LOCAL SINGULARITIES, FILTRATIONS AND TANGENTIAL FLATNESS

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Abstract

We consider local rings (B_0, n_0) all of whose deformations are tangentially flat when endowed with suitably constructed filtrations. Various examples of such singularities are constructed and applications to Lech-Hironaka type inequalities between the Hilbert functions of base and total space are discussed.

Introduction

Let (B_0, n_0) be a local ring. By a deformation of it we will mean a flat local homomorphism $f : (A, m) \longrightarrow (B, n)$ of local rings whose special fiber B/mB is isomorphic to (B_0, n_0). As $A/m \hookrightarrow B/mB$ this concept makes sense only in the case (B_0, n_0) contains a field.

In 1959 C. Lech [13] stated the problem whether the multiplicities of local rings (A, m) and (B, n) being base, respectively total space, of a deformation satisfy the inequality

$$e_0(A) \leq e_0(B). \tag{1}$$

A generalization of this is the analogous inequality

$$H_A^{d+i} \leq H_B^i \tag{2}$$

between sum transforms of the Hilbert functions, where d denotes the dimension of B_0. Recall that these are defined inductively by $H_A^j(l) := \sum_{k=0}^{l} H_A^{j-1}(k)$ where H_A^0 is the usual Hilbert function given by

$$H_A^0(l) := \dim_{A/m} m^l/m^{l+1}.$$

By an inequality $H \leq H'$ between two functions $H, H' : \mathbb{N} \longrightarrow \mathbb{N}$ we always mean the inequality in its total sense, i.e. $H(l) \leq H'(l)$ for all l. In 1970 H. Hironaka [9] asked whether (2) is always true for $i = 1$ as that would simplify his proof of the existence of a resolution of singularities in characteristic zero [8].

In a remarkable paper Larfeldt and Lech [12] showed that the problem of Hironaka is equivalent to the following statement: For every local ring A and every coheight one prime P in A the inequality $H^1_{A_P} \leq H^0_A$ is true. This one and its immediate corollaries are usually referred as Bennett's inequality. Note that it generalizes Serre's result that the localization

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of a regular local ring by a prime ideal is again regular. It is proven in the case A is excellent [2, 16].

Unfortunately, Lech-Hironaka type inequalities are established in very few cases, only. The most interesting result in that direction is due to Lech himself. It says that $H_A^1 \leq H_B^1$ in the case that the special fiber B_0 is a zero dimensional complete intersection [14]. B. Herzog generalized this to the situation that B_0 corresponds to a regular point $[B_0]$ of the Hilbert scheme [5]. This includes all complete intersections and all singularities with embedding dimension less than 3.

A completely different approach is given in the paper [6]. B. Herzog asks the question for which local rings B_0 all deformations are tangentially flat, by which one means that the induced homomorphism $\mathrm{G}(A) \longrightarrow \mathrm{G}(B)$ between the associated graded rings is flat, where A and B are equipped with the canonical filtrations by powers of their maximal ideals. He proves a criterion expressing this property in terms of the normal module of B_0. Thus he is able to give a large class of local rings for all whose deformations Hironaka's inequality is true even for $i = 0$.

The aim of this paper is to give a generalization of the second approach to more general filtrations. Recall that a homomorphism of filtered local rings $f : (A, m, F_A) \longrightarrow (B, n, F_B)$ is a local homomorphism of local rings satisfying $f(F_A^i) \subseteq F_B^i$. Such a homomorphism is called tangentially flat, if the associated map $\mathrm{G}_{F_A}(A) \longrightarrow \mathrm{G}_{F_B}(B)$ is flat.

We emphasize explicitly that we mainly consider the more general filtrations as a tool to obtain results on the usual Hilbert functions. For that we need a comparison between the Hilbert functions of a local ring induced by two different filtrations. One possibility to do that is given in Lemma 3.2.

The paper will be organized as follows. In section 1 we introduce the concept of a monomially filtered local ring and prove the basic features of them being necessary for what follows. Section 2 contains the main theorem of the paper. It expresses the property of a monomially filtered local ring to allow tangentially flat deformations only in terms of normal modules. In section 3 we give the application to Lech-Hironaka type inequalities. We include several examples which were not covered by the results known before. Section 4 is a little more technical. It deals with the problem that the normal modules we are forced to consider as a consequence of the main theorem can be highly complicated to compute. The experience says that it is possible practically only for singularities defined by monomials. By some modification of the deformation to the normal cone we show that in some sense it is actually possible to reduce the question to that case.

Throughout the paper we follow the standard assumptions and notations of commutative algebra as in [15] (unless stated otherwise). We will make extensive use of the volume [7], which is fundamental concerning tangential flatness for general filtrations. All local rings will be assumed Noetherian.

1. Fundamental Concepts

1.1 Definition. By a *monomially filtered local ring* we will mean a triple (B_0, y_0, E), where

i) B_0 is a local ring

ii) $y_0 = (y_{10}, \ \ldots \ , y_{r0})$ is an r-tuple of elements generating n_0, the maximal ideal of B_0.

iii) $E = (E_d)_{d \in \mathbb{N}}$ is a *separated filtrating family*, i.e. a sequence of subsets $E_d \subseteq \mathbb{N}^r$ satisfying the following conditions.

a) $\mathbb{N}^r \setminus E_1$ is a finite set.

b) $(a_1, \ \ldots \ , a_r) \in E_d$ implies $(b_1, \ \ldots \ , b_r) \in E_d$ if $b_i \geq a_i$ for every i.

c) $E_i \supseteq E_{i+1}$ for all $i \in \mathbb{N}$.

d) $E_i + E_j \subseteq E_{i+j}$ for arbitrary $i, j \in \mathbb{N}$.

e) $(0, \ \ldots \ , 0) \in E_0 \setminus E_1$.

f) $\bigcap_d E_d = \emptyset$ \qquad (*separatedness*).

1.2 Let (B_0, y_0, E) be a monomially filtered local ring. Associated to the data given, one has a filtration F_{B_0} on B_0 defined by

$$F^i_{B_0} := \ < y_{10}^{a_1} \cdot \ \ldots \ \cdot y_{r0}^{a_r} \mid (a_1, \ \ldots \ , a_r) \in E_i > .$$

F_{B_0} is cofinite, i.e. the modules $B/F^i_{B_0}$ are of finite length. A filtration generated by a separated filtrating family induces the canonical (n_0-adic) topology.

We will call a filtrating family *finitely generated*, if there is some d_0 such that

$$E_d = \bigcup_{i_1 + \ \ldots \ + i_r = d, i_j \le d_0} E_{i_1} + \ \ldots \ + E_{i_r}.$$

Finitely generated filtrating families give rise to finitely generated filtrations.

1.3 Definition. Let (B_0, y_0, E) be a monomially filtered local ring. By a *deformation* of (B_0, y_0, E) we will mean a homomorphism $f : (A, m, F_A) \longrightarrow (B, n, F_B)$ of filtered local rings such that

i) $B/mB \cong B_0$.

ii) The filtration F_B is the sum [7] of a filtration defined by E and some lift y of the r-tuple y_0 to B with the direct image $f_*(F_A)$, i.e.

$$F^i_B = \sum_{j+k=i, (a_1, \ \ldots \ , a_r) \in E_k} F^j_A y_1^{a_1} \ \ldots \ y_r^{a_r}$$

iii) The homomorphism $f \otimes A/F^1_A : A/F^1_A \longrightarrow B/F^1_A B$ of filtered local rings is tangentially flat.

1.4 Remarks. a) We are interested in monomially filtered local rings *having only tangentially flat* deformations.

b) Without losing any of the results of this paper, condition i) could be weakened. It would be sufficient to require that only the completions of the local rings in question should be isomorphic. Then condition ii) has to be modified in the obvious manner.

c) For questions concerning tangential flatness by [7, 3.16] one is easily reduced to the case the base A is Artin. But in that case F_A induces the m-adic (discrete) topology on A and F_B defines the n-adic topology on B by construction. For that reason, from now on we are going to assume the following

1.5 Convention. *All filtrations on local rings induce the canonical topology, i.e. the same topology as the filtration by powers of the maximal ideal.* This is equivalent to say we restrict ourself to cofinite *Artin-Rees filtrations* with $1 \notin F^1$ [7, 1.12, 1.14].

Note that Artin-Rees filtrations are not necessarily finitely generated, so the associated graded rings may be non-Noetherian. Weight filtrations with weight vectors having irrational coordinates are simple counterexamples. Nevertheless every *standard base* of an ideal, i.e. every system of elements whose initial forms generate the initial ideal, is a system of generators.

1.6 The following technical lemma is a generalization of [7, 5.9].

Lemma. *Let (A, m) and (B, n) be complete filtered local rings and $f : A \longrightarrow B$ be a deformation of the complete monomially filtered local ring (B_0, y_0, E). Then there exists a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow_g \ \nearrow^h & \\ & R & \end{array}$$

of homomorphisms of filtered complete local rings, where g is tangentially flat, h is surjective and $F_B = h_* F_R$. The diagram can be adjusted such that $h_0 := h \otimes_A A/m : R/mR =: R_0 \longrightarrow B_0$ is isomorphic to the natural surjection $L[[Y_1, \ \dots \ , Y_r]] \longrightarrow B_0$, where $Y_i \mapsto y_{i0}$ and L is a coefficient field for B_0.

Proof. Using the theory of Cohen rings [4, Ch. 0, §19] one sees there is a commutative diagram of local homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ C_A[[X_1, \ \dots, \ X_s]] & \xrightarrow{q} & C_B[[X_1, \ \dots, \ X_s, Y_1, \ \dots, \ Y_r]], \end{array}$$

where C_A and C_B are Cohen rings, X_i and Y_j are indeterminates, q induces the identity on X_i, the vertical arrows are surjective and Y_j is mapped to y_{j0} under the canonical homomorphism onto B_0. Note that for C_A and C_B there are two cases. If the residue characteristic is p, they are both discrete valuation rings whose maximal ideals are generated by the prime number p. Otherwise they are fields of characteristic 0. We identify A with an appropriate factor ring of $C_A[[X_1, \ \dots, \ X_s]]$ and obtain the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & & \uparrow h \\ C_A[[X_1, \ \dots, \ X_s]]/I & \xrightarrow{g} & (C_B[[X_1, \ \dots, \ X_s]]/(q(I)))\,[[Y_1, \ \dots, \ Y_r]] =: R. \end{array}$$

Clearly the filtrating family E defines a prefiltration F'_R on R. We put $F_R = F'_R + g_* F_A$. One checks easily that this construction fulfills the assertions of the lemma.

The only point that requires a little bit work is to show g is tangentially flat. For that consider the natural factorization

$$A = C_A[[X_1, \ \dots \ , X_s]]/I \xrightarrow{i} C_B[[X_1, \ \dots \ , X_s]]/(q(I)) =: \ \tilde{R} \ \xrightarrow{j} \tilde{R}[[Y_1, \ \dots \ , Y_r]] = R$$

and equip $\tilde{R}$ with $F_{\tilde{R}} = i_* F_A$. Then $F^d_R = \sum_{k=1}^d F^k_{\tilde{R}} < Y_1^{a_1} \cdot \ldots \cdot Y_r^{a_r} >_{(a_1,\dots,a_r) \in E_{d-k}}$, hence

$$G(R)(d) = \bigoplus_{k=1}^{d} G(\tilde{R})(k) < Y_1^{a_1} \cdot \ldots \cdot Y_r^{a_r} >_{(a_1,\dots,a_r) \in E_{d-k} \setminus E_{d-k+1}} .$$

In particular we get the identity $H^0_R = H^0_{\tilde{R}} H^0_{L[[Y_1,\dots,Y_r]]}$ such that j is tangentially flat [7, 6.13]. On the other hand C_B is flat over C_A in any of the two cases above, thus $\tilde{R}$ is flat over A. Therefore $F^k_{\tilde{R}} = F^k_A \otimes_A \tilde{R}$ and $G(\tilde{R}) = G(A) \otimes_A \tilde{R}$. Consequently, i is tangentially flat, too. $\square$

2. Tangential flatness of all deformations and the normal module

2.1 Let R be a filtered local ring and $I \subseteq R$ be an ideal. Then the normal module $N_I = \mathrm{Hom}_R(I, R/I)$ carries a natural filtration $F_{N_I} = (F^d_{N_I})_{d \in \mathbf{Z}}$ given by

$$F^d_{N_I} := \{f \in N_I \mid \forall k \in \mathbf{N} : f(I \cap F^k_R) \subseteq F^{k+d}_R + I/I\}.$$

Note also, if R is a graded ring and $I \in R$ is a homogeneous ideal, then $N_I = \bigoplus_{g \in \mathbf{Z}} N_I(d)$ admits a natural grading.

2.2 Definition. Let $(B_0, y_0, E) = (B_0, n_0)$ be a complete monomially filtered local ring containing a field. Choose a coefficient field $B_0/n_0 =: L \hookrightarrow B_0$, let $p : L[[Y_1, \dots, Y_r]] =: R_0 \longrightarrow B_0$ be the canonical surjection and denote its kernel by I_0. We will call

$$N_{B_0} := N_{I_0}$$

the *normal module* of B_0 and

$$N_{G(B_0)} := N_{\mathrm{in}(I_0)}$$

the *normal module of the associated graded ring* to B_0. A standard argument gives that N_{B_0} and $N_{G(B_0)}$ are determined uniquely up to (non canonical) isomorphism respecting filtrations respectively gradings.

2.3 Proposition. *Let $f : (A, m, F_A) \longrightarrow (R, M, F_R)$ be a tangentially flat homomorphism of filtered local rings, $t \in A$ an element in the socle, and I an ideal in R such that $B := R/I$ is flat over A. Suppose $\overline{B} := B/tB$ is tangentially flat over $\overline{A} := A/tA$ and consider the following conditions.*

i) $N_{\mathrm{in}(I_0)}(< -\mathrm{ord}_{F_A}(t)) = 0$. Here $R_0 := R/mR$ and $I_0 := IR_0$.

ii) For every ideal I' in R such that $I + tR = I' + tR$ flatness of $B' := R/I'$ over A implies tangential flatness.

Then i) implies ii).

Proof. We have to show that $B' = R/I'$ is tangentially flat over A. For this by [7, 4.6] it suffices to show that there is a standard base of I_0 which can be lifted to I' preserving the orders. So let $r_0 = (r_{0\lambda})_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} R_0$ be a standard base of I_0. We put $\overline{R} := R/tR$ and $\overline{I} := I\overline{R} = I'\overline{R}$. As $\overline{B} = \overline{R}/\overline{I}$ is tangentially flat over $\overline{A}$, r_0 can be lifted to a standard base $\overline{r} \in \prod_{\lambda\in\Lambda} \overline{R}$ satisfying $\mathrm{ord}(\overline{r}) = \mathrm{ord}(r_0)$. In particular the coordinates of $\overline{r}$ generate $\overline{I}$. Further B' is flat over A, so there is a lift $r' \in \prod_{\lambda\in\Lambda} R$ of $\overline{r}$ whose coordinates generate I'. Write

$$r' = r + ts_0, \qquad r \in \prod_{\lambda\in\Lambda} R,\ s_0 \in \prod_{\lambda\in\Lambda} R_0,$$

where ts_0 is well-defined since t annihilates mR.

Let $e := \mathrm{ord}_{F_A}(t)$. If the lift r' and its decomposition $r' = r + ts_0$ are such that

$$\mathrm{ord}(s_0) \geq \mathrm{ord}(r_0) - (e), \tag{3}$$

then $\mathrm{ord}(r') = \mathrm{ord}(r_0)$, i.e. I' satisfies condition (iii) of [7, 4.6] and B' is tangentially flat over A.

So assume (3) is wrong and choose m to be the minimal number greater than e such that there exist $\lambda \in \Lambda$ for which

$$\mathrm{ord}(s_{0\lambda}) = \mathrm{ord}(r_{0\lambda}) - m.$$

Denote the set of such indices by Λ_m. We are going to prove that in this situation the decomposition $r' = r + ts_0$ can be replaced by another one, say $r'' = r + ts_0'$, which satisfies

$$\mathrm{ord}(s'_{0\lambda}) \geq \mathrm{ord}(r''_{0\lambda}) - m + 1$$

for $\lambda \in \Lambda_m$ and $s'_{0\lambda} = s''_{0\lambda}$ otherwise.

Note that this will suffice to conclude the proof. Indeed after a finite number of steps the decomposition $r' = r + ts_0$ will be replaced by one which satisfies (3) for all indices $\lambda \in \Lambda_m$ while the other coordinates remain unchanged. We continue by doing this process for the next m and obtain a limit for $m \to \infty$ as every coordinate will eventually become stationary.

We shall use the following notations.

$$\begin{array}{rcl} d_\lambda & := & \operatorname{ord}(r_{0\lambda}) \\ S_{0\lambda} & := & \left\{ \begin{array}{ll} s_{0\lambda} + F_{R_0}^{d_\lambda - m+1} & \text{if } \lambda \in \Lambda_m \\ 0 & \text{otherwise} \end{array} \right. \end{array}$$

We aim at showing that the system $S_0 \in \prod_{\lambda\in\Lambda} G(R_0)$ defines an element

$$< H_0, \operatorname{in}(r_0) > \quad \mapsto \quad < H_0, S_0 > \bmod \operatorname{in}(I_0) \tag{4}$$

of the normal module

$$N_{\operatorname{in}(I_0)} = \operatorname{Hom}_{G(R_0)}(\operatorname{in}(I_0), G(R_0)/\operatorname{in}(I_0)).$$

Note r_0 is a standard base of I_0, so the elements of $\operatorname{in}(I_0)$ can be written $< H_0, \operatorname{in}(r_0) >$ with $H_0 \in \bigoplus_{\lambda\in\Lambda} G(R_0)$.

The element (4), if existing, is homogeneous of degree $-m$ as $\deg(S_{0\lambda}) - \deg(\operatorname{in}(r_{0\lambda})) = -m$ for all λ where $S_{0\lambda} \neq 0$. By assumption i) the element (4) has to be zero, i.e. all the coordinates of S_0 must be in $\operatorname{in}(I_0)$. Since r_0 is a standard base of I_0 there exists a linear map $A : \prod_{\lambda\in\Lambda} R_0 \longrightarrow \bigoplus_{\lambda\in\Lambda_m} R_0$ such that the system $s'_0 := s_0 - Ar_0$ satisfies

$$\operatorname{ord}(s'_{0\lambda}) \geq \operatorname{ord}(r_{0\lambda}) - m + 1$$

for $\lambda \in \Lambda_m$. From the decomposition $r' = r + ts_0$ one obtains $r' = r + tAr_0 + ts'_0$. hence

$$r'' := (E - tA)r' = r + ts'_0.$$

Obviously $r'' \bmod t\prod_{\lambda\in\Lambda} R = r' \bmod t\prod_{\lambda\in\Lambda} R$ and $r'' \in \prod_{\lambda\in\Lambda} I$ by construction. So r'' is another lift of $\overline{r}$ to a system of generators of I'. We may replace the decomposition $r' = r + ts_0$ by $r'' = r + ts'_0$. The proof is reduced to the assertion that (4) gives a well-defined element in $N_{\operatorname{in}(I_0)}$.

So we have to show $< H_0, S_0 > \ \in \ \operatorname{in}(I_0)$ for every system $H_0 \in \bigoplus_{\lambda\in\Lambda} G(R_0)$ satisfying $< H_0, \operatorname{in}(r_0) > \ = \ 0$. We may assume the relation $< H_0, \operatorname{in}(r_0) > \ = \ 0$ is homogeneous, i.e.

$$\deg(H_{0\lambda}) + \deg(\operatorname{in}(r_{0\lambda})) = d, \tag{5}$$

whenever $H_{0\lambda} \neq 0$. [7, 3.8] gives $G(R_0) = G(\overline{R}/\overline{m}\overline{R}) = G(\overline{R})/\operatorname{in}(\overline{m}\overline{R}) = G(\overline{R})/\operatorname{in}(\overline{m})G(\overline{R})$ as $\overline{R}$ is tangentially flat over $\overline{A}$. So we may consider $< H_0, \operatorname{in}(r_0) > \ = \ 0$ as a relation of $\operatorname{in}(\overline{r})$ modulo $\operatorname{in}(\overline{m})G(\overline{R})$. But $G(\overline{B}) = G(\overline{R})/\operatorname{in}(\overline{I}) = G(\overline{R})/\operatorname{in}(\overline{r})G(\overline{R})$ is flat over $G(\overline{A})$, so a standard argument shows the relation above can be lifted to one of $\operatorname{in}(\overline{r})$. There is a system $H \in \bigoplus_{\lambda\in\Lambda} G(R)$ of homogeneous elements satisfying

$$< H, \operatorname{in}(r) > \ \equiv \ 0 \bmod \operatorname{in}(tR) \quad \text{and} \quad H_0 = (H \bmod \operatorname{in}(m) \bigoplus_{\lambda\in\Lambda} G(R)).$$

Choose $h \in \bigoplus_{\lambda\in\Lambda} R$ such that $\operatorname{in}(h) = H$. Then $< h, r > \ \in F_R^{d+1} + tR$ and, by homogeneity (5),

$$\operatorname{ord}(h_\lambda) + \operatorname{ord}(r_{0\lambda}) \geq d$$

for all λ. As $\overline{r}$ is a standard base of $\overline{I}$ there is a system $h' \in \bigoplus_{\lambda\in\Lambda} R$ such that

$$< h, r > \ \equiv \ < h', r > \bmod tR \tag{6}$$

and

$$\operatorname{ord}(h'_\lambda) + \operatorname{ord}(r_{0\lambda}) \geq d + 1. \tag{7}$$

Let $x_0 \in R_0$ be such that $< h-h', r > \; = tx_0$. Then $\mathrm{ord}_{F_R}(tx_0) = \mathrm{ord}_{F_R}(< h-h', r >) \geq d$ and, since R is tangentially flat over A, [7, 5.10] implies

$$\mathrm{ord}_{F_{R_0}}(x_0) \geq d - e > d - m.$$

In the congruence (6) we may replace r by r', i.e. $< h - h', r' > \; \equiv 0 \bmod tR$. As $B' = R/r'R$ is flat over A this relation modulo tR comes from a relation of r' in R. There is a system $y_0 \in \bigoplus_{\lambda \in \Lambda} R_0$ such that

$$< h - h' + ty_0, r' > \; = 0.$$

That means, when indicating residue classes modulo mR by subscripts "0",

$$\begin{aligned} 0 &= \; < h - h', r' > + t < y_0, r_0 > \\ &= \; < h - h', r > + < h - h', ts_0 > + t < y_0, r_0 > \\ &= \; t(x_0 + < h_0 - h'_0, s_0 > + < y_0, r_0 >). \end{aligned}$$

As R is flat over A this means nothing but $x_0 + < h_0 - h'_0, s_0 > + < y_0, r_0 > \; = 0$ and

$$< h_0, s_0 > - < h'_0, s_0 > + x_0 \in r_0 R_0 = I_0.$$

Taking initial forms on both sides will give the desired relation $< H_0, S_0 > \; \in \mathrm{in}(I_0)$. To show this we have to estimate the orders of the terms $< h_0, s_0 >$, $< h'_0, s_0 >$ and x_0. Recall for $\lambda \in \Lambda_m$

$$\begin{aligned} \mathrm{ord}(h_{0\lambda}) + \mathrm{ord}(s_{0\lambda}) &\geq \mathrm{ord}(h_{0\lambda}) + \mathrm{ord}(r_{0\lambda}) - m \\ &\geq \mathrm{ord}(h_\lambda) + \mathrm{ord}(r_\lambda) - m \\ &\geq d - m \qquad \text{(by (5))}, \\ \mathrm{ord}(h'_{0\lambda}) + \mathrm{ord}(s_{0\lambda}) &\geq \mathrm{ord}(h'_{0\lambda}) + \mathrm{ord}(r_{0\lambda}) - m \\ &\geq \mathrm{ord}(h'_\lambda) + \mathrm{ord}(r_\lambda) - m \\ &\geq d + 1 - m \qquad \text{(by (7))}. \end{aligned}$$

As $S_{0\lambda} = 0$ for $\lambda \notin \Lambda_m$ we found

$$\mathrm{ord}_{F_{R_0}} < h_0, s_0 > \; \geq d - m, \quad \mathrm{ord}_{F_{R_0}} < h'_0, s_0 > \; \geq d + 1 - m, \quad \mathrm{ord}_{F_{R_0}} x_0 \geq d + 1 - m$$

and, therefore

$$< h_0, s_0 > + F_{R_0}^{d+1-m} \in \mathrm{in}\ (I_0).$$

But this is just the relation $< H_0, S_0 > \; \in \mathrm{in}\ (I_0)$ as for $\lambda \in \Lambda_m$

$$\begin{aligned} H_{0\lambda} &= h_{0\lambda} + F_{R_0}^{d-d_\lambda+1} \quad \text{and} \\ S_{0\lambda} &= s_{0\lambda} + F_{R_0}^{d_\lambda - m+1}. \end{aligned}$$

□

2.4 Proposition. *Let $f : (A, m, F_A) \longrightarrow (R, M, F_R)$ be a tangentially flat homomorphism of filtered local rings, $t \in A$ an element in the socle, and I an ideal in R such that $B := R/I$ is flat over A. We consider the following conditions.*

i) For every ideal I' in R such that $I + tR = I' + tR$ flatness of $B' := R/I'$ over A implies tangential flatness.

ii) $G_{F_{N_{I_0}}}(N_{I_0})(< -\mathrm{ord}_{F_A}(t)) = 0$. Here $R_0 := R/mR$ and $I_0 := IR_0$.

Then i) implies ii).

Proof. For simplicity denote $\mathrm{ord}_{F_A}(t)$ by e. We have to show $F^{-e}_{N_{I_0}} = N_{I_0}$, i.e. that every element g of the normal module $N_{I_0} = \mathrm{Hom}_{R_0}(I_0, R_0/I_0)$ satisfies the relation

$$g(I_0 \cap F^k_{R_0}) \subseteq F^{k-e}_{R_0} + I_0/I_0 \tag{8}$$

for each $k \in \mathbb{N}$.

For this let $r_0 = (r_{0\lambda})_{\lambda\in\Lambda}$ be a standard base of I_0 and $(s_{0\lambda})_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} R$ be a system representing g.

$$g(< h_0, r_0 >) \;=\; < h_0, s_0 >$$

By assumption $B = R/I$ is tangentially flat over A. So by [7, 4.6] there exists a lift r of r_0 to a standard base of I such that $\mathrm{ord}(r) = \mathrm{ord}(r_0)$. We put $r' := r + ts_0$ and let $I' := r'R$ be the ideal generated by the entries of r'. Then $B' = R/I'$ is flat over A by the description of first-order deformation in terms of the normal module [1] and even tangentially flat by assumption. The ring $B' = R/I'$ is constructed such that there are the identities $t\ (s_{0\lambda} \bmod I_0) = -r_\lambda \bmod I'$. But then [7, 5.10] implies that

$$\begin{aligned} e + \mathrm{ord}_{F_{R_0/I_0}}\ (s_{0\lambda} \bmod I_0) &= \mathrm{ord}_{F_{R/I'}}\ (r_\lambda \bmod I') \\ &\geq \mathrm{ord}_{F_R}\ (r_\lambda) \\ &= \mathrm{ord}_{F_{R_0}}\ (r_{\lambda 0}) \end{aligned}$$

giving the desired relation (8) as r_0 is a standard base for I_0. □

2.5 Theorem. *Let (B_0, y_0, E) be a monomially filtered local ring containing a field. Consider the following statements.*

i) $N_{G(B_0^\wedge)}(< -1) = 0$

ii) (B_0, y_0, E) *has only tangentially flat deformations.*

iii) $G(N_{B_0^\wedge})(< -1) = 0$

Then ii) implies iii). If E is finitely generated, then i) implies ii).

Proof. i) $\Longrightarrow$ ii) Let $f : (A, m, F_A) \longrightarrow (B, n, F_B)$ be a deformation of (B_0, y_0, E). We have to show f is tangentially flat. For this we may assume that A, B and B_0 are complete. By [7, 3.16] it is sufficient to show that $f \otimes_A A/J : A/J \longrightarrow B/JB$ is tangentially flat for every cofinite ideal J. We will proceed using induction on the length $l := \mathrm{l}(A/J)$. If $l = 1$ A/J is a field and $f \otimes A/J$ is obviously tangentially flat. So let $l > 1$.

First case: $F^1_{A/J} = 0$. Then $F^1_A \subseteq J$ and $f \otimes_A A/J$ is a surjective base change of $f \otimes_A A/F^1_A$ being tangentially flat by assumption.

Second case: $F^1_{A/J} \neq 0$. Then there is an element $0 \neq t \in F^1_{A/J}$ in the socle of A/J. By induction hypothesis $f \otimes_A (A/J)/t(A/J) : (A/J)/t(A/J) \longrightarrow (B/JB)/t(B/JB)$ is tangentially flat. By Lemma 1.6 there is a commutative diagram of homomorphisms of filtered local rings

$$\begin{array}{ccc} A/J & \overset{f\otimes_A A/J}{\longrightarrow} & B/JB \\ & \searrow_g \quad \nearrow_h & \\ & R, & \end{array}$$

where g is tangentially flat, h is surjective, $h_0 : R_0 \longrightarrow B_0$ is isomorphic to the natural surjection $L[[Y_1, \ \ldots\ , Y_r]] \longrightarrow B_0$ and $F_{B/JB} = h_* F_R$. Write $I := \ker(h)$ and identify R_0 with $L[[Y_1, \ \ldots\ , Y_r]]$. So we are in the situation of Proposition 2.3 having

$$\begin{aligned} N_{\mathrm{in}(I_0)}(< -\mathrm{ord}_{F_{A/J}}(t)) &= N_{G(B_0)}(< -\mathrm{ord}_{F_{A/J}}(t)) \\ &\subseteq N_{G(B_0)}(< -1) \\ &= 0. \end{aligned}$$

$f \otimes_A A/J : A/J \longrightarrow B/JB$ is tangentially flat in the second case, too.

ii) $\Longrightarrow$ iii) Assume as above B_0 to be complete. Suppose condition iii) is wrong. We start with the trivial first-order deformation

$$L[[\varepsilon]]/(\varepsilon)^2 =: A \xrightarrow{f} A \otimes_L B_0 =: B,$$

where $L \hookrightarrow B_0$ is a coefficient field. Equip A with any filtration and B with a filtration as described in Definition 1.3 ii). We have a natural commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow_g \quad \nearrow_h & \\ & R := A \otimes_L L[[Y_1, \ \dots \ , Y_r]] & \end{array}$$

and equip R with the sum of the filtration defined by E and $Y_1, \ \dots \ , Y_r$ with $g_*(F_A)$. Then g is tangentially flat as a residually rational base change [7, 3.17] of $L \longrightarrow L[[Y_1, \ \dots \ , Y_r]]$ being tangentially flat for trivial reason. Further h is surjective, $h_* F_R = F_B$ and the induced homomorphism $h_0 : R_0 \longrightarrow B_0$ is isomorphic to the natural map $L[[Y_1, \ \dots \ , Y_r]] \longrightarrow B_0$. Proposition 2.4 proves there must be a factor ring of R giving rise to a non-tangentially flat deformation of B_0. □

2.6 Remark. Theorem 2.5 is the main result of this paper. Is is a natural generalization of [6, Thm. (2.5)]. The following examples are singularities allowing non-tangentially flat deformations when equipped with the canonical filtration. Nonetheless they admit this rather strong property with respect to some other filtrations which are not too artificial. In the next section we are going to give applications of this phenomenon to Lech-Hironaka type inequalities.

2.7 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and $I_0 = (X^2, XY, Z^2)$, and let E be the filtrating family for the (X,Y,Z^2)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations.
Proof. We have to show $N_{G(B_0)}(< -1) = 0$. Obviously $G(R_0) = L[X,Y,Z]$ as an L-vector space, where the multiplication is given by

$$X^{a_1}Y^{b_1}Z^{c_1} \cdot X^{a_2}Y^{b_2}Z^{c_2} = \begin{cases} X^{a_1+a_2}Y^{b_1+b_2}Z^{c_1+c_2} & \text{if (9) is true} \\ 0 & \text{otherwise} \end{cases}$$

with

$$\operatorname{ord}(X^{a_1}Y^{b_1}Z^{c_1}) + \operatorname{ord}(X^{a_2}Y^{b_2}Z^{c_2}) = \operatorname{ord}(X^{a_1+a_2}Y^{b_1+b_2}Z^{c_1+c_2}). \tag{9}$$

The last condition is easily reduced to $[\frac{c_1}{2}] + [\frac{c_2}{2}] < 1$. So we see $(X^2, XY, Z^2) =: F$ is standard base for I_0. Therefore a homogeneous element of $N_{G(B_0)}$ of degree d is given by a triple (G_1, G_2, G_3) of elements of R_0 of degrees $d+2, d+2, d+1$, respectively, satisfying

$$< R, F > \ = 0 \quad \Longrightarrow \quad < R, G > \ \in \operatorname{in}(I_0).$$

The proof is then concentrated in the following table.

generators	F	X^2	XY	Z^2
a syzygy	R	Y	$-X$	
$N_{G(B_0)}(< -1)$	G	$A_1 + A_2 Z$ 0	$B_1 + B_2 Z$ 0	0

Tab. 1

The polynomials in the first row of the table are the generators F_i of $\operatorname{in}(I_0)$. Below that we have written down a syzygy of these generators needed for the proof. The third row contains a presentation of all polynomials of degree 0, respectively -1 in $G(R_0)$. Here A_i and B_i are coefficients from the ground field L. The syzygy requires $A_1Y + A_2YZ - B_1X - B_2XZ = 0$ such that all G_i need to be zero. □

2.8 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and

$$I_0 = (X^2YZ + XY^2Z, X^2YZ + XYZ^2, X^2YZ - Y^2Z^2),$$

and let E be the filtrating family for the $(X,Y,Z)^2$-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations.
Proof. For $N_{G(B_0)}(< -1)$ we get the following table.

generators	F	$X^2YZ + XY^2Z$	$X^2YZ + XYZ^2$	$X^2YZ - Y^2Z^2$
degrees		2	2	2
some syzygies	R_1	XZ	$-YZ$	$-XZ$
	R_2	Z^2	$-XZ$	XZ
$N_{G(B_0)}(< -1)$	G	$A_1+B_1X+C_1Y+D_1Z$	$A_2+B_2X+C_2Y+D_2Z$	$A_3+B_3X+C_3Y+D_3Z$

Tab. 2

Here R_1 gives $A_2 = C_2 = D_2 = 0$ and R_2 implies $A_1 = C_1 = D_1 = 0$. But then we get $A_3 = D_3 = 0$ from R_1 and $C_3 = 0$ from R_2. Now R_1 requires $B_2 = 0$ while R_2 implies $B_1 = 0$. Finally $B_3 = 0$ by R_1 or R_2. □

2.9 We can mix the situation of a complete intersection being simple from the point of view of deformation theory with situations being trivial from the point of view of tangential flatness.
Proposition. *a) Let $X_1, \dots, X_t$ be sets of indeterminates consisting of at least two elements each and $Y_1, \dots, Y_r$ be indeterminates. Put $B_0 = R_0/I_0$, where $R_0 = L[[X_1, \dots, X_t, Y_1, \dots, Y_r]]$ and $I_0 = (X_1^{e_1}, \dots, X_t^{e_t}, Y_1^{d_1}, \dots, Y_r^{d_r})$. Let E be the filtrating family for the $(X_1, \dots, X_t, Y_1^{d_1}, \dots, Y_r^{d_r})$-adic filtration. Then $(B_0, (\underline{X}, \underline{Y}), E)$ admits only tangentially flat deformations.*

b) Instead of all monomials in X_i of a given degree one could have used t systems of power products as described in [6, Prop. (3.1)] and put

$$I_0 = (s_1^1, \dots, s_1^{N_1}, \dots, s_t^1, \dots, s_t^{N_t}, Y_1^{d_1}, \dots, Y_r^{d_r}).$$

Proof. The system of generators given for I_0 is a standard base. The monomials in $\underline{Y}$ do not cause any difficulty as they are of degree 1. But for the monomials in $\underline{X}$ one can argue as in [6, Prop. (3.1)]. □

2.10 Theorem 2.5 gives a necessary and a sufficient condition for a monomially filtered local ring to have tangentially flat deformations only. Unfortunately, they are not equivalent in general as the following example shows.
Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and $I_0 = (X^2, XY, XZ, Y^2, YZ, Z^3)$, and let E be the filtrating family for the (X,Y,Z^2)-adic filtration. Then $N_{G(B_0)}(< -1) \neq 0$ but $G(N_{B_0})(< -1) = 0$.
Proof. Let us show $N_{G(B_0)}(< -1) \neq 0$ first. $G(B_0)$ is as in Example 2.7. So in order to get a standard base of I_0 we have to add certain generators.

number	i	1	2	3	4	5	6	7	8	9
standard base	F_i	X^2	XY	XZ	XZ^2	Y^2	YZ	YZ^2	Z^3	Z^4
degree		2	2	1	2	2	1	2	1	2
syzygies	R_{ij}									
an element from $\mathrm{N}_{\mathrm{G(B_0)}}(-2)$	G_i	0	0	0	0	0	0	0	0	Z

Tab. 3

We have to show $R_{j9} \cdot Z = \; < R_j, G > \; \in \mathrm{in}(I_0)$ for every syzygy. But for $a + b + c \geq 2$ the product $X^aY^bZ^c \cdot Z$ is zero or a monomial whose sum of exponents is at least 3. In both cases it is in $\mathrm{in}(I_0)$. Further $X \cdot Z \in \mathrm{in}(I_0)$, $Y \cdot Z \in \mathrm{in}(I_0)$ and $Z \cdot Z = 0 \in \mathrm{in}(I_0)$. So the only critical case is when R_{j9} contains a constant term. But this is impossible as $Z \cdot F_8 = Z \cdot Z^3 = 0$.

Consider $G(N_{B_0})(< -1)$ now. We have to start with a system F of generators for I_0. For simplicity we choose the standard base above. An element in N_{B_0} is given by a 9-tuple $G \in R_0^9$ satisfying $< R, G > \ \in I_0$ for every syzygy of F. We obtain the following table.

number	i	1	2	3	4	5	6	7	8	9
system of generators	F_i	X^2	XY	XZ	XZ^2	Y^2	YZ	YZ^2	Z^3	Z^4
order		2	2	1	2	2	1	2	1	2
some syzygies	R_{1i}	Z		$-X$						
	R_{2i}		Z				$-X$			
	R_{3i}					Z	$-Y$			
	R_{4i}				Z				$-X$	
	R_{5i}							Z	$-Y$	
	R_{6i}								Z	-1
	R_{7i}			$-Z^2$					X	
N_{B_0}	G_i									

Tab. 4

We want to show $\mathrm{ord}(G_i) \geq \mathrm{ord}(F_i) - 1$. This is trivial for $i = 3, 6, 8$. For the other i we need G_i cannot contain terms of the type C or CZ. For $i = 1, 2, 4, 5$ and 7 this clear from the syzygies R_1, R_2, R_4, R_3, respectively R_5. Consider $i = 9$. R_6 would allow only that G_9 contains CZ while C is in G_8. But this is forbidden by syzygy R_7. As F was chosen to be a standard base we have shown $N_{B_0} = F^{-1}_{N_{B_0}}$ as desired. □

3. Lech-Hironaka type inequalities

3.1 Singularities with only tangentially flat deformations with respect to the canonical filtration give rise to a large class of homomorphisms for which the Lech-Hironaka problem has trivially a positive answer. We will show here that the local rings admitting only tangentially flat deformations for some other filtration give inequalities of exactly the same kind. The only difference is that one possibly gets some coefficients into the play.

3.2 Lemma. *Let (R, M) be a local ring, $f_1, \ \ldots \ , f_r \in R[X]$ be polynomials without constant term and $x \in M$. Then for any $d \in \mathbb{N}$*

$$\mathrm{l}\,(R/(f_1(x^d), \ \ldots \ , f_r(x^d))) \leq d \cdot \mathrm{l}\,(R/(f_1(x), \ \ldots \ , f_r(x)))$$

as soon as the right hand side is finite.

Proof. It is sufficient to show the desired inequality for all factor rings R/M^i. So assume without restriction R to be Artin. Equip R with the (x)-adic filtration. Then $S := G_{(x)}(R)$ is a factor ring of $R_0[X]$ by some homogeneous ideal I, where $R_0 := R/(x)$ is an Artin local ring. Further we have

$$\begin{aligned} \mathrm{l}\,(R/(f_1(x), \ \ldots \ , f_r(x))) &= \mathrm{l}\,(G(R/(f_1(x), \ \ldots \ , f_r(x)))) \\ &= \mathrm{l}\,(G(R)/\mathrm{in}(f_1(x), \ \ldots \ , f_r(x))) \\ &= \mathrm{l}\,(R_0[X]/(I, H)) \end{aligned}$$

and the analogous result

$$\mathrm{l}\,(R/(f_1(x^d), \ \ldots \ , f_r(x^d))) = \mathrm{l}\,(R_0[X]/(I, H^{(d)}))$$

for the other ideal in question.

We claim $H^{(d)}$ is generated by $\{F(X^d)|F \in H\}$. Clearly, if $F \in H$, i.e. if F is the homogeneous initial form of some element from $(f_1(x), \ \dots \ , f_r(x))$, then $F(X^d) \in H^{(d)}$. For the other direction let $G \in H^{(d)}$, i.e. the initial form of some element $g \in (f_1(x^d), \ \dots \ , f_r(x^d))$. Then

$$\begin{array}{rcl} g & = & h_1^0(x^d)f_1(x^d) + \ \dots \ + h_r^0(x^d)f_r(x^d) \\ & + & x \ [h_1^1(x^d)f_1(x^d) + \ \dots \ + h_r^1(x^d)f_r(x^d)] \\ & & \vdots \\ & + & x^{d-1}[h_1^{d-1}(x^d)f_1(x^d) + \ \dots \ + h_r^{d-1}(x^d)f_r(x^d)], \end{array}$$

whose initial form G is obviously in the ideal generated by the set given above.

We note, as everything is Artin, we could have dealt with power series in X instead of polynomials. As I is a homogeneous ideal we have a local homomorphism of local rings

$$R_0[[X]]/(I,H) \overset{X \mapsto X^d}{\longrightarrow} R_0[[X]]/(I,H^{(d)})$$

whose special fiber is of length d. Consequently, $\mathrm{l}(R_0[[X]]/(I,H^{(d)})) \leq d \cdot \mathrm{l}(R_0[[X]]/(I,H))$. □

3.3 Definition. Let (B_0, n_0) be a local ring of dimension d and I be an n_0-primary ideal in it. By the Hilbert-Serre theorem the Hilbert series of B_0 with respect to I can be written in the form

$$H^1_{B_0,I}(T) = \frac{P(T)}{(1-T)^{d+1}} + \sum_{i=1}^{d} \frac{p_i(T)}{(1-T)^i},$$

where P and p_i are polynomials with non-negative coefficients. Define the *vage multiplicity* $\nu_I(B_0)$ of B_0 with respect to I as the maximal value of $P(0)$ occurring in decompositions of the type above. Note, if B_0 is Artin, then $\nu_I(B_0) = \mathrm{l}(B_0/I)$. In any case $\nu_I(B_0) \geq 1$ as $H^1_{B_0,I} \geq H^1_{B_0,n_0}$ and $\frac{1}{(1-T)^{d+1}}$ is the minimal Hilbert series possible in dimension d.

3.4 Theorem. *Let $B_0 = R_0/I_0$, where $R_0 = L[[X_1, \ \dots \ , X_r]]$, be a d-dimensional local ring and let E be a filtrating family such that $(B_0, (\overline{X_1}, \ \dots \ , \overline{X_r}), E)$ admits only tangentially flat deformations. Assume $(d_1, 0, \ \dots \ , 0), \ \dots \ , (0, \ \dots \ , 0, d_r) \in E_1$ (i.e. $(X_1^{d_1}, \ \dots \ , X_r^{d_r}) \subseteq F^1_{R_0}$). Further let $I \in B_0$ be some ideal such that $F^i_{B_0} \subseteq I^i$ for all i. Then for any deformation $f : (A,m) \longrightarrow (B,n)$ of B_0*

$$e(B) \ \geq \ \frac{e_I(B_0)}{d_1 \cdot \ldots \cdot d_r} e(A) \quad \text{and} \tag{10}$$

$$H^1_B(i) \ \geq \ \frac{\nu_I(B_0)}{d_1 \cdot \ldots \cdot d_r} H^{d+1}_A(i) \tag{11}$$

for all i. If B_0 is Artin one has even

$$H^1_B(i) \geq \frac{\mathrm{l}(B_0/F^1_{B_0})}{d_1 \cdot \ldots \cdot d_r} H^1_A(i). \tag{12}$$

Proof. Equip A with the m-adic filtration and B with one of the filtrations described in Definition 1.3 ii). Clearly we obtain a deformation of the monomially filtered local ring $(B_0, (\overline{X_1}, \ \dots \ , \overline{X_r}), E)$, which is tangentially flat by assumption. [7, 6.13] implies the equality $H^1_{B,F_B} = H^0_A H^1_{B_0,F_{B_0}}$ between Hilbert series.

The maximal ideal n in B is generated by some $z_1, \ \dots \ , z_s \in f(m)$ and the lift $(y_{10}, \dots, y_{r0})$ chosen. We obtain $F_B^i \supseteq (z_1, \ \dots \ , z_s, y_{10}^{d_1}, \ \dots \ , y_{r0}^{d_r})^i$ and

$$\begin{array}{rcl} H^1_{B,F_B}(i) & \leq & \mathrm{l}(B/(z_1, \ \dots \ , z_s, y_{10}^{d_1}, \ \dots \ , y_{r0}^{d_r})^{i+1}) \\ & \leq & d_1 \cdot \ldots \cdot d_r \cdot \mathrm{l}(B/(z_1, \ \dots \ , z_s, y_{10}, \ \dots \ , y_{r0})^{i+1}) \\ & = & d_1 \cdot \ldots \cdot d_r \cdot H^1_B(i), \end{array}$$

where the middle inequality is nothing but an r-fold application of Lemma 3.2. Consequently,

$$H^1_B \geq \frac{H^1_{B_0,F_{B_0}}}{d_1 \cdot \ldots \cdot d_r} H^0_A.$$

Furthermore $H^1_{B_0,F_{B_0}}(T) \geq H^1_{B_0,I}(T) \geq \frac{P(T)}{(1-T)^{d+1}}$. So the estimate $P(T) \geq \nu_I(B_0)$ implies assertion (11). On the other hand assertion (10) comes the fact that $e_I(B_0) = P(1)$ and there are analogous formulas for $e(A)$ and $e(B)$. For (12) in the case B_0 is Artin we simply use $H^1_{B_0,F_{B_0}}(T) \geq \mathrm{l}(B_0/F^1_{B_0})\frac{1}{1-T}$. □

3.5 Remarks. a) In Example 2.7 one has $H^1_B \geq H^2_A$. Indeed

$$\begin{array}{rcl} \mathrm{l}((L[[X,Y,Z]]/(X^2,XY,Z^2))/(X,Y,Z^2)) & = & 2 \quad \text{and} \\ \mathrm{l}((L[[X,Y,Z]]/(X^2,XY,Z^2))/(X,Y,Z^2)^{i+1}) & = & 2i+4. \end{array}$$

for $i \geq 1$. Consequently $H^1_{B_0,F_{B_0}}(T) = \frac{2}{(1-T)^2} + \frac{4T}{1-T}$.

b) For Example 2.8 we get $H^1_B \geq H^3_A$. In fact

$$\mathrm{l}(n_0^i/n_0^{i+1}) = \begin{cases} 2i+2 & \text{if } i \leq 2 \\ 2i+5 & \text{otherwise} \end{cases}$$

as the monomials X^aY^b and X^aZ^b are not affected by the relations given and among the $(X^aY^bZ^c)YZ$ there are the relations $X^2(YZ) = -XY(YZ)$, $X^2(YZ) = -XZ(YZ)$ and $X^2(YZ) = YZ(YZ)$ bringing every monomial into the form $X^i(YZ)$, $Y^i(YZ)$ or $Z^i(YZ)$. One easily computes $H^1_{B_0,F_{B_0}}(T) = \frac{8}{(1-T)^3} + \frac{6+3T+3T^2}{(1-T)^2}$.

c) For deformations of the singularities from 2.9 we get $H^1_B \geq \nu(L[[\underline{X}]]/(\underline{s})) \cdot H^{d+1}_A$ and $e(B) \geq e(L[[\underline{X}]]/(\underline{s})) \cdot e(A)$.

3.6 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and $I_0 = (X^2, XY, Y^2, XZ^2, YZ^2, Z^3)$, and let E be the filtrating family for the (X,Y,Z^2)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. In particular, for every deformation $f : A \longrightarrow B$ of B_0 one has $e(B) \geq \frac{7}{2}e(A)$ and $H^1_B \geq H^1_A$.

Proof. We easily see $\mathrm{l}(B_0) = 7$ and $\mathrm{l}(B_0/F^1_{B_0}) = 2$. Tangential flatness of all deformations is shown in the table below.

standard base	X^2	XY	Y^2	XZ^2	YZ^2	Z^3	Z^4
degrees	2	2	2	2	2	1	2
some syzygies	Y	$-X$					
	Z^2			$-X$			
		Y	$-X$				
		Z^2			$-X$		
				Z^2			$-X$
$N_{G(B_0)}(<-1)$	A_1+A_2Z 0	B_1+B_2Z 0	C_1+C_2Z 0	D_1+D_2Z 0	E_1+E_2Z 0	0	F_1+F_2Z 0

Tab. 5

□

3.7 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and $I_0 = (X^2, XY, Y^2, XZ^2, Z^3)$, and let E be the filtrating family for the (X,Y,Z^2)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. In particular, for every deformation of B_0 one has $e(B) \geq 4e(A)$ and $H^1_B \geq H^1_A$.
Proof. We have $\mathrm{l}(B_0) = 8$ and $\mathrm{l}(B_0/F^1_{B_0}) = 2$ and get the following table.

standard base	X^2	XY	Y^2	XZ^2	Z^3	Z^4
degrees	2	2	2	2	1	2
some syzygies	Y	$-X$				
	Z^2			$-X$		
		Y	$-X$			
				Z^2		$-X$
$N_{G(B_0)}(< -1)$	$A_1 + A_2Z$ 0	$B_1 + B_2Z$ 0	$C_1 + C_2Z$ 0	$D_1 + D_2Z$ 0	0	$E_1 + E_2Z$ 0

Tab. 6

□

3.8 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and $I_0 = (X^3, XZ, Y^3, Y^2Z, Z^2)$, and let E be the filtrating family for the (X^2, XY, Y^2, Z)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. For every deformation of B_0 the inequalities $e(B) \geq \frac{11}{4}e(A)$ and $H^1_B \geq \frac{3}{4}H^1_A$ are true.
Proof. To get a standard base we have to add further generators. The proof is concentrated in the table below.

standard base		X^3	XZ	Y^3	X^4	X^3Y	X^2Z	XYZ	XY^3	Y^4	Y^2Z	Z^2
number		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
degrees		1	1	1	2	2	2	2	2	2	2	2
some syzygies	R_1						XY	$-X^2$				
	R_2							XY			$-X^2$	
	R_3										Z	$-Y^2$
	R_4						$-Z$					X^2
	R_5				XY	$-X^2$						
	R'_5				Y^2	$-XY$						
	R_6								$-Y^2$	XY		
	R'_6								$-XY$	X^2		
	R_7					Z	$-XY$					
	R_8									Z	$-Y^2$	
$N_{G(B_0)}(< -1)$		0	0	0	0	0	0	0	0	0	0	0

Tab. 7

In principal for generators of degree 2 linear combinations of type $A + BX + CY$ would be possible. R_1 and R_2 give 0 at number (6), R_2 and R_3 0 at (7). Further, one considers R_2 and R_8 for number (10), R_3 and R_4 for (11), R_5, R'_5 and R_7 for numbers (4) and (5) and R_6, R'_6 and R_8 for (8) and (9). □

3.9 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X,Y,Z]]$ and $I_0 = (X^3, XY^2, XZ, Y^3, Y^2Z, Z^2)$, and let E be the filtrating family for the (X^2, XY, Y^2, Z)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. For every deformation of B_0 one has the inequalities $e(B) \geq \frac{9}{4}e(A)$ and $H^1_B \geq \frac{3}{4}H^1_A$.

Proof. We obtain the table below.

standard base		X^3	XZ	XY^2	Y^3	X^2Z	XYZ	Y^2Z	X^4	X^3Y	X^2Y^2	XY^3	Y^4	Z^2
number		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
degrees		1	1	1	1	2	2	2	2	2	2	2	2	2
some syzygies						Y^2	$-XY$							
						XY	$-X^2$							
							Y^2	$-XY$						
							XY	$-X^2$						
									Y^2	$-XY$				
									XY	$-X^2$				
										Y^2	$-XY$			
										XY	$-X^2$			
											Y^2	$-XY$		
											XY	$-X^2$		
												Y^2	$-XY$	
												XY	$-X^2$	
	R_1					X^2			$-Z$					
	R_2						X^2			$-Z$				
	R_3					Z								$-X^2$
	R_4						Z							$-XY$
$N_{G(B_0)}(<-1)$		0	0	0	0	0	0	0	0	0	0	0	0	0

Tab. 8

Trivially we have zeros at numbers (1) to (4). At the other generators there could be linear combinations like $A+BX+CY$. R_1 and R_2 yield that the coefficient of Y disappears at (5), (6), (8) and (9). The pairs of syzygies $((Y^2,-XY),(XY,-X^2))$ allow for the normal module only linear combinations of (X,Y) and $(Y,0)$. Consequently, we have 0 everywhere, maybe except at number (13). But there the zero comes from R_3 and R_4. □

3.10 Example. Let $B_0=R_0/I_0$, where $R_0=L[[X,Y,Z]]$ and $I_0=(X^3,X^2Y,Y^3,XYZ,Y^2Z,Z^2)$, and let E be the filtrating family for the (X^2,XY,Y^2,Z)-adic filtration. Then $(B_0,(\overline{X},\overline{Y},\overline{Z}),E)$ admits only tangentially flat deformations. For every deformation $f:A\longrightarrow B$ of B_0 one has $e(B)\geq\frac{11}{4}e(A)$ and $H_B^1\geq\frac{3}{4}H_A^1$.
Proof. Here we obtain the following table.

standard base		X^3	X^2Y	Y^3	XYZ	Y^2Z	X^4	X^3Y	X^2Y^2	XY^3	Y^4	Z^2
number		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
degrees		1	1	1	1	2	2	2	2	2	2	2
some syzygies							Y^2	$-XY$				
							XY	$-X^2$				
								Y^2	$-XY$			
								XY	$-X^2$			
									Y^2	$-XY$		
									XY	$-X^2$		
										Y^2	$-XY$	
										XY	$-X^2$	
	R_1					Z						$-Y^2$
	R_2				Z							$-XY$
	R_3					Y^2					$-Z$	
	R_4					XY				$-Z$		
$N_{G(B_0)}(<-1)$		0	0	0	0	0	0	0	0	0	0	0

Tab. 9

Here the pairs $((Y^2, -XY), (XY, -X^2))$ of syzygies allow only linear combinations of (X, Y) and $(0, X)$. But R_3 and R_4 yield that the coefficients of X at (9) and (10) disappear. We get everything is zero except maybe the entries at numbers (4), (5) and (11). Considering the syzygies R_1 and R_2 completes the proof. □

3.11 Proposition (ramified coverings). *Let $e := (e_1, \ldots, e_r)$ be some r-tuple of positive integers and consider the homomorphism $R_0 := L[[X_1, \ldots, X_r]] \longrightarrow R_0^e := L[[X_1, \ldots, X_r]]$ of local rings given by $X_i \mapsto X_i^{e_i}$. For every R_0-module M denote the tensor product $M \otimes_{R_0} R_0^e$ by $M^{(e)}$.*

For some factor ring $B_0 = R_0/I_0$ and a filtrating family E assume $G(R_0)$ to be Noetherian and $N_{G_{F_{B_0}}(B_0)}(< -1) = 0$. Then $N_{G_{F_{B_0^{(e)}}}(B_0^{(e)})}(< -1) = 0$. In particular $B_0^{(e)}$ admits only tangentially flat deformations.

Proof. $R_0^{(e)}$ is flat over R_0. Therefore $G_{F^{(e)}}(R_0^{(e)}) = G_F(R_0) \otimes_{R_0} R_0^{(e)}$ and $R_0^{(e)}$ is even tangentially flat. Consequently,

$$G_{F^{(e)}}(B_0^{(e)}) = G_{F^{(e)}}(R_0^{(e)})/\mathrm{in}_F(I_0) \cdot G_{F^{(e)}}(R_0^{(e)}) = G_F(B_0) \otimes_{R_0} R_0^{(e)}$$

and, in particular, $\mathrm{in}_{F^{(e)}}(I_0^{(e)}) = \mathrm{in}_F(I_0) \otimes_{R_0} R_0^{(e)}$. For the normal modules one gets

$$\begin{aligned} N_{G(B_0^{(e)})}(B_0^{(e)}) &= \mathrm{Hom}_{G(R_0) \otimes_{R_0} R_0^{(e)}}(\mathrm{in}_F(I_0) \otimes_{R_0} R_0^{(e)}, G_F(B_0) \otimes_{R_0} R_0^{(e)}) \\ &= N_{G(B_0)}(B_0) \otimes_{R_0} R_0^{(e)} \end{aligned}$$

as $\mathrm{in}_F(I_0)$ is finitely generated. The isomorphism constructed respects the gradings. □

3.12 Corollary. *Let $B_0 = R_0/I_0$, where $R_0 = L[[X_1, \ldots, X_r]]$, be a d-dimensional local ring and let E be a finitely generated filtrating family such that $N_{G(B_0)}(< -1) = 0$. Assume $(d_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_r) \in E_1$. Further let $I \subseteq B_0$ be some ideal such that $F_{B_0}^i \subseteq I^i$ for all i. Then for any deformation $f : (A, m) \longrightarrow (B, n)$ of one of the $B_0^{(e)}$*

$$\begin{aligned} e(B) &\geq \frac{e_I(B_0)}{d_1 \cdot \ldots \cdot d_r} e(A) \quad \textit{and} \\ H_B^1(i) &\geq \frac{\nu_I(B_0)}{d_1 \cdot \ldots \cdot d_r} H_A^{d+1}(i) \end{aligned}$$

for all i. If B_0 is Artin one has even

$$H_B^1(i) \geq \frac{\mathrm{l}(B_0/F_{B_0}^1)}{d_1 \cdot \ldots \cdot d_r} H_A^1(i).$$

4. Initial monomials

4.1 In the previous sections we mainly considered examples $B_0 = L[[X_1, \ldots, X_r]]/I_0$, where I_0 was generated by monomials. That is because in other cases the computation of the normal module will usually become hopelessly complicated. Nevertheless based on the deformation to the normal cone there is a method to reduce the question whether the normal module of the associated graded ring to some $B_0 = R_0/I_0$ has non-zero elements in degrees less than -1 to the analogous question for some other ring $B_0' = R_0/I_0'$ where I_0' is generated by monomials.

4.2 Definition. Let G be an ordered abelian group and $H \subseteq G$ be a semigroup. An *H-filtration* on a ring R is a system $F = (F^g)_{g \in H}$ of ideals satisfying

i) $F^{g_1} \supseteq F^{g_2}$ if $g_1 \preceq g_2$,
ii) $F^{g_1} \cdot F^{g_2} \subseteq F^{g_1+g_2}$.

The *associated H-graded ring* to R is given by

$$\mathrm{G}_F(R) := \bigoplus_{g\in H} F^g/(\sum_{g\prec g'} F^{g'}).$$

We note that the concept of an *initial ideal* carries over directly to the situation of an H-filtered ring. Any element in R has a well-defined *order* in $H \cup \{\infty\}$.

On R let F' be a filtration and F an H-filtration. F is said to *refine* F' if for every $d \in \mathbb{N}$ there is a $k \in H$ such that $F'^d = F^k$.

By a *filtration* on an R-module M we always mean a G-filtration, i.e. a system $(F_M^g)_{g\in G}$ of submodules satisfying
i) $F_M^{g_1} \supseteq F_M^{g_2}$ if $g_1 \preceq g_2$,
ii) $F^g \cdot F_M^{g'} \subseteq F_M^{g+g'}$.

Then $G_{F_M}(M) := \bigoplus_{g\in G} F_M^g/(\sum_{g\prec g'} F_M^{g'})$ is endowed with a natural structure of a graded module over $G(R)$.

4.3 Example. Let $R_0 := L[[X_1, \ \dots \ ,X_r]]$. Then put $G := \mathbb{Z}^r$ and $H := \mathbb{N}^r$ and equip G with an ordering inducing the order type $\omega \cong (\mathbb{N}, \leq)$ on H. We get an H-filtration given by

$$F^g := \Big((y_{01}^{g_1} \cdot \ \dots \ \cdot y_{0r}^{g_r}) \mid g \preceq (g_1, \dots, g_r) \Big).$$

If $\mathrm{l}(F^g/\sum_{g\prec g'} F^{g'}) = 1$ for each $g \in H$ the direct image p_*F under the natural map to a monomially filtered local ring B_0 will be called a *full monomial filtration.* Note initial ideals with respect to full monomial filtrations are generated by monomials.

Let F' be the weight filtration defined by $(q_1, \ \dots \ ,q_r)$ on (B_0, y_0). Then there exists a full monomial filtration F refining F'. Indeed put

$$(g_1, \ \dots \ ,g_r) \prec (g_1', \ \dots \ ,g_r') \ := \ \begin{cases} g_1q_1 + \ \dots \ + g_rq_r < g_1'q_1 + \ \dots \ + g_r'q_r & \text{or} \\ g_1q_1 + \ \dots \ + g_rq_r = g_1'q_1 + \ \dots \ + g_r'q_r & \text{and} \\ (g_1, \ \dots \ ,g_r) \text{ precedes } (g_1', \ \dots \ ,g_r') \text{ lexicographically.} \end{cases}$$

Analogously for the $(y_{01}^{a_1}, \ \dots \ ,y_{0r}^{a_r})$-adic filtration one can choose the ordering

$$(g_1, \ \dots \ ,g_r) \prec (g_1', \ \dots \ ,g_r') \ := \ \begin{cases} \lfloor \frac{g_1}{a_1} \rfloor + \ \dots \ + \lfloor \frac{g_r}{a_r} \rfloor < \lfloor \frac{g_1'}{a_1} \rfloor + \ \dots \ + \lfloor \frac{g_r'}{a_r} \rfloor & \text{or} \\ \lfloor \frac{g_1}{a_1} \rfloor + \ \dots \ + \lfloor \frac{g_r}{a_r} \rfloor = \lfloor \frac{g_1'}{a_1} \rfloor + \ \dots \ + \lfloor \frac{g_r'}{a_r} \rfloor & \text{and} \\ (g_1, \ \dots \ ,g_r) \text{ precedes } (g_1', \ \dots \ ,g_r') \text{ lexicographically.} \end{cases}$$

4.4 Proposition. *Let $R_0 = (L[[X_1, \ \dots \ ,X_r]]$ and $(R_0/I_0, (X_1, \ \dots \ ,X_r), E)$ be a monomially filtered local ring, where $G(R_0)$ is supposed Noetherian. Assume F is a full monomial filtration refining F_{R_0}.*

Abusing notation we consider $G_F(R_0) \cong L[X_1, \ \dots \ ,X_r]$ as a subring of R_0. Let $I_0' := \mathrm{in}_F(I_0)R_0$ be the ideal generated by all initial forms of the elements $i_0 \in I_0$. The following implication is true:

$$N_{\mathrm{in}_{F_{R_0}}}(I_0')(< -1) = 0 \Longrightarrow N_{\mathrm{in}_{F_{R_0}}}(I_0)(< -1) = 0.$$

Proof. Apply the Lemma below to $R := G_F(R_0)$. $\square$

4.5 Lemma. *Let G be an ordered abelian group and $H \subseteq G$ a semigroup, which generates G and is isomorphic to $\mathbb{N}$ as an ordered set. Consider an H-filtered Noetherian ring R and an ideal $I \subseteq R$. Assume $G_{F_R}(R)$ is Noetherian, too. Then the normal module N_I carries a filtration given by $F^g_{N_I} := \{f \in N_I \mid \forall k \in \mathbf{H} : f(I \cap F^k_R) \subseteq \sum_{k+g \preceq l}(F^l_R + I/I)\}$.*

There is a graded injective homomorphism of degree zero

$$\mathrm{G}(N_I) \longrightarrow N_{\mathrm{in}_{F_R}(I)}. \tag{13}$$

Moreover, if $R = \bigoplus_{k\in\mathbb{N}} R(k)$ is an $\mathbb{N}$-graded ring, $I = \bigoplus_{k\in\mathbb{N}} I(k)$ a homogeneous ideal and the filtration F_R is compatible with the graded structure

$$F^d_R = \bigoplus_{k\in\mathbb{N}} F^d_R(k), \quad F^d_R(k) := F^d_R \cap R(k),$$

then the $\mathbb{N}$-graded structure on R induces $\mathbb{Z}$-graded structures on $G_{F_R}(R)$, $\mathrm{in}_{F_R}(I)$, $G(N_I)$, and $N_{\mathrm{in}_{F_R}(I)}$, respectively, such that (13) is also graded of degree zero with respect to these additional graded structures. In particular, if

$$N_{\mathrm{in}_{F_R}(I)}(< -1) = 0$$

and F_R as well as the filtration induced by F_R on R/I are separated, then $N_I(< -1) = 0$.

Proof. This is a generalized version of [6, Lemma (4.7)]. Define a G-filtration on R by $F^g_R := \sum_{g' \in H, g \preceq g'} F^{g'}_R$. Then the construction of (13) and the proof of injectivity carry over immediately when using the successor of some element d in G is replaced by the consideration of a sum indexed by all elements strictly succeeding d. $N_{\mathrm{in}_{F_R}(I)}(< -1) = 0$ immediately implies $\mathrm{G}(N_I(< -1)) = 0$. In order to deduce $N_I(< -1) = 0$ we need every non-zero element $g \in N_I$ admits a non-zero initial form.

As R and $G_{F_R}(R)$ are Noetherian, there exists a finite standard base $(r_1, \dots, r_s)$ generating I (and not containing 0). The orders of its elements will be denoted $d_1, \dots, d_s \in H$. Since H is a system of generators for G there exists one $a \in G$ such that $a - d_i \in H$ for every i. Thus the filtration on N_I lives on $-a + H$ only, which is isomorphic to $\mathbb{N}$ as an ordered set. As R/I is separated, $0 \neq g \in N_I$ admits a well-defined order in $-a + H$ and a non-zero initial form. □

4.6 Remark. In the special case of a weight filtration with rational weights $(q_1, \dots, q_r)$, which is in fact the most interesting one for applications, there is another proof for Proposition 4.4, which is more geometric in nature and therefore may be more enlightening. Consider the following weighted version of the deformation to the normal cone (cf. [3, Ch. 5]).

Let $I_0 = (f_1, \dots, f_s)$. We choose an integer M such that all Mq_i become natural numbers. Define $I_{(\lambda)} := (f^{(\lambda)}_1, \dots, f^{(\lambda)}_s)$, where $f^{(\lambda)}_i(X_1, \dots, X_r) := f_i(\lambda^{Mq_1}X_1, \dots, \lambda^{Mq_r}X_r)$. We obtain an algebraic family

$$\mathcal{X} = \mathrm{Spec}\ L[\lambda, X_1, \dots, X_r]/\mathcal{I} \longrightarrow \mathbf{A}^1 = \mathrm{Spec}\ L[\lambda],$$

whose fibers are isomorphic to Spec R_0/I'_0 if $\lambda = 0$ and to Spec R_0/I_0 otherwise. Going over to the associated graded objects defines a family of weighted affine cones over $\mathbf{A}^1$. Then the semicontinuity argument from [6, Proposition (4.5)] carries over immediately.

4.7 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X, Y, Z]]$ and $I_0 = (X^2, XY, Y^2, Z^3 + XZ)$, and let E be the filtrating family for the (X, Y, Z^3)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. In particular, for every deformation $f : A \longrightarrow B$ of B_0 one has $e(B) \geq 3e(A)$ and $H^1_B \geq H^1_A$.

Proof. We deal with the weight filtration defined by $(1, 1, \frac{1}{3})$. So consider the refining full monomial filtration described in Example 4.3. By Proposition 2.9 it is sufficient to show $I_0' = (X^2, XY, Y^2, Z^3)$, i.e. that $(X^2, XY, Y^2, Z^3 + XZ)$ is a standard base for I_0. Consider the table below.

generators of I_0	s	X^2	XY	Y^2	$Z^3(+XZ)$
generators of the syzygy module	r^1	Y	$-X$		
(except trivial syzygies)	r^2		Y	$-X$	

Tab. 10

We get $< r^1, s > \ = \ < r^2, s > \ = 0$ and see s is a standard base for I_0 [6, Lemma (4.3)]. □

4.8 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X, Y, Z]]$ and

$$I_0 = (X^2 + YZ^2, XY, Y^2 + YZ^2, XZ^2, Z^3 + YZ^2),$$

and let E be the filtrating family for the (X, Y, Z^2)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. In particular, for every deformation $f : A \longrightarrow B$ of B_0 one has $e(B) \geq 4e(A)$ and $H^1_B \geq H^1_A$.
Proof. Here we deal with the weight filtration defined by $(1, 1, \frac{1}{3})$. So consider the refining full monomial filtration from Example 4.3. We aim at showing that the system of generators given for I_0 is a standard base. Note the monomials written first in the sums above always precede the other summand (Z comes after X and Y). We get the following table.

generators of I_0	s	$X^2(+YZ^2)$	XY	$Y^2(+YZ^2)$	XZ^2	$Z^3(+YZ^2)$	$< r^j, s >$
generators of the syzygy module	r^1	Y	$-X$				Y^2Z^2
(except trivial syzygies)	r^2	Z^2			$-X$		YZ^4
	r^3		Y	$-X$			$-XYZ^2$
	r^4		Z^2		$-Y$		0
	r^5				Z	$-X$	$-XYZ^2$

Tab. 11

The error terms occurring can be expressed by linear combinations in s such that $\mathrm{ord}_F(h^j_i) + \mathrm{ord}_F(s_i)$ succeeds (in $(\mathbb{Z}^3, \prec)$) the syzygy order of r^j.

generators of I_0	s	$X^2(+YZ^2)$	XY	$Y^2(+YZ^2)$	XZ^2	$Z^3(+YZ^2)$	$< r^j, s >$
coefficients	h^1			$Z^2 \cdot \frac{1}{1-Z}$		$-YZ \cdot \frac{1}{1-Z}$	Y^2Z^2
	h^2			$-Z^3 \cdot \frac{1}{1-Z}$		$YZ \cdot \frac{1}{1-Z}$	YZ^4
	h^3				$-Y$		$-XYZ^2$
	h^4						0
	h^5				$-Y$		$-XYZ^2$

Tab. 12

By Example 3.7 the proof is complete. □

4.9 Example. Let $B_0 = R_0/I_0$, where $R_0 = L[[X, Y, Z]]$ and

$$I_0 = (X^3 + X^2Y + YZ, XY^2 + YZ, XZ + YZ, Y^3 + YZ, Y^2Z, Z^2),$$

and let E be the filtrating family for the (X^2, XY, Y^2, Z)-adic filtration. Then $(B_0, (\overline{X}, \overline{Y}, \overline{Z}), E)$ admits only tangentially flat deformations. In particular, for every deformation $f : A \longrightarrow B$ of B_0 one has $e(B) \geq \frac{9}{4}e(A)$ and $H^1_B \geq \frac{3}{4}H^1_A$.
Proof. We have to consider the weight filtration defined by $(\frac{1}{2}, \frac{1}{2}, 1)$ and the corresponding

full monomial filtration F. In order to show we have already a standard base for I_0 one obtains the table below.

generators of I_0	s	X^3 $(+X^2Y+YZ)$	XY^2 $(+YZ)$	XZ $(+YZ)$	Y^3 $(+YZ)$	Y^2Z	Z^2	$< r^j, s >$
generators of the syzygy module	r^1	Y^2	$-X^2$					$X^2Y^3-X^2YZ$ $+Y^3Z$
(except trivial syzygies)	r^2	$-Z$		X^2				$-YZ^2$
	r^3		Z	$-Y^2$				$-Y^3Z+YZ^2$
	r^4		Y		$-X$			$-XYZ+Y^2Z$
	r^5		Z			$-X$		YZ^2
	r^6			Y^2		$-X$		Y^3Z
	r^7			Z			$-X$	YZ^2
	r^8				Z	$-Y$		YZ^2
	r^9					Z	$-Y^2$	0

Tab. 13

The error terms are expressed by linear combinations of s as follows.

generators of I_0	s	X^3 $(+X^2Y+YZ)$	XY^2 $(+YZ)$	XZ $(+YZ)$	Y^3 $(+YZ)$	Y^2Z	Z^2	$< r^j, s >$
coefficients	h^1		XY	$-XY$		Y		$X^2Y^3-X^2YZ$ $+Y^3Z$
	h^2						$-Y$	$-YZ^2$
	h^3					$-Y$	Y	$-Y^3Z+YZ^2$
	h^4			$-Y$		2		$-XYZ+Y^2Z$
	h^5						Y	YZ^2
	h^6					Y		Y^3Z
	h^7						Y	YZ^2
	h^8						Y	YZ^2
	h^9							0

Tab. 14

We are reduced to the situation of Example 3.9. □

4.10 Proposition. *Let (B_0, y_0) be a local ring with a system of generators for its maximal ideal n_0. There exists some integer d with the following property: Let $d' \geq d$ and E be the filtrating family for the $n_0^{d'}$-adic filtration. Then (B_0, y_0, E) admits only tangentially flat deformations.*

Proof. Write down the canonical factorization $B_0 = L[[X_1, \ldots, X_r]]/I_0$. There is one full monomial filtration being a refinement for all filtrations in question. We obtain an ideal I'_0 generated by monomials and apply the Lemma below. □

4.11 Lemma. *Let $B_0 = R_0/I_0$, where $R_0 = L[[X_1, \ldots, X_r]]$ and I_0 is generated by monomials of degrees at most d, and let E be the filtrating family for the $(X_1, \ldots, X_r)^d$-adic filtration. Then $N_{G(B_0)}(< -1) = 0$. In particular, $(B_0, (\overline{X_1}, \ldots, \overline{X_r}), E)$ admits only tangentially flat deformations.*

Proof. In order to get a standard base to the given set of generators for I_0 we have to add all their multiples of degrees up to d. So we find a system of generators for $\mathrm{in}(I_0) \subseteq G(R_0)$ consisting of homogeneous elements in degrees 0 and 1 only. Consequently, the normal module $N_{G(B_0)}$ has non-zero elements in degrees ≥ -1 only. □

4.12 Question. For some monomially filtered local ring B_0 consider the set of all points $(q_1, \ldots, q_r) \in \mathbb{R}^d_+$ such that B_0 admits only tangentially flat deformations with respect to the filtrating family for the weight filtration defined by $(q_1, \ \ldots \ , q_r)$. This set is non-empty. What is its geometric nature?

4.13 Corollary. *Let B_0 be some d-dimensional local ring containing a field. Then there exists some $\varepsilon > 0$ such that every deformation $F : A \longrightarrow B$ of B_0 satisfies $H^1_B \geq \varepsilon H^{d+1}_A$.*

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