}) = L(1, \chi_P)$$

where ζ denotes the Riemann zeta function and P is a representation which does not contain trivial components. [Mu, Corollary 11.5 and Corollary 11.4] show that $L(s, \chi_P)$ has neither a pole nor a zero at $s = 1$.

Finally, τ_H is the *Tamagawa measure* on the set $S(\mathbb{A}_{\mathbb{Q}})$ of adelic points on S and $S(\mathbb{A}_{\mathbb{Q}})^{\text{Br}} \subseteq S(\mathbb{A}_{\mathbb{Q}})$ denotes the part which is not affected by the Brauer-Manin obstruction.

8.3. — As S is projective, we have $S(\mathbb{A}_{\mathbb{Q}}) = \prod_{\nu \in \operatorname{Val}(\mathbb{Q})} S(\mathbb{Q}_\nu)$. Then, the Tamagawa measure τ_H is defined to be the product measure $\tau_H := \prod_{\nu \in \operatorname{Val}(\mathbb{Q})} \tau_\nu$.

Here, for a prime number p, the local measure τ_p on $S(\mathbb{Q}_p)$ is given as follows. Let $a \in S(\mathbb{Z}/p^k\mathbb{Z})$ and put $\mathfrak{U}_a^{(k)} := \{x \in S(\mathbb{Q}_p) \mid x \equiv a \pmod{p^k}\}$. Then,

$$\tau_p(\mathfrak{U}_a^{(k)}) := \det(1 - p^{-1}\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}) \cdot \lim_{m \to \infty} \frac{\#\{y \in S(\mathbb{Z}/p^m\mathbb{Z}) \mid y \equiv a \pmod{p^k}\}}{p^{m \dim S}}.$$

$\operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}$ denotes the fixed module under the inertia group.

τ_∞ is described in [Pe, Lemme 5.4.7]. In the case of a cubic surface, defined by the equation $f = 0$, this yields

$$\tau_\infty(U) = \frac{1}{2} \int\limits_{\substack{CU \\ |x_0|, \dots, |x_3| \leq 1}} \omega_{\text{Leray}}$$

for $U \subset S(\mathbb{R})$. Here, ω_{Leray} is the *Leray measure* on the cone $CS(\mathbb{R})$. It is related to the usual hypersurface measure by the formula $\omega_{\text{Leray}} = \frac{1}{\|\operatorname{grad} f\|}\,\omega_{\text{hyp}}$.

8.4. — Using [EJ2, Algorithm 6.7], we constructed many examples of smooth cubic surfaces over $\mathbb{Q}$ with a Galois invariant double-six. For each of them, one may apply Strategy 7.4 to compute the effect of the Brauer-Manin obstruction. Then, the method described in [EJ1] applies for the computation of Peyre's constant.

From the ample supply, the examples below were chosen in the hope that they indicate the main phenomena. The Brauer-Manin obstruction may work at many primes simultaneously but examples where few primes are involved are the most interesting. We show that the fraction of the Tamagawa measure excluded by the obstruction can vary greatly. We also show that there may be an obstruction at the infinite prime.

8.5. Example. — The polynomial

$$f := T^6 - 390T^4 - 10\,180T^3 + 10\,800T^2 + 2\,164\,296T + 13\,361\,180 \in \mathbb{Q}[T]$$

yields the cubic surface S given by the equation

$$-x^2z - x^2w - 3xy^2 + xz^2 + 14xzw + 8xw^2 - 2y^3 - 11y^2z \\ + y^2w + 4yz^2 + 4yzw + 10yw^2 + 4z^3 - 11z^2w + 9zw^2 - 6w^3 = 0\,.$$

S has bad reduction at 2, 3, 5, 11, and 9 265 613 761. The Galois group operating on the 27 lines is S_6 acting in such a way that the orbit structure is $[12,15]$. Therefore, we have $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathbb{Z}/2\mathbb{Z}$. The quadratic field splitting the double-six is $\mathbb{Q}(\sqrt{10})$.

The primes 3 and 9 265 613 761 split in $\mathbb{Q}(\sqrt{10})$. The local H^1-criterion excludes the prime 5. Further, it turns out that the local evaluation map at 11 is constant. Hence, the Brauer-Manin obstruction works only at the prime 2. From the whole of $S(\mathbb{Q}_2)$ which is of measure 4 only a subset of measure $\frac{9}{4}$ is allowed.

Using this, for Peyre's constant, we find $\tau(S) \approx 1.7005$. There are actually 6641 $\mathbb{Q}$-rational points of height at most 4000 in comparison with a prediction of 6802.

8.6. Example. — The polynomial

$$f := T^6 + 60T^4 - 40T^3 - 900T^2 + 15\,072T - 27\,860 \in \mathbb{Q}[T]$$

yields the cubic surface S given by the equation

$$5x^3 - 9x^2y + x^2z + 6x^2w + 3xy^2 + xyz + 6xyw - 2xz^2 \\ - 4xzw - y^3 - 3y^2z + 2yz^2 + 2yzw + 4z^3 + 2z^2w + 2zw^2 = 0\,.$$

S has bad reduction at 2, 3, 5, and 73. The Galois group operating on the 27 lines is $A_6 \times \mathbb{Z}/2\mathbb{Z}$ and the orbit structure is $[12,15]$. The quadratic field $\mathbb{Q}(\sqrt{2})$ splits the double-six.

The prime 73 splits in $\mathbb{Q}(\sqrt{2})$. Further, the local H^1-criterion excludes the prime 5. Hence, the Brauer-Manin obstruction works only at the primes 2 and 3. At 2, the local evaluation map decomposes $S(\mathbb{Q}_2)$ into two sets of measures 1 and $\frac{1}{2}$, respectively. At 3, the corresponding measures are $\frac{70}{81}$ and $\frac{28}{81}$. An easy calculation shows $\tau_H(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}) = \frac{4}{7}\tau_H(S(\mathbb{A}_\mathbb{Q}))$.

Using this, for Peyre's constant, we find $\tau(S) \approx 5.0879$. Up to a search bound of 4000, there are actually 19 302 $\mathbb{Q}$-rational points in comparison with a prediction of 20 352.

8.7. Example. — The polynomial $f := T(T^5 - 5T - 2) \in \mathbb{Q}[T]$ yields the cubic surface S given by the equation

$$2x^3 + x^2y - 4x^2z - x^2w + 2xy^2 + 2xyz + 2xyw - 2xz^2 - 4xzw \\ - 2xw^2 + 2y^2z - y^2w + yz^2 + 2yzw - 5yw^2 - 3z^2w + 6zw^2 + 9w^3 = 0\,.$$

S has bad reduction at 2, 3, and 5. Further, $S(\mathbb{R})$ consists of two connected components. The Galois group operating on the 27 lines is isomorphic to $S_5 \times \mathbb{Z}/2\mathbb{Z}$ and the orbit structure is $[12,15]$. $\mathbb{Q}(\sqrt{-15})$ is the field splitting the double-six.

The prime 2 splits in $\mathbb{Q}(\sqrt{-15})$. Further, the local H^1-criterion excludes the prime 5. Hence, the Brauer-Manin obstruction works only at 3 and the infinite prime. At 3, the local evaluation map decomposes $S(\mathbb{Q}_2)$ into two sets of measures $\frac{2}{3}$ and $\frac{4}{9}$, respectively. At the infinite prime, the corresponding measures are approximately 1.9179 and 1.1673. An easy calculation shows $\tau_H(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}) \approx 0.5243 \cdot \tau_H(S(\mathbb{A}_\mathbb{Q}))$.

Using this, for Peyre's constant, we find $\tau(S) \approx 3.7217$. Up to a search bound of 4000, there are actually 14 249 $\mathbb{Q}$-rational points in comparison with a prediction of 14 887.

8.8. Example. — The polynomial

$$f := T(T^5 - 60T^3 - 90T^2 + 675T + 810) \in \mathbb{Q}[T]$$

yields the cubic surface S given by the equation

$$3x^3 + 2x^2z + xy^2 - 2xyz - 2xyw - xzw + 2xw^2 - yzw - yw^2 - z^3 + z^2w = 0\,.$$

S has bad reduction at 2, 3, and 5. The Galois group operating on the 27 lines is isomorphic to S_5 and the orbit structure is $[12,15]$. The quadratic field $\mathbb{Q}(\sqrt{-3})$ splits the double-six.

The local H^1-criterion excludes the prime 5. Further, the local evaluation maps turn out to be constant on $S(\mathbb{Q}_2)$ and $S(\mathbb{R})$. At the real prime, the reason is simply that $S(\mathbb{R})$ is connected. Consequently, the Brauer-Manin obstruction works only at the prime 3. From the whole of $S(\mathbb{Q}_3)$ measuring $\frac{2}{3}$ a subset of measure $\frac{4}{9}$ is allowed.

Using this, for Peyre's constant, we find $\tau(S) \approx 2.2647$. Up to a search bound of 4000, there are actually 8886 $\mathbb{Q}$-rational points in comparison with a prediction of 9059.

8.9. Example. — The polynomial $f := T(T^5 + 20T + 16) \in \mathbb{Q}[T]$ yields the cubic surface S given by the equation

$$\begin{aligned} -3x^3 - 7x^2y - 4x^2z + 5x^2w + 4xy^2 + 10xyz - 4xyw - 2xz^2 \\ + 2xzw + xw^2 - 4y^2z + yz^2 - 4yzw - 16yw^2 + z^2w - 5zw^2 = 0\,. \end{aligned}$$

S has bad reduction at 2 and 5. The Galois group operating on the 27 lines is isomorphic to $A_5 \times \mathbb{Z}/2\mathbb{Z}$. The orbit structure is $[12,15]$. The quadratic field $\mathbb{Q}(\sqrt{-5})$ splits the double-six.

The local H^1-criterion excludes the prime 5. At the infinite prime, the local evaluation map is constant since $S(\mathbb{R})$ is connected. Hence, the Brauer-Manin obstruction works only at the prime 2. It allows a subset of measure $\frac{17}{16}$ out of $S(\mathbb{Q}_2)$ measuring $\frac{5}{4}$.

Using this, for Peyre's constant, we find $\tau(S) \approx 2.4545$. Up to a search bound of 4000, there are actually 9736 $\mathbb{Q}$-rational points in comparison with a prediction of 9818.

8.10. Example. — The polynomial

$$f := T^6 - 456T^4 - 904T^3 + 102\,609T^2 + 1\,041\,060T + 2\,935\,300 \in \mathbb{Q}[T]$$

yields the cubic surface S given by the equation

$$-2x^3+3x^2z+9x^2w-4xy^2-8xyz-10xzw+4xw^2-4y^3-3y^2z \\ -4y^2w-2yz^2-2yzw+8yw^2-z^3+z^2w-6zw^2-2w^3=0\,.$$

S has bad reduction at 2, 3, 5, 31, and 11071. The Galois group operating on the 27 lines is isomorphic to $(S_3\times S_3)\ltimes \mathbb{Z}/2\mathbb{Z}$ of order 72. The orbit structure is $[6,6,6,9]$. There is a triple of Galois invariant double-sixes. Therefore, we have that $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\mathrm{Pic}(S_{\overline{\mathbb{Q}}}))\cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. The quadratic field splitting the double-sixes is $\mathbb{Q}(\sqrt{2})$.

The primes 31 and 11071 split in $\mathbb{Q}(\sqrt{2})$. Further, the local evaluation maps turn out to be constant on $S(\mathbb{Q}_3)$. Consequently, the Brauer-Manin obstruction works only at the primes 2 and 5.

The local evaluation maps decompose $S(\mathbb{Q}_2)$ into four sets of measures $\frac{7}{16}$, $\frac{5}{16}$, $\frac{1}{4}$, and $\frac{1}{4}$, respectively. At the prime 5, the corresponding measures are $\frac{516}{625}$, 0, $\frac{96}{625}$, and 0. Observe, for one of the three non-zero Brauer classes, the local evaluation map is constant on $S(\mathbb{Q}_5)$.

A simple calculation shows that $\tau_H(S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}})=\frac{111}{340}\tau_H(S(\mathbb{A}_\mathbb{Q}))$. Using this, for Peyre's constant, we find $\tau(S)\approx 1.8532$. Up to a search bound of 4000, there are actually 6994 $\mathbb{Q}$-rational points in comparison with a prediction of 7413.

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