ON THE DISTRIBUTION OF THE PICARD RANKS OF THE REDUCTIONS OF A $K3$ SURFACE

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Abstract. We report on our results concerning the distribution of the geometric Picard ranks of $K3$ surfaces under reduction modulo various primes. In the situation that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ is even, we introduce a quadratic character, called the jump character, such that $\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_\mathfrak{p}}} > \operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ for all good primes at which the character evaluates to (-1).

1. Introduction

Let S be a $K3$ surface over a number field K. It is a well-known fact that the geometric Picard rank of S may not decrease under reduction modulo a good prime $\mathfrak{p}$ of S. I.e., one always has

$$\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_\mathfrak{p}}} \geqslant \operatorname{rk}\operatorname{Pic} S_{\overline{K}}\,. \tag{1}$$

It would certainly be interesting to understand the sequence $(\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_\mathfrak{p}}})_\mathfrak{p}$, or at least the set of *jump primes*

$$\Pi_{\text{jump}}(S) := \{\, \mathfrak{p} \text{ prime of } K \mid \mathfrak{p} \text{ good for } S, \operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_\mathfrak{p}}} > \operatorname{rk}\operatorname{Pic} S_{\overline{K}} \,\},$$

for a given surface. In an ideal case, one would be able to give a precise reason why the geometric Picard rank jumps at a given good prime. There are two well-known such reasons.

i) According to the Tate conjecture [KM], the left hand side is always even. Thus, in the case that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ is odd, inequality (1) is always strict and every good prime is a jump prime.

ii) Generalising this, if S has real multiplication (RM) by an endomorphism field E and the integer $(22 - \operatorname{rk}\operatorname{Pic} S_{\overline{K}})/[E:\mathbb{Q}]$ is odd, then again every good prime is a jump prime [Ch14, Theorem 1(2)].

It is known due to F. Charles [Ch14, Theorem 1] that these are the only cases in which every good prime is a jump prime.

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In this article, we describe a third reason for a prime to jump, the *jump character*. It was observed experimentally by the first author together with Yu. Tschinkel [CT] that, in the even rank case, one seems to have $\liminf_{B\to\infty} \gamma(S,B) \geqslant \frac{1}{2}$, for

$$\gamma(S,B) := \frac{\#\{\mathfrak{p} \in \Pi_{\mathrm{jump}}(S) \mid |\mathfrak{p}| \leqslant B\}}{\#\{|\mathfrak{p}| \leqslant B\}}\,.$$

Theorem A shows that this observation is indeed true, except for some corner cases, which are simple to describe. Moreover, $\Pi_{\mathrm{jump}}(S)$ contains an entirely regular subset of exact density one half.

In some sense, this result is complementary to those of F. Charles [Ch14]. In fact, [Ch14, Theorem 1] shows, except for the cases i) and ii), that there are always infinitely many *non-jump* primes.

Theorem A (Theorem 2.4.4). *Let K be a number field and S a $K3$ surface over K. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of good reduction and residue characteristic $\neq 2$.*

a) *Then there are two quantities, the* discriminant $\Delta_{H^2}(S)$ *of S and the* discriminant of the Picard representation *or* algebraic part of the discriminant $\Delta_{\mathrm{Pic}}(S)$ *of S (cf. 2.2.2.ii) and 2.3.2.a.ii) for precise definitions), such that the following equations hold,*

$$\det(\mathrm{Frob}_{\mathfrak{p}}\colon H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \circlearrowleft) = \left(\frac{\Delta_{H^2}(S)}{\mathfrak{p}}\right) \qquad \textit{and}$$

$$\det(\mathrm{Frob}_{\mathfrak{p}}\colon T \circlearrowleft) = \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\mathfrak{p}}\right).$$

Here, $T := H^{\perp}_{\mathrm{alg}} \subset H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1))$ denotes the transcendental part of the cohomology, and $(\frac{\cdot}{\mathfrak{p}})$ is the quadratic residue symbol modulo $\mathfrak{p}$ [Ne, Chapter V, §3].

b) *If* $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ *is even then*

$$\left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\mathfrak{p}}\right) = -1 \quad \Longrightarrow \quad \operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_{\mathfrak{p}}}} \geqslant \operatorname{rk}\operatorname{Pic} S_{\overline{K}} + 2\,. \tag{2}$$

In other words, if $K(\sqrt{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)})/K$ is indeed a quadratic extension then

$$\{\,\mathfrak{p} \mid \mathfrak{p} \text{ inert in } K(\sqrt{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)})/K\,\} \subseteq \Pi_{\mathrm{jump}}(S)\,.$$

The quadratic character τ_S, given by

$$\mathfrak{p} \mapsto \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\mathfrak{p}}\right)$$

might be called the *transcendental character* of the $K3$ surface S. Nevertheless, having implication (2) in mind, we prefer to call it the *jump character* of S, at least in the even rank case. It may happen that the quadratic extension, and hence the jump character, are trivial. We provide particular surfaces of this kind, defined over $\mathbb{Q}$, in Examples 2.6.10.a) and 2.6.12. These are the corner cases mentioned above.

One might think about the jump character also as follows. The $\mathbb{Q}_l$-vector space T is equipped with the non-degenerate cup product pairing and acted upon by $\mathrm{Gal}(\overline{K}/K)$. Moreover, the action is orthogonal with respect to the pairing, so that one has a continuous group homomorphism

$$\tau\colon\ \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{O}(T)\,.$$

There is, however, no reason for $\operatorname{im}\tau$ to be contained in $\mathrm{SO}(T)$, in general, so the group homomorphism $\det\tau\colon\ \mathrm{Gal}(\overline{K}/K) \to \{1,-1\}$ is usually non-trivial. Moreover, $\det\tau$ turns out to be independent of l, cf. Proposition 2.2.1.b). The article thus in essence describes the effects of $\det\tau$ being non-trivial.

The criterion for non-triviality.
Due to its construction, the jump character is unramified at every prime of good reduction, cf. Corollary 2.5.7.a). On the other hand, it may ramify at bad primes. We show this to be always the case when the singular reduction is of the mildest possible type.

Theorem B (Corollary 2.5.7.b)). *Let K be a number field and S a $K3$ surface over K. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of residue characteristic different from 2. Suppose that S has a regular, projective model $\underline{S}$ over $\mathscr{O}_{K,\mathfrak{p}}$, the geometric fibre $\underline{S}_{\overline{\mathfrak{p}}}$ of which has exactly one singular point, and assume this to be an ordinary double point. Then the jump character $\tau_S = \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\cdot}\right)$ ramifies at $\mathfrak{p}$.*

In addition, we present algorithms to compute the two characters for a given surface S over $\mathbb{Q}$, a deterministic one for $\Delta_{H^2}(S)$ and a statistical one for the jump character. We can deterministically compute the jump character in two particular situations, which together cover many, but not all examples. These situations occur when Corollary 2.5.7.b) applies to at least a single bad prime, and when $\operatorname{Pic} S_{\overline{\mathbb{Q}}}$ is known as a Galois module to such an extent that $\bigwedge^{\max}(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})$ can be determined. When neither of these circumstances occurs, the statistical algorithm may still be used.

An application. Rational curves on even rank $K3$ surfaces.
It has been a long standing conjecture that every $K3$ surface over an algebraically closed field contains infinitely many rational curves. In full generality, it has been settled only recently by X. Chen, F. Gounelas, and C. Liedtke [CGL]. As an application of our results, we show that the existence of infinitely many rational curves may be obtained rather easily in the situation that the jump character is non-trivial and the surface is otherwise generic.

Theorem C (Theorem 3.1). *Let K be a number field and S a $K3$ surface over K. Assume that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ is even, that $S_{\overline{K}}$ has neither real nor complex multiplication, and that $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)$ is a non-square in K.*

Then $S_{\overline{K}}$ contains infinitely many rational curves.

Notation. i) *Abstract Algebra, Algebraic Geometry, and Algebraic Number Theory.* For K a field, we write $\overline{K}$ for a fixed algebraic closure of K. When K is a number field then by $\mathscr{O}_K$ we denote the ring of algebraic integers in K. For $\mathfrak{p} \subset \mathscr{O}_K$ a prime ideal, we let $K_\mathfrak{p}$ be the completion of K with respect to $\mathfrak{p}$, $\mathscr{O}_{K,\mathfrak{p}}$ the ring of integers in $K_\mathfrak{p}$, and $\mathbb{F}_\mathfrak{p}$ the residue field.

Moreover, we always let $\mathrm{Frob} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the *geometric Frobenius* automorphism $x \mapsto x^{1/q}$, cf. [DeWI, (1.15)]. Similarly, when K is a number field and $\mathfrak{p} \subset \mathscr{O}_K$ a prime ideal, $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(\overline{K}/K)$ denotes an arbitrary lift of the geometric Frobenius. We write Frob and $\mathrm{Frob}_\mathfrak{p}$, too, for the automorphisms of schemes and their cohomology groups, induced by Frob, respectively $\mathrm{Frob}_\mathfrak{p}$, via functoriality. Let us note that $\mathrm{Frob}\colon \mathbf{P}^N_{\mathbb{F}_q} \to \mathbf{P}^N_{\mathbb{F}_q}$ maps the geometric point $(x_0 : \cdots : x_N)$ to $(x_0^q : \cdots : x_N^q)$.

Perhaps deviating from a certain standard, we say that a proper variety S over K has *good reduction* at $\mathfrak{p}$ when there exists a proper model of S over $\mathscr{O}_K$ that has good reduction at $\mathfrak{p}$ in the usual sense.

ii) *Quadratic extensions.* Let K be a field of characteristic $\neq 2$ and L/K an at most quadratic field extension. Then, according to Kummer theory, there exists a unique class $\Delta_L \in K^*/(K^*)^2$ such that $L = K(\sqrt{u})$ for any $u \in \Delta_L$. In this situation, we shall also write $K(\sqrt{\Delta_L})$ for $K(\sqrt{u})$.

Assume that K is a number field and $\mathfrak{p} \subset \mathscr{O}_K$ a prime ideal of residue characteristic $\neq 2$, at which L/K is unramified. Then the quadratic residue symbol $(\frac{u}{\mathfrak{p}})$ [Ne, Chapter V, Proposition (3.5)] is independent of the choice of a $\mathfrak{p}$-adic unit $u \in \Delta_L$. We will therefore write $(\frac{\Delta_L}{\mathfrak{p}})$ instead of $(\frac{u}{\mathfrak{p}})$.

iii) *Characters.* By a *character,* we always mean a continuous homomorphism from a topological group to a discrete abelian group. A *quadratic character* is a character to $\{1, -1\}$.

We often describe a quadratic character $\chi\colon \mathrm{Gal}(\overline{K}/K) \to \{1, -1\}$, when K is a number field, in the form

$$\mathfrak{p} \mapsto \left(\tfrac{\Delta}{\mathfrak{p}}\right),$$

or simply $(\frac{\Delta}{\cdot})$, for $\Delta \in K^*/(K^*)^2$. This is supposed to mean that $\chi(\mathrm{Frob}_\mathfrak{p}) = (\frac{\Delta}{\mathfrak{p}})$ for every prime ideal $\mathfrak{p} \subset \mathscr{O}_K$ of residue characteristic $\neq 2$ of the kind that Δ is representable by a $\mathfrak{p}$-adic unit. Note that, $\{1, -1\}$ being abelian, $\chi(\mathrm{Frob}_\mathfrak{p})$ is well defined for every quadratic character. Moreover, the values at the Frobenii $\mathrm{Frob}_\mathfrak{p}$ determine χ uniquely, due to the Chebotarev density theorem.

When V is a one-dimensional vector space over a field F, equipped with the discrete topology and acted upon continuously by a topological group G, we denote by $[V]\colon G \to F^*$ the character given by $[V](g) = a$, for $a \in F^*$ the scalar satisfying $g \cdot v = av$ for every $v \in V$. This notation is due to T. Saito [Sa].

Computations.

All computations are done using `magma` [BCP], `sage` [St], and `C++`, including the libraries `FLINT` [HJP] and `NTL` [Sh]. For point counting on the examples being quartic surfaces, we used the software developed by the first author, which is publicly available at `https://github.com/edgarcosta/controlledreduction`.

2. The jump character

2.1. **The determinant of Frob and the relationship with the sign in the functional equation.**

Let S be a smooth, proper variety over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. Then Frob acts linearly on the l-adic cohomology modules $H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(j))$. The characteristic polynomial $\Phi_j^{(i)}$ of Frob is independent of the choice of $l \neq p$ and has in fact rational coefficients [DeWI, Théorème (1.6)]. In particular, the determinant of Frob is a rational number and independent of $l \neq p$.

In this section, we discuss the behaviour of det Frob. The theorem below, which is essentially a summary of results of P. Deligne and J. Suh, shows that each statement on det Frob may be translated into a statement about the sign in the functional equation.

Theorem 2.1.1 (Deligne, Suh)**.** a) *The polynomial* $\Phi_j^{(i)} \in \mathbb{Q}[T]$ *fulfils the functional equation*

$$T^N \Phi(q^{i-2j}/T) = \pm q^{\frac{N}{2}(i-2j)} \Phi(T)\,, \tag{3}$$

for $N := \operatorname{rk} H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(j))$.

b) *The sign in the functional equation (3) is that of*

$$\det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))\circlearrowleft) = (-1)^N \det(\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))\circlearrowleft)\,.$$

This is a rational number, the sign of which is independent of the Tate twist, i.e. of the choice of j*.*

c.i) *If* i *is even, then* $\det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(i/2))\circlearrowleft)$ *is either* $(+1)$ *or* (-1)*. In other words, the determinant gives the sign in (3) exactly.*

ii) *If* i *is odd then* N *is even and in (3), the plus sign always holds.*

Proof. a) and b) Let us write Φ for $\Phi_j^{(i)}$. The polynomials on both sides of (3) then have the same roots as, with z, the number $\overline{z} = \frac{q^{i-2j}}{z}$ is a root of Φ, too [DeWII, Corollaire (3.3.9)], and has the same multiplicity. To show that they perfectly agree, let us adopt the convention that Φ is monic. Then the leading coefficient of the polynomial on the left hand side is equal to the constant term of Φ. By definition, this is the determinant of $(-\operatorname{Frob})$ on $H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$, which is known to be a rational number and of absolute value $q^{\frac{N}{2}(i-2j)}$ [DeWII, Corollaire (3.3.9)]. Thus, a) follows, together with the first assertion of b). The final claim is clear, since

$$\det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))\circlearrowleft) = q^{-Nj} \cdot \det(-\operatorname{Frob}\colon H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)\circlearrowleft)\,.$$

c.i) As $\det \operatorname{Frob} = (-1)^N \Phi(0)$ for Φ monic, this can be read off the functional equation $T^N \Phi(1/T) = \pm \Phi(T)$.

ii) If S is projective then, by Poincaré duality and the hard Lefschetz theorem [DeWII, Théorème (4.1.1)], there is a non-degenerate pairing

$$H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \times H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \to \mathbb{Q}_l(2j - i)$$

that is compatible with the action of Frob. It is, moreover, alternating since i is supposed to be odd, cf. [Sp, Chapter 5, Section 6, §11]. The assertion follows directly from this [DeWI, (2.6)]. Cf. the remarks after [DeWII, Corollaire (4.1.5)]. The proper non-projective case has only recently been settled by J. Suh [Su, Corollary 2.2.3 and Corollary 3.3.5]. □

Remark 2.1.2. The polynomials $\Phi_j^{(i)} \in \mathbb{Q}[T]$ occurring as characteristic polynomials of Frob have remarkable properties, which were established mainly by P. Deligne, B. Mazur, and A. Ogus. Details are summarised in the article [EJ15] of the second and third authors. In the proof above, the only property that was used is that every complex root of $\Phi_j^{(i)}$ is of absolute value $q^{i/2-j}$. This was first proven by P. Deligne in [DeWI, Théorème (1.6)] for the projective case and later in [DeWII, Corollaire (3.3.9)], in general. The assertion had been formulated by A. Weil as a part of his famous conjectures.

2.2. **The discriminant of the H^i-representation.**
Let us start by recalling some facts on l-adic cohomology.

Proposition 2.2.1. *Let K be a field and S a smooth and proper K-scheme.*
a) *Then, for all prime numbers $l \neq \operatorname{char} K$ and all integers i and j, associated with the one-dimensional $\mathbb{Q}_l$-vector space $\bigwedge^{\max} H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(j))$, there is the character*

$$[\det H^i(S_{\overline{K}}, \mathbb{Q}_l(j))] \colon \operatorname{Gal}(\overline{K}/K) \longrightarrow \mathbb{Q}_l^*$$

of the absolute Galois group of K.
b) *Suppose that i is even and that S is pure of dimension i. Then the character $[\det H^i(S_{\overline{K}}, \mathbb{Q}_l(i/2))]$ has values in $\{1, -1\} \subset \mathbb{Q}_l^*$ and is independent of l.*

Proof. a) This follows from the functoriality of l-adic cohomology, together with the fact that every $\sigma \in \operatorname{Gal}(\overline{K}/K)$ induces an automorphism of schemes of $S_{\overline{K}}$.
b) By Poincaré duality [SGA4, Exp. XVIII, Théorème 3.2.5], there is a canonical non-degenerate pairing $s \colon H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) \times H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) \to \mathbb{Q}_l$ that is compatible with the action of $\operatorname{Gal}(\overline{K}/K)$. Here, i is assumed even, so the pairing s is symmetric. According to a standard fact from linear algebra [War, Def. 2.9], s induces another symmetric pairing

$$\textstyle\bigwedge^{\max}(s) \colon \bigwedge^{\max} H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) \times \bigwedge^{\max} H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) \longrightarrow \mathbb{Q}_l$$

that is again non-degenerate. The action of $\operatorname{Gal}(\overline{K}/K)$ is orthogonal with respect to s, which implies that the character $[\det H^i(S_{\overline{K}}, \mathbb{Q}_l(i/2))]$ must have values in $\{1, -1\} \subset \mathbb{Q}_l^*$.

For the case that K is a number field, independence of l is easily reduced to the Weil conjectures, proven by P. Deligne [DeWII, Corollaire (3.3.9)], using the Chebotarev density theorem together with the smooth specialisation theorem for cohomology groups [SGA4, Exp. XVI, Corollaire 2.3]. In general, the result has been established by T. Saito [Sa, Corollary 3.3].

N.B. In the notation for the characters, we write det instead of $\bigwedge^{\max}$. This convention follows [Sa]. □

Definition 2.2.2. i) In the situation of part b), we denote by L_S the extension field of K that corresponds to $\ker\left[\det H^i(S_{\overline{K}}, \mathbb{Q}_l(i/2))\right]$ under the Galois correspondence. By construction, L_S/K is an at most quadratic extension.

ii) If $\operatorname{char} K \neq 2$ then we denote the class in $K^*/(K^*)^2$ that yields the field extension L_S/K by $\Delta_{H^i}(S)$ and call it the *discriminant of the H^i-representation* of S.

Lemma 2.2.3. *Let K be a number field and S a smooth and proper K-scheme. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime at which S has good reduction.*

a) *If l is a prime different from the residue characteristic of $\mathfrak{p}$ then, for every $j \in \mathbb{Z}$, the $\mathrm{Gal}(\overline{K}/K)$-representation $H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(j))$ is unramified at $\mathfrak{p}$.*

b) *Suppose that S is pure of dimension i, for an even integer i. Then the quadratic character $\left(\frac{\Delta_{H^i}(S)}{\cdot}\right) = \left[\det H^i(S_{\overline{K}}, \mathbb{Q}_l(i/2))\right]$ is unramified at $\mathfrak{p}$. Equivalently, the splitting field L_S is unramified at $\mathfrak{p}$.*

Proof. a) For this, it suffices to consider the restriction of the representation to the decomposition group $D_\mathfrak{p} \cong \mathrm{Gal}(\overline{K}_\mathfrak{p}/K_\mathfrak{p})$. This coincides with the natural action of $\mathrm{Gal}(\overline{K}_\mathfrak{p}/K_\mathfrak{p})$ on $H^i_{\text{ét}}(S_{\overline{K}_\mathfrak{p}}, \mathbb{Q}_l(j)) \cong H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(j))$, according to invariance of étale cohomology under extensions of separably closed fields [SGA4, Exp. XII, Corollaire 5.4]. Moreover, by the smooth specialisation theorem for cohomology groups [SGA4, Exp. XVI, Corollaire 2.2], $H^i_{\text{ét}}(S_{\overline{K}_\mathfrak{p}}, \mathbb{Q}_l(j)) \cong H^i_{\text{ét}}(S_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathbb{Q}_l(j))$, which shows that this cohomology vector space is acted upon via the quotient $\mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p}) \cong D_\mathfrak{p}/I_\mathfrak{p}$. I.e., the inertia group $I_\mathfrak{p}$ fixes $H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(j))$ pointwise, as required.

b) This is a direct consequence of a). □

Remarks 2.2.4. i) In particular, for every exponent $k \in \mathbb{N}$, the splitting field of $H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}/l^k\mathbb{Z}(j))$ is unramified at every prime ideal $\mathfrak{p}$ of good reduction.

ii) If the residue characteristic is $\neq 2$ then one has

$$\det(\mathrm{Frob}_\mathfrak{p} \colon H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) \circlearrowleft) = \left(\frac{\Delta_{H^i}(S)}{\mathfrak{p}}\right). \tag{4}$$

iii) For any number field $K' \supseteq K$, one has $\Delta_{H^i}(S_{K'}) = \Delta_{H^i}(S) \cdot (K'^*)^2 \in K'/(K'^*)^2$.

iv) The name "discriminant" has not been chosen at random. Indeed, Proposition 2.2.1 allows a generalisation to families $\pi \colon F \to X$ over a general base scheme. The quadratic field extension then goes over into a twofold covering $\varrho_\pi \colon Y \to X$, ramified at most over the discriminant locus. If π is sufficiently reasonable then ϱ_π is given by $w^2 = \Delta$, for Δ a normalised version of the discriminant of the family. Furthermore, for every non-singular member $S = F_x$, the value $\Delta(x)$ belongs to the class $\Delta_{H^i}(S)$.

We plan to report about the interpretation of the quantities $\Delta_{H^i}(S)$ as actual discriminants, as well as some applications thereof, in a forthcoming paper.

2.3. **Surfaces—The discriminant of the Néron–Severi representation.**

Proposition 2.3.1. *Let K be a field and S a smooth projective surface over K.*
a) *Then, associated with the one-dimensional $\mathbb{Q}$-vector space $\bigwedge^{\max}(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})$, there is the character*

$$[\det(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})] \colon \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathbb{Q}^*$$

of the absolute Galois group of K.
b) *The character $[\det(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})]$ takes values only in $\{1, -1\} \subset \mathbb{Q}^*$.*

Proof. a) This follows from the functoriality of the Néron–Severi group, together with the fact that every $\sigma \in \mathrm{Gal}(\overline{K}/K)$ induces an automorphism of schemes of $S_{\overline{K}}$.
b) Every $\sigma \in \mathrm{Gal}(\overline{K}/K)$ induces an automorphism of the Néron–Severi group $\mathrm{NS}(S_{\overline{K}})$, in particular one of $\mathrm{NS}(S_{\overline{K}})_{\mathrm{tors}}$, and consequently one of the torsion-free $\mathbb{Z}$-module $\mathrm{NS}(S_{\overline{K}})/\mathrm{NS}(S_{\overline{K}})_{\mathrm{tors}}$. As that is a full rank lattice in $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, it induces a lattice in the one-dimensional vector space $\bigwedge^{\max}(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})$, which must be respected by the action of $\mathrm{Gal}(\overline{K}/K)$. The assertion immediately follows from this. □

Definition 2.3.2. a.i) In the situation of part b), we denote by $L_{S,\mathrm{alg}}$ the extension field of K that corresponds to $\ker [\det(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})]$ under the Galois correspondence.
ii) If $\mathrm{char}\, K \neq 2$ then we denote the class in $K^*/(K^*)^2$ that yields the field extension $L_{S,\mathrm{alg}}/K$ by $\Delta_{\mathrm{NS}}(S)$ and call it the *discriminant of the Néron–Severi representation* or the *algebraic part of the discriminant* of S.
b) For surfaces such that $H^1(S_{\overline{K}}, \mathbb{Q}_l) = 0$, one has $\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. In this case, one may write $\Delta_{\mathrm{Pic}}(S)$ instead of $\Delta_{\mathrm{NS}}(S)$ and speak of the *discriminant of the Picard representation*. Similarly, let us then write $[\det(\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})]$ instead of $[\det(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})]$.

Remark 2.3.3. The algebraic part of the discriminant of S should not be confused with the discriminants of $\mathrm{NS}(S)$ and $\mathrm{NS}(S_{\overline{K}})$ as lattices. Instead, one might think about it as follows.

Since $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space, there is a smallest finite field extension of K, over which all elements of $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ are defined, the splitting field L of $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\mathrm{Gal}(\overline{K}/K)$ acts on $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ via its quotient $\mathrm{Gal}(L/K)$ and the action of this quotient is faithful.

Therefore, the action of $\mathrm{Gal}(\overline{K}/K)$ on $\bigwedge^{\max}(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})$ factors via $\mathrm{Gal}(L/K)$, too. On the other hand, $\bigwedge^{\max}(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})$ is one-dimensional, so the action of the finite group $\mathrm{Gal}(L/K)$ must factor via $(\mathbb{Q}^*)_{\mathrm{tors}} = \{1, -1\}$. The stabiliser is of index at most 2 in $\mathrm{Gal}(\overline{K}/K)$ and hence the splitting field of $\bigwedge^{\max}(\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q})$ is an at most quadratic extension $K(\sqrt{\Delta_{\mathrm{NS}}(S)}) = L_{S,\mathrm{alg}} \subseteq L$.

Lemma 2.3.4. *Let K be a number field and S a smooth projective surface over K. Moreover, let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime at which S has good reduction.*

a) *Then the splitting field of* $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is unramified at* $\mathfrak{p}$.

b) *The character* $\left(\frac{\Delta_{\mathrm{NS}}(S)}{\cdot}\right) = [\det \mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}]$ *is unramified at* $\mathfrak{p}$. *Equivalently, the splitting field* $L_{S,\mathrm{alg}}$ *is unramified at* $\mathfrak{p}$.

Proof. a) The first Chern class homomorphism factors via the Néron–Severi group, i.e. via algebraic equivalence,

$$\begin{array}{ccc} \mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l & \xrightarrow{\quad c_1 \quad} & H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \\ \downarrow & \nearrow & \\ \mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l\,. & & \end{array}$$

Indeed, c_1 factors via numerical equivalence, since the intersection pairing on $\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ is compatible with the cup product pairing on $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1))$. Moreover, Matsusaka's theorem [Ma, Theorem 4], cf. [An, paragraph 3.2.7], shows that algebraic equivalence coincides with numerical equivalence, already on $\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Now write L for the splitting field of $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and assume that L would ramify at $\mathfrak{p}$. By definition, $\mathrm{Gal}(L/K)$ acts faithfully on $\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Choose a prime $\mathfrak{q}$ of L lying above $\mathfrak{p}$ and a non-trivial element $\sigma \in \mathrm{Gal}(L_{\mathfrak{q}}/K^{\mathrm{n}}_{\mathfrak{p}}) \subseteq \mathrm{Gal}(L_{\mathfrak{q}}/K_{\mathfrak{p}})$, for $K^{\mathrm{n}}_{\mathfrak{p}}$ the maximal unramified subfield of $L_{\mathfrak{q}}$. Then σ acts non-trivially on the image of the first Chern class homomorphism

$$c_1 \colon \mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1))\,.$$

This, however, is in contradiction with the smooth specialisation theorem for cohomology groups, as seen before.

b) This follows immediately from a). □

Remarks 2.3.5. i) If the residue characteristic is not 2 then one has

$$\det(\mathrm{Frob}_{\mathfrak{p}} \colon (\mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}) \circlearrowleft) = \left(\frac{\Delta_{\mathrm{NS}}(S)}{\mathfrak{p}}\right)\,. \tag{5}$$

ii) If $K' \supseteq K$ is a number field extending K, then one has

$$\Delta_{\mathrm{NS}}(S_{K'}) = \Delta_{\mathrm{NS}}(S) \cdot (K'^*)^2 \in K'/(K'^*)^2.$$

iii) It is worthwhile to observe that, for $K3$ surfaces, the assertion of Lemma 2.3.4 is still true when S has bad reduction at $\mathfrak{p}$ of the mildest possible form. Cf. Corollary 2.5.4, below.

2.4. $K3$ surfaces.

There is a strong relation, which is established for $K3$ surfaces, but relies on Tate's and Serre's conjectures in general, between the Galois action on l-adic cohomology and the variation of the geometric Picard ranks under reduction modulo various primes [EJ12b, CT]. From our point of view, this is in fact the main application of the constructions presented so far.

Facts 2.4.1. *Let* S *be a* $K3$ *surface over a base field* K.

a) *Then* $\operatorname{Pic} S_{\overline{K}}$ *is a free abelian group of rank at most* 22. *If* K *is of characteristic zero then the rank is at most* 20. *If* K *is finite then the rank is even.*

b) *If* K *is finite then* $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ *is equal to the number [counted with multiplicities] of all eigenvalues of* Frob *on* $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1))$ *that are roots of unity.*

Proof. a) The first statement is found, e.g., in [Hu, Chapter 17, first formula of Section 2]. The second one is [Hu, Chapter 17, formula (1.1)] in the case that $K = \mathbb{C}$, while the general case follows from this in view of [Hu, Chapter 17, Lemma 2.2]. The final claim is a direct consequence of b).

b) See [Hu, Chapter 17, Corollary 2.9 and the arguments given before]. Note that this result is an application of the Tate conjecture, which has been shown for $K3$ surfaces over finite fields by to the combined work of several people, most notably F. Charles [Ch13], M. Lieblich, D. Maulik, and A. Snowden [LMS], K. Madapusi Pera [MP], as well as W. Kim and K. Madapusi Pera [KM]. □

For K an arbitrary field and $l \neq \operatorname{char} K$ a prime number, there is a canonical orthogonal decomposition

$$H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) = H_{\text{alg}} \oplus T\,. \tag{6}$$

Here, $H_{\text{alg}} = c_1(\operatorname{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ is clearly $\operatorname{Gal}(\overline{K}/K)$-invariant. Moreover, $T := H_{\text{alg}}^{\perp}$ is $\operatorname{Gal}(\overline{K}/K)$-invariant, too, as the Galois action is orthogonal.

In the particular case that K is a number field, let $\mathfrak{p} \subset \mathscr{O}_K$ be any prime of good reduction and of residue characteristic different from l. Then $\operatorname{Frob}_{\mathfrak{p}} \in \operatorname{Gal}(\overline{K}/K)$ is determined only up to conjugation. But this suffices to have well-defined eigenvalues and a well-defined determinant of $\operatorname{Frob}_{\mathfrak{p}}$, associated with any vector space being acted upon by $\operatorname{Gal}(\overline{K}/K)$. In particular,

$$\det(\operatorname{Frob}_{\mathfrak{p}} \colon T \circlearrowleft) = \frac{\det(\operatorname{Frob}_{\mathfrak{p}} \colon H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \circlearrowleft)}{\det(\operatorname{Frob}_{\mathfrak{p}} \colon \operatorname{Pic} S_{\overline{K}} \circlearrowleft)}\,. \tag{7}$$

Our main theoretical observation on the distribution of the Picard ranks of the reductions is then as follows.

Proposition 2.4.2 (Rank jumps)**.** *Let* S *be a* $K3$ *surface over a number field* K *and* $\mathfrak{p} \subset \mathscr{O}_K$ *a prime of good reduction. Assume that* $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ *is even. Then the following is true:*

If $\det(\operatorname{Frob}_{\mathfrak{p}} \colon T \circlearrowleft) = -1$, *then* $\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}}_{\mathfrak{p}}} \geqslant \operatorname{rk}\operatorname{Pic} S_{\overline{K}} + 2$.

Proof. Choose a prime number l, different from the residue characteristic of $\mathfrak{p}$. Then one has $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \cong H^2_{\text{ét}}(S_{\overline{K}_{\mathfrak{p}}}, \mathbb{Q}_l(1))$ [SGA4, Exp. XII, Corollaire 5.4], as well as $H^2_{\text{ét}}(S_{\overline{K}_{\mathfrak{p}}}, \mathbb{Q}_l(1)) \cong H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_l(1))$ [SGA4, Exp. XVI, Corollaire 2.2]. Consequently, by transport of structure, the orthogonal decomposition (6) carries over into

$$H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_{\mathfrak{p}}}, \mathbb{Q}_l(1)) = H_{\text{alg}} \oplus T\,. \tag{8}$$

Note that, as a consequence of its construction, T may well contain algebraic classes.

Moreover, under the first isomorphism, the action of $\mathrm{Gal}(\overline{K}_\mathfrak{p}/K_\mathfrak{p})$ is compatible with that of the decomposition group $D_\mathfrak{p} \subset \mathrm{Gal}(\overline{K}/K)$, while the second isomorphism shows that $\mathrm{Gal}(\overline{K}_\mathfrak{p}/K_\mathfrak{p})$ acts via its quotient $\mathrm{Gal}(K_\mathfrak{p}^{\mathrm{nr}}/K_\mathfrak{p})$. In particular, the action of $\mathrm{Frob} \in \mathrm{Gal}(\overline{\mathbb{F}}_\mathfrak{p}/\mathbb{F}_\mathfrak{p})$ on $H^2_{\text{ét}}(S_{\overline{\mathbb{F}}_\mathfrak{p}}, \mathbb{Q}_l(1))$ agrees with that of any $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(\overline{K}/K)$ on $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1))$. For instance, Frob and $\mathrm{Frob}_\mathfrak{p}$ have the same eigenvalues on H_{alg}, as well as on T.

By Lemma 2.3.4.a), the splitting field of $\mathrm{Pic}\, S_{\overline{K}}$ is a number field unramified at $\mathfrak{p}$. Therefore, $\mathrm{Gal}(K_\mathfrak{p}^{\mathrm{nr}}/K_\mathfrak{p})$ acts on H_{alg} via a finite quotient group. In particular, there exists an integer $e > 0$ such that Frob^e acts trivially. Consequently, all eigenvalues of Frob on H_{alg} are roots of unity.

In view of Fact 2.4.1.b), we need to show that Frob acts on T with at least two eigenvalues being roots of unity. For this, let us observe that each eigenvalue is of absolute value 1, so that those different from 1 and (-1) come in pairs $\{z, \overline{z}\}$ of complex conjugates. As $z\overline{z} = 1$ and $\det(\mathrm{Frob}_\mathfrak{p} \colon T \circlearrowleft) = -1$, one of the eigenvalues must be equal to (-1). Finally, as $\dim T = 22 - \mathrm{rk}\,\mathrm{Pic}\, S_{\overline{K}}$ is even, a further eigenvalue 1 is enforced. This completes the proof. □

Remarks 2.4.3. i) The proof given above, shows that, in addition to the specialisations of the invertible sheaves from $\mathrm{Pic}\, S_{\overline{K}}$, the Picard group of $\mathrm{Pic}\, S_{\overline{\mathbb{F}}_\mathfrak{p}}$ has (at least) two further generators. One of them may be chosen to be defined over $\mathbb{F}_\mathfrak{p}$, the other over its quadratic extension.

ii) Without the hypothesis on the determinant of the Frobenius, the argument simply reproves the standard fact that $\mathrm{rk}\,\mathrm{Pic}\, S_{\overline{\mathbb{F}}_\mathfrak{p}} \geqslant \mathrm{rk}\,\mathrm{Pic}\, S_{\overline{K}}$.

Theorem 2.4.4. *Let K be a number field and S a $K3$ surface over K. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of residue characteristic $\neq 2$ and good reduction.*

a) *Then the following two equations hold,*

$$\det(\mathrm{Frob}_\mathfrak{p} \colon H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \circlearrowleft) = \left(\frac{\Delta_{H^2}(S)}{\mathfrak{p}}\right) \qquad \textit{and}$$

$$\det(\mathrm{Frob}_\mathfrak{p} \colon T \circlearrowleft) = \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\mathfrak{p}}\right).$$

b) *If $\mathrm{rk}\,\mathrm{Pic}\, S_{\overline{K}}$ is even, then*

$$\left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\mathfrak{p}}\right) = -1 \quad \Longrightarrow \quad \mathrm{rk}\,\mathrm{Pic}\, S_{\overline{\mathbb{F}}_\mathfrak{p}} \geqslant \mathrm{rk}\,\mathrm{Pic}\, S_{\overline{K}} + 2\,.$$

In other words, if $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)$ is not a square in K then

$$\{\, \mathfrak{p} \mid \mathfrak{p} \text{ inert in } K(\sqrt{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}) \,\} \subseteq \Pi_{\mathrm{jump}}(S)\,.$$

Proof. a) The first formula is a particular case of formula (4). The second one is a consequence of the first together with formulae (5) and (7).

b) This follows from a), together with Proposition 2.4.2. □

Corollary 2.4.5. *Let K be a number field and S a $K3$ surface over K. Assume that $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)$ is a non-square in K. Then*

$$\liminf_{B\to\infty} \gamma(S,B) \geqslant \tfrac{1}{2}\,.$$

Definition 2.4.6. For K a number field and S a $K3$ surface over K, we call the quadratic character

$$\tau_S := \bigl[\det H^2(S_{\overline{K}}, \mathbb{Q}_l(2))\bigr]\cdot\bigl[\det \mathrm{Pic}(S_{\overline{K}})\otimes_{\mathbb{Z}}\mathbb{Q}\bigr]\colon\ \mathrm{Gal}(\overline{K}/K) \longrightarrow \{1,-1\}$$

the *jump character* of S.

Remarks 2.4.7. i) The jump character τ_S is given by

$$\mathfrak{p} \mapsto \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\mathfrak{p}}\right),$$

for all good primes $\mathfrak{p}$.

ii) Proposition 2.4.2 shows that, for S a $K3$ surface of even geometric Picard rank, $\tau_S(\mathfrak{p}) = -1$ implies $\mathrm{rk}\,\mathrm{Pic}\, S_{\overline{\mathbb{F}}_{\mathfrak{p}}} \geqslant \mathrm{rk}\,\mathrm{Pic}\, S_{\overline{K}} + 2$.

In this section, the assumption on the surface to be of type $K3$ was used only in referring to the Tate conjecture. We actually showed the following.

Theorem 2.4.8. *Let K be a number field and S a smooth and proper surface over K, for which the Tate conjecture holds. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of good reduction and suppose that the Tate conjecture holds for $S_{\mathbb{F}_{\mathfrak{p}}}$, too. Then, in the situation that $\mathrm{rk}\,\mathrm{NS}\, S_{\overline{K}} \equiv \dim H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \pmod 2$, one has*

$$\bigl[\det H^2(S_{\overline{K}}, \mathbb{Q}_l(2))\bigr]\cdot\bigl[\det \mathrm{Pic}(S_{\overline{K}})\otimes_{\mathbb{Z}}\mathbb{Q}\bigr](\mathfrak{p}) = -1 \Longleftrightarrow$$
$$\left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{NS}}(S)}{\mathfrak{p}}\right) = -1 \Longrightarrow \mathrm{rk}\,\mathrm{NS}\, S_{\overline{\mathbb{F}}_{\mathfrak{p}}} \geqslant \mathrm{rk}\,\mathrm{NS}\, S_{\overline{K}} + 2\,.$$

□

2.5. **The criterion for non-triviality.**

Lemma 2.5.1. *Let $p \neq 2$ be a prime number and M a free $\mathbb{Z}_p$-module of finite rank r, equipped with a non-degenerate symmetric pairing. Let an orthogonal map $V\colon M \to M$ be given with characteristic polynomial $(T-1)^{r_1}(T+1)^{r_2}$. Then M is the orthogonal direct sum of the two generalised eigenspaces,*

$$M = \ker(1-V)^{r_1} \oplus \ker(1+V)^{r_2}\,.$$

Proof. It is well known that the generalised eigenspaces for two distinct eigenvalues only have the zero element in common. Moreover, $\frac{1+V}{2} + \frac{1-V}{2} = \mathrm{id}$ yields

$$\sum_{k=0}^{r} \tbinom{r}{k} \bigl(\tfrac{1+V}{2}\bigr)^k \bigl(\tfrac{1-V}{2}\bigr)^{r-k}(x) = x\,,$$

for every $x \in M$. According to Cayley–Hamilton, the summands for $k \geqslant r_2$ are contained in $\ker(1-V)^{r_1}$, while those for $k \leqslant r_2$, i.e. $r-k \geqslant r_1$, are contained

in $\ker(1+V)^{r_2}$. Thus, the sum is the whole of M. Finally, it is a classical result for orthogonal maps that the generalised eigenspaces for two eigenvalues λ_1, λ_2 with $\lambda_1\lambda_2 \neq 1$ are perpendicular. An argument is given, e.g., in [Wal, Proposition 10.4.1]. □

Theorem 2.5.2 (The vanishing cycle)**.** *Let K be a number field and S a $K3$ surface over K. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of residue characteristic $\neq 2$ such that S has a regular, projective model $\underline{S}$ over $\mathscr{O}_{K,\mathfrak{p}}$, the geometric fibre $\underline{S}_{\overline{\mathfrak{p}}}$ of which has exactly one singular point $\mathfrak{z}$. Assume $\mathfrak{z}$ to be an ordinary double point.*

Then, for every prime number l, different from the residue characteristic of $\mathfrak{p}$, the vanishing cycle [SGA7, Exp. XV, Théorème 3.4.(i)] *associated with $\mathfrak{z}$, fulfils*

$$\delta_{\mathfrak{z},l} \in H^{\perp}_{\mathrm{alg},l}\,.$$

Proof. *First step.* Generalities.
Let us denote the residue characteristic of $\mathfrak{p}$ by p. On the scheme $S_{\overline{K}}$, there is a monodromy automorphism [SGA7, Exp. XV, Proposition 3.2.1.(ii)], which is induced by a particular non-trivial element $\nu \in I_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{K}/K)$ of the inertia group. For every prime number l, including $l = p$, the induced map on l-adic cohomology is called the monodromy operator $V\colon H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_l(1)) \circlearrowleft$. This is an orthogonal map with respect to the cup product pairing. By a slight abuse of notation, we denote the map induced by ν on $\mathrm{Pic}\, S_{\overline{K}}$ by V, too.

If $l \neq p$ then the action of V is described by the Picard–Lefschetz formula [SGA7, Exp. XV, Théorème 3.4.(iii)]

$$V(c) = c + \langle c, \delta_{\mathfrak{z},l}\rangle \delta_{\mathfrak{z},l}\,. \tag{9}$$

Note here that V, being induced by an element from the inertia group, acts trivially on $\mathbb{Z}_l(1)$ itself. The class $\delta_{\mathfrak{z},l} \in H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_l(1))$ is the so-called vanishing cycle. It is known that $\langle \delta_{\mathfrak{z},l}, \delta_{\mathfrak{z},l}\rangle = -2$ [SGA7, Exp. XV, Théorème 3.4.(i)]. In particular, the Picard–Lefschetz formula shows $V(\delta_{\mathfrak{z},l}) = -\delta_{\mathfrak{z},l}$. Moreover, the operator V acts with characteristic polynomial $(T-1)^{21}(T+1)$.

When the action on $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_p(1))$ is concerned, the characteristic polynomial is the same [Oc, Theorem 3.1], cf. [Fo, §§2.3 and 2.4]. It seems, however, not to be known whether the action of V is semisimple. In particular, no Picard–Lefschetz formula is available.

Second step. If, for some $l_0 \neq p$, $\delta_{\mathfrak{z},l_0} \notin H^{\perp}_{\mathrm{alg},l_0}$ then V acts non-trivially on $\mathrm{Pic}\, S_{\overline{K}}$. The assumption means that $\langle c, \delta_{\mathfrak{z},l_0}\rangle \neq 0$, for a certain class $c \in H_{\mathrm{alg},l_0}$. Then formula (9) immediately implies that

$$\delta_{\mathfrak{z},l_0} = \frac{1}{\langle c, \delta_{\mathfrak{z},l_0}\rangle}[V(c) - c] \in H_{\mathrm{alg},l_0}\,.$$

I.e., the operator V acts on $H_{\mathrm{alg},l_0} \cong \mathrm{Pic}\, S_{\overline{K}} \otimes_{\mathbb{Z}} \mathbb{Q}_{l_0}$ non-trivially with one eigenvector (-1), while all others are equal to 1. As $\pm 1 \in \mathbb{Q}$, the eigenspaces are defined in $\mathrm{Pic}\, S_{\overline{K}} \otimes_{\mathbb{Z}} \mathbb{Q}$ already, which implies the claim.

Third step. Let $d \in \mathrm{Pic}(S_{\overline{K}}) \subset \mathrm{Pic}\, S_{\overline{K}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be a generator of the (-1)-eigenspace, which is minimal, i.e. not divisible by any integer $\neq \pm 1$. Then $\langle d, d\rangle = -2$.
For every prime number l, the inclusion $\mathrm{Pic}\, S_{\overline{K}}/l \stackrel{\bar{c}_1}{\hookrightarrow} H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_l(1))/l$ coming from the Kummer sequence shows that $c_1(d) \in H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_l(1))$ is not divisible by l. Hence, $c_1(d)$ is actually a generator of the (-1)-eigenspace in $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_l(1))$.

Claim: $\langle d, d\rangle$ is an l-adic unit, for every prime $l \neq 2$.
Indeed, for $l \neq 2, p$, the $\mathbb{Z}_p$-module $H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_l(1))$ is the orthogonal direct sum of the 1- and (-1)-eigenspaces, since $\frac{1}{2} \in \mathbb{Z}_l$ and every cohomology class c may be written in the form $\frac{1}{2}(c + V(c)) + \frac{1}{2}(c - V(c))$. For $l = p$, the (-1)-eigenspace is a direct summand, too, due to Lemma 2.5.1. As the pairing on the total space is perfect, the same is true for the direct summands, which implies the claim.

Thus, $\langle d, d\rangle = \pm 2^k$ for some non-negative integer k. On the other hand, $c_1(d) \in H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Z}_2(1))$ is a generator of the (-1)-eigenspace and $\delta_{\mathfrak{z},2}$ is another. Hence, $c_1(d) = u \cdot \delta_{\mathfrak{z},2}$, for a certain unit $u \in \mathbb{Z}_2^*$. Consequently,

$$\langle d, d\rangle = \langle c_1(d), c_1(d)\rangle = u^2 \langle \delta_{\mathfrak{z},2}, \delta_{\mathfrak{z},2}\rangle = -2u^2\,,$$

which immediately shows that $k = 1$. Moreover, the minus sign is correct, since (-1) is not a square in $\mathbb{Q}_2$.

Fourth step. Conclusion.
In the particular case of a $K3$ surface, it is well known that, for a class $d \in \mathrm{Pic}(S_{\overline{K}})$ with $\langle d, d\rangle = -2$, either d or $(-d)$ is represented by an effective divisor [BHPV, Chap. VIII, Proposition 3.7]. But then V cannot interchange the two, a contradiction. □

Remarks 2.5.3. i) The regularity of the model $\underline{S}$ implies that the singular point on $\underline{S}_{\overline{\mathfrak{p}}}$ does not lift to a $\mathscr{O}^{\mathrm{nr}}_{K_{\mathfrak{p}}}$-rational point on $\underline{S}$.

ii) When there are two singular points instead of one, then the argument above then only shows that a non-trivial linear combination of $\delta_{\mathfrak{z}_1}$ and $\delta_{\mathfrak{z}_2}$ lies in $H^{\perp}_{\mathrm{alg}}$. The splitting field of $\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ may well ramify then. Cf. Corollary 2.5.4.a), below.

iii) There does not seem to be an obvious generalisation to other types of surfaces. For example, for rational surfaces one has $H^{\perp}_{\mathrm{alg}} = 0$, but $\delta_{\mathfrak{z},l}$ is clearly nonzero. Also, the argument heavily relies on the fact that, for $d \in \mathrm{Pic}(S_{\overline{K}})$ with $\langle d, d\rangle = -2$, either d or $(-d)$ is effective, which seems to be rather specific for $K3$ surfaces.

Corollary 2.5.4. *Let K be a number field and S a $K3$ surface over K. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of residue characteristic $\neq 2$ such that S has a regular, projective model $\underline{S}$ over $\mathscr{O}_{K,\mathfrak{p}}$, the geometric fibre $\underline{S}_{\overline{\mathfrak{p}}}$ of which has exactly one singular point $\mathfrak{z}$. Assume $\mathfrak{z}$ to be an ordinary double point.*

a) *Then the splitting field of* $\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is unramified at* $\mathfrak{p}$.

b) *The character* $\left(\frac{\Delta_{\mathrm{NS}}(S)}{\cdot}\right) = [\det \mathrm{NS}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}]$ *is unramified at* $\mathfrak{p}$. *Equivalently, the splitting field $L_{S,\mathrm{alg}}$ is unramified at* $\mathfrak{p}$.

Proof. a) Choose a prime number l, different from the residue characteristic of $\mathfrak{p}$. Then there is the short exact sequence [SGA7, Exp. XV, Théorème 3.4.(ii)]

$$0 \longrightarrow H^2_{\text{ét}}(\underline{S}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l(1)) \longrightarrow H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \longrightarrow \mathbb{Q}_l \longrightarrow 0\,, \qquad c \;\mapsto\; \langle c, \delta_{\mathfrak{z},l}\rangle\,,$$

provided by the theory of vanishing cycles. Together with the result of Theorem 2.5.2, it shows that every invertible sheaf on $S_{\overline{K}}$ extends to $\underline{S}_{\overline{\mathfrak{p}}}$. In other words, the splitting field of $\mathrm{Pic}(S_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in $K^{\mathrm{nr}}_{\mathfrak{p}}$.

b) This is a direct consequence of a). □

Proposition 2.5.5 (Reduction to one ordinary double point)**.** *Let K be a number field and S a proper K-scheme that is pure of even dimension i. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of residue characteristic $\neq 2$ such that S has a regular, projective model $\underline{S}$ over $\mathscr{O}_{K,\mathfrak{p}}$, the geometric fibre $\underline{S}_{\overline{\mathfrak{p}}}$ of which has exactly one singular point $\mathfrak{z}$. Assume $\mathfrak{z}$ to be an ordinary double point.*

a) *Then, for any prime l different from the residue characteristic of $\mathfrak{p}$, the $\mathrm{Gal}(\overline{K}/K)$-representation $H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2))$ is tamely ramified at $\mathfrak{p}$. The $\mathfrak{p}$-adic valuation of its conductor is equal to 1.*

b) *The quadratic character $\left(\frac{\Delta_{H^i(S)}}{\cdot}\right) = [\det H^i(S_{\overline{K}}, \mathbb{Q}_l(i/2))]$ is ramified at $\mathfrak{p}$. Equivalently, the splitting field L_S is ramified at $\mathfrak{p}$.*

Proof. a) In this generality, the short exact sequence provided by the theory of vanishing cycles reads [SGA7, Exp. XV, Théorème 3.4.(ii)]

$$0 \longrightarrow H^i_{\text{ét}}(\underline{S}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l(i/2)) \longrightarrow H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) \longrightarrow \mathbb{Q}_l \longrightarrow 0\,, \qquad c \;\mapsto\; \langle c, \delta_{\mathfrak{z},l}\rangle\,,$$

with $(\delta_{\mathfrak{z},l}, \delta_{\mathfrak{z},l}) = (-1)^{i/2}\cdot 2$. The $\mathfrak{p}$-adic valuation of the conductor is determined by the restriction of the representation to the decomposition group $D_{\mathfrak{p}} \cong \mathrm{Gal}(\overline{K}_{\mathfrak{p}}/K_{\mathfrak{p}})$. The exact sequence shows that the subspace $\delta^{\perp}_{\mathfrak{z},l} \subset H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2))$ is acted upon via the quotient $\mathrm{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \cong D_{\mathfrak{p}}/I_{\mathfrak{p}}$. I.e., the inertia group $I_{\mathfrak{p}}$ fixes $\delta^{\perp}_{\mathfrak{z},l}$ pointwise. As one has $V(\delta_{\mathfrak{z},l}) = -\delta_{\mathfrak{z},l}$ for $V \in I_{\mathfrak{p}}$, due to the Picard–Lefschetz formula, this yields

$$H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2))^{I_{\mathfrak{p}}} = \delta^{\perp}_{\mathfrak{z},l}\,.$$

Moreover, the action of $I_{\mathfrak{p}}$ respects orthogonality and cup product pairing, so $\delta_{\mathfrak{z},l}$ can be mapped only to $\pm\delta_{\mathfrak{z},l}$. Thus, there is a subgroup $I' \subset I_{\mathfrak{p}}$ of index two acting trivially. Since the residue characteristic of $\mathfrak{p}$ is $\neq 2$, this yields tameness.

In this case, the $\mathfrak{p}$-adic valuation of the conductor is given by [Se, formulae (11) and (8)]

$$\dim_{\mathbb{Q}_l} H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) - \dim_{\mathbb{Q}_l} H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2))^{I_{\mathfrak{p}}} = \dim_{\mathbb{Q}_l} H^i_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(i/2)) - \dim_{\mathbb{Q}_l} \delta^{\perp}_{\mathfrak{z},l} = 1\,.$$

b) This is very easily shown directly. The monodromy operator $V \in I_{\mathfrak{p}}$ fixes all cohomology classes perpendicular to $\delta_{\mathfrak{z},l}$ and sends $\delta_{\mathfrak{z},l}$ to $(-\delta_{\mathfrak{z},l})$. Therefore, $\det(V\colon H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1)) \circlearrowleft) = -1$. In particular, $\ker[\det H^2(S_{\overline{K}}, \mathbb{Q}_l(1))]$ does not

include all of $I_{\mathfrak{p}}$, and hence the field corresponding under the Galois correspondence to $\ker\left[\det H^2(S_{\overline{K}}, \mathbb{Q}_l(1))\right]$ is not contained in $K_{\mathfrak{p}}^{\mathrm{nr}}$. □

Remark 2.5.6. Suppose that, for some prime $\mathfrak{p}$, S has a model of bad reduction of the kind as described above. Then S does not have a model of good reduction at $\mathfrak{p}$. Indeed, the conclusions of Lemma 2.2.3.b) and Proposition 2.5.5.b) are independent of the model. The existence of a model of good reduction implies that $\left(\frac{\Delta_{H^i}(S)}{\cdot}\right)$ is unramified, while the existence of a model of the type above enforces ramification.

Corollary 2.5.7 (The jump character)**.** *Let K be a number field and S a $K3$ surface over K. Moreover, let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime of residue characteristic different from 2.*

a) *If S has good reduction at $\mathfrak{p}$, then $\tau_S = \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\cdot}\right)$ is unramified at $\mathfrak{p}$.*

b) *Suppose that S has a regular, projective model $\underline{S}$ over $\mathscr{O}_{K,\mathfrak{p}}$, the geometric fibre $\underline{S}_{\overline{\mathfrak{p}}}$ of which has exactly one singular point, and assume this to be an ordinary double point. Then the jump character $\tau_S = \left(\frac{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)}{\cdot}\right)$ ramifies at $\mathfrak{p}$.*

Proof. a) is clear from Lemmata 2.2.3.b) and 2.3.4.b), while the assertion of part b) follows from Corollary 2.5.4.b) together with Proposition 2.5.5.b). □

2.6. **Examples and experimental results.**

Algorithm 2.6.1 (Computing $\Delta_{H^2}(S)$)**.** Given a proper surface S over $\mathbb{Q}$, the set $\{q_1, \ldots, q_m\}$ of all bad primes of S, and an oracle for $\det(\mathrm{Frob}_p \colon H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(1)) \circlearrowleft)$ for any $p \neq q_j$, this algorithm computes $\Delta_{H^2}(S)$.

i) Add $q_0 := -1$ to the list of bad primes.

ii) Build a matrix A, the entries of which are the Legendre symbols $(\frac{q_j}{p_i})$, for good primes p_i chosen at random. Keep adding rows until the matrix has rank $m+1$ over $\{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$.

iii) Put $b_i = \det(\mathrm{Frob}_{p_i} \colon H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(1)) \circlearrowleft)$ and solve the linear system $Ax = b$ of equations. If the solution vector is $(x_0, \ldots, x_m) \in (\mathbb{Z}/2\mathbb{Z})^m$ then $\Delta_{H^2}(S)$ is the class of $(-1)^{e_0} q_1^{e_1} \cdots q_m^{e_m}$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, for $e_i \in \{0, 1\} \subset \mathbb{Z}$ representing the residue class $x_i \in \mathbb{Z}/2\mathbb{Z}$.

Remarks 2.6.2. i) The oracle for $\det(\mathrm{Frob}_p \colon H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(1)) \circlearrowleft)$ is, of course, provided by counting the points on S that are defined over $\mathbb{F}_p$ and some of its extensions.

ii) Dirichlet's Theorem on primes in arithmetic progressions ensures that there exist primes so that the matrix A has rank $m+1$.

iii) (An improvement.) In the case that 2 is a good prime, step i) of Algorithm 2.6.1 may be omitted. The sign of $\Delta_{H^2}(S)$ is then determined by the condition that the character be unramified at 2.

iv) (A further improvement.) Assume that the surface S is $K3$. Then, for some or many of its bad primes $p \neq 2$, it may happen that Proposition 2.5.5.b) applies. At such a prime, the jump character necessarily ramifies, which means that $\Delta_{H^2}(S)$ must be of odd p-adic valuation. Thus, the solution vector is bound to have a component $1 \in \mathbb{Z}/2\mathbb{Z}$ at the corresponding coordinates.

If S has a model in some $\mathbf{P}^N_{\mathbb{Z}}$ that is given by explicit equations, one may compute the set of all bad primes using Gröbner bases and integer factorisation, and finally analyse the singular points. Having found several primes to which Proposition 2.5.5.b) applies, this information may be used in order to get by with a linear system of equations of smaller size. In other words, less point counting is necessary.

It is our experience that this improvement of Algorithm 2.6.1 often leads to an enormous gain for a "random" surface, while for the constructed examples, which we present below, it would not help much.

v) There is an obvious modification of Algorithm 2.6.1 to directly determine the jump character.

Algorithm 2.6.3 (Statistical algorithm computing the jump character)**.** Given a $K3$ surface S over $\mathbb{Q}$ of even geometric Picard rank, the set $\{q_1, \dots, q_m\}$ of all bad primes of S, and a list of good non-jump primes, this algorithm determines a finite subgroup, containing the jump character, of the group of all characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in $\{1, -1\}$.

i) Add $q_0 := -1$ to the list of bad primes.

ii) Build a matrix A, the entries of which are the Legendre symbols $(\frac{q_j}{p_i})$, for the non-jump primes p_i.

iii) Determine the kernel of A. From each kernel vector, calculate a candidate for the jump character in the same way as in Algorithm 2.6.1.iii).

Remarks 2.6.4. i) If the kernel is the zero space, then this proves the jump character to be trivial. If the kernel is one-dimensional, then there are two possible answers. A non-trivial character, which is directly computed from a kernel vector, and the trivial one.

ii) If the kernel is still one-dimensional when the system of equations is rather overdetermined, then this gives strong evidence for the jump character to be non-trivial. In practice, we work with at least $4(m + 1)$ non-jump primes.

iii) The trivial character is unramified at every prime. Thus, as soon as it applies, Corollary 2.5.7.b) excludes the trivial character, and therefore makes the outcome of Algorithm 2.6.3 usually unique. Corollary 2.5.7.b) is useful as well to accelerate the calculations.

Example 2.6.5. Let S be the diagonal quartic in $\mathbf{P}^3_{\mathbb{Q}}$, given by $X_0^4+X_1^4+X_2^4+X_3^4 = 0$. Then the geometric Picard rank of S is 20 and the jump character is given by $(\frac{-1}{\cdot})$.

Proof. The model $\mathscr{S}$ of S that is given in $\mathbf{P}^3_{\mathbb{Z}}$ by the same equation has bad reduction only at 2. Hence, $\Delta_{H^2}(S) = \pm 1$ or ± 2. Counting points on the reductions $S_{\mathbb{F}_3}$ and $S_{\mathbb{F}_5}$, one finds that $\det(\mathrm{Frob}_p\colon H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(1))\circlearrowleft) = 1$ for both $p = 3$ and 5. Thus, Algorithm 2.6.1 shows that $\Delta_{H^2}(S) = 1$.

On the other hand, it is classically known that the 48 lines on $S_{\overline{\mathbb{Q}}}$ generate the geometric Picard group, which is of rank 20. In particular, $\mathrm{Pic}\, S_{\overline{\mathbb{Q}}}$ is defined over

$\mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2})$. Moreover, [Br, Appendix A, Examples A62, B33, C27, and D27] show that the Galois representation $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ splits into characters as

$$\chi_{\mathrm{triv}}^5 \oplus \chi_{\mathbb{Q}(i)}^3 \oplus \chi_{\mathbb{Q}(\sqrt{2})}^6 \oplus \chi_{\mathbb{Q}(\sqrt{-2})}^6 \,.$$

Here, for K a quadratic number field, $\chi_K \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \{1, -1\}$ denotes the character that becomes trivial after restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and defines the non-trivial quadratic character on $\mathrm{Gal}(K/\mathbb{Q})$. Consequently,

$$\bigwedge^{\max} \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{C} = \chi_{\mathrm{triv}}^{\otimes 5} \otimes \chi_{\mathbb{Q}(i)}^{\otimes 3} \otimes \chi_{\mathbb{Q}(\sqrt{2})}^{\otimes 6} \otimes \chi_{\mathbb{Q}(\sqrt{-2})}^{\otimes 6} = \chi_{\mathbb{Q}(i)} \,,$$

$\Delta_{\mathrm{Pic}}(S) = -1$ and $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S) = -1$. □

Remark 2.6.6. It is known at least since 1963 [T] that, in this example, there are no rank jumps, except for those explained by the jump character. I.e., one has $\operatorname{rk} \mathrm{Pic}\, S_{\overline{\mathbb{F}_p}} = 20$ for all primes $p \equiv 1 \pmod 4$. In fact, the eigenvalues of Frob_p on $H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(1))$ may be determined using Jacobi sums [IR, Chapter 8, Theorem 5] and it turns out that two of them are $\frac{\pi^2}{p}$ and its conjugate, for $p = \pi\overline{\pi}$ a factorisation in $\mathbb{Q}(i)$. Cf. [PS, particularly formulae (12) and (13)] for more details.

Example 2.6.7. Let S be the double cover of $\mathbf{P}^2_{\mathbb{Q}}$, given by $w^2 = X_0^6 + X_1^6 + X_2^6$. Then the geometric Picard rank of S is 20 and the jump character is given by $(\frac{-3}{\cdot})$.

Proof. The double cover $\mathscr{S}$ of $\mathbf{P}^2_{\mathbb{Z}}$ that is given by the same equation has bad reduction only at the primes 2 and 3. Hence, $\Delta_{H^2}(S) = \pm 1$, ± 2, ± 3 or ± 6. Furthermore, counting points on the reductions $S_{\mathbb{F}_5}$, $S_{\mathbb{F}_7}$, and $S_{\mathbb{F}_{13}}$, one finds that $\det(\mathrm{Frob}_p \colon H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(1)) \circlearrowleft) = 1$, for both $p = 5$ and $p = 13$, and (-1) for $p = 7$. Thus, Algorithm 2.6.1 shows $\Delta_{H^2}(S) = -1$.

The ramification sextic allows 18 tritangent lines of the type $X_i + \zeta_{12}^m X_j = 0$, for m odd. Furthermore, the 18 conics of type $X_i X_j + \frac{\zeta_6^m}{\sqrt[3]{2}} X_k^2 = 0$ are six times tangent to the ramification sextic. The irreducible components of the preimages of these curves together generate the geometric Picard group up to finite index. Working with the tritangents alone, one would end up with a sublattice that is not of full rank.

We implemented in `magma` a function to compute intersection numbers on S and, starting with 14 tritangent lines and six conics being six times tangent, found a non-degenerate 20×20 intersection matrix. Using this, it turns out that the splitting field of $\mathrm{Pic}\, S_{\overline{\mathbb{Q}}}$ is in fact $\mathbb{Q}(\zeta_3, \sqrt[3]{2}, i)$, having Galois group $S_3 \times \mathbb{Z}/2\mathbb{Z}$. Moreover, the Galois representation $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ splits into irreducible components as

$$\chi_{\mathrm{triv}}^4 \oplus \chi_{\mathbb{Q}(i)}^4 \oplus \chi_{\mathbb{Q}(\sqrt{3})}^3 \oplus \chi_{\mathbb{Q}(\sqrt{-3})}^3 \oplus V^3 \,,$$

where the characters are defined as above and V denotes the irreducible two-dimensional representation of the factor group $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$. Consequently,

$$\bigwedge^{\max} \mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C} = \chi_{\mathrm{triv}}^{\otimes 4} \otimes \chi_{\mathbb{Q}(i)}^{\otimes 4} \otimes \chi_{\mathbb{Q}(\sqrt{3})}^{\otimes 3} \otimes \chi_{\mathbb{Q}(\sqrt{-3})}^{\otimes 3} \otimes \chi_{\mathbb{Q}(\sqrt{-3})}^{\otimes 3} = \chi_{\mathbb{Q}(\sqrt{3})} \,,$$

which implies that $\Delta_{\mathrm{Pic}}(S) = 3$ and $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S) = -3$. □

Remark 2.6.8. Again, there are no rank jumps, except for those explained by the jump character. I.e., one has $\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_p}} = 20$ for all primes $p \equiv 1 \pmod 3$. The eigenvalues of Frob_p may again be determined using Jacobi sums. Here, it turns out that two of them are $J(\omega,\omega,\omega)/p$ [IR, Proposition 8.5.1] and its conjugate, for ω a primitive sextic character on $\mathbb{F}_p^*$. A short calculation, using [IR, Chapter 8, Theorem 3] and [BE, Theorems 3.1 and 3.4)] shows that these quantities evaluate to $(\frac{-1}{p})\frac{\pi^2}{p}$ and its conjugate, for π a primary element [IR, Proposition 9.3.5] in $\mathbb{Z}(\zeta_3)$ of norm p.

Remarks 2.6.9. i) The surfaces described in Examples 2.6.5 and 2.6.7 are $K3$ surfaces defined over $\mathbb{Q}$ of geometric Picard rank 20, and, as such, very particular objects. Due to the work of R. Livné [Lv], it is known that they are modular. Moreover, there is a formula describing the determinant of Frobenius on the transcendental part of cohomology in terms of the discriminant of the Picard lattice alone [Lv, Example 1.6]. Our calculations presented above are supposed to illustrate our method to compute the jump character in a situation where the Picard group is completely known.

ii) The surface from Example 2.6.7 is, up to isogeny, the Kummer surface associated with a product of an elliptic curve with itself. (Over $\mathbb{C}$, this is classically known, cf. [SI, Proof of Theorem 4]. Over $\mathbb{Q}$, an argument is given, e.g., in [BCFNW, Proposition 2.3].) Its jump character may also be determined that way.

Example 2.6.10. Let K be a number field and S the Kummer surface of an abelian surface over K that geometrically splits into a product $E_1 \times E_2$ of elliptic curves. Assume that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}} = 18$. Then there are two cases.

a) If the elliptic curves E_1 and E_2 are defined over K then the jump character of S is trivial.

b) If the elliptic curves E_1 and E_2 are defined over a quadratic extension $K(\sqrt{d})$ and conjugate to each other, then the jump character of S is $(\frac{d}{\cdot})$

Proof. a) The transcendental part $T \subset H^2_{\text{ét}}(S_{\overline{K}}, \mathbb{Q}_l(1))$ is isomorphic to

$$T \cong H^1_{\text{ét}}(E_1, \mathbb{Q}_l) \otimes H^1_{\text{ét}}(E_2, \mathbb{Q}_l(1))\,,$$

hence

$$\begin{aligned}\textstyle\bigwedge^{\max} T &\cong \textstyle\bigwedge^2 H^1_{\text{ét}}(E_1, \mathbb{Q}_l)^{\otimes 2} \otimes \bigwedge^2 H^1_{\text{ét}}(E_2, \mathbb{Q}_l(1))^{\otimes 2} \\ &\cong H^2_{\text{ét}}(E_1, \mathbb{Q}_l)^{\otimes 2} \otimes H^2_{\text{ét}}(E_2, \mathbb{Q}_l(2))^{\otimes 2} \\ &\cong H^2_{\text{ét}}(E_1, \mathbb{Q}_l(1))^{\otimes 2} \otimes H^2_{\text{ét}}(E_2, \mathbb{Q}_l(1))^{\otimes 2}\,,\end{aligned}$$

and both factors are acted upon trivially by $\mathrm{Gal}(\overline{K}/K)$.

b) Let $\sigma \in \mathrm{Gal}(\overline{K}/K)$ be any automorphism that changes the sign of $\sqrt{d}$. Then σ interchanges the components of $H^1_{\text{ét}}(E_1 \times E_2, \mathbb{Q}_l) = H^1_{\text{ét}}(E_1, \mathbb{Q}_l) \oplus H^1_{\text{ét}}(E_2, \mathbb{Q}_l)$. I.e., σ acts with eigenvalues (-1) and 1, both of multiplicity 2. Hence, on $H^2_{\text{ét}}(E_1 \times E_2, \mathbb{Q}_l) \cong \bigwedge^2 H^1_{\text{ét}}(E_1 \times E_2, \mathbb{Q}_l)$, one has the eigenvalues (-1), of multiplicity 4, and 1, of multiplicity 2.

However, under σ, the two algebraic classes in $H^2_{\text{ét}}(E_1, \mathbb{Q}_l) \oplus H^0_{\text{ét}}(E_2, \mathbb{Q}_l)$ and $H^0_{\text{ét}}(E_1, \mathbb{Q}_l) \oplus H^2_{\text{ét}}(E_2, \mathbb{Q}_l)$ are interchanged, so that the eigenvalues (-1) and 1 occur on the algebraic part. Therefore, the eigenvalues on T are (-1), with multiplicity 3, and 1, with multiplicity 1. Hence, every $\sigma \in \mathrm{Gal}(\overline{K}/K)$ as chosen above acts as (-1) on $\bigwedge^{\max} T$, which is enough to imply the claim. □

Example 2.6.11. In [CT, Examples 3.3, 3.4, and 3.5], Yu. Tschinkel and the first author reported numerical evidence for $\liminf_{B\to\infty} \gamma(S,B) \geqslant 1/2$, in the case of three $K3$ surfaces over $\mathbb{Q}$ of geometric Picard rank two.

This indeed follows from Corollary 2.4.5, once one proves that $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)$ is not a square in $\mathbb{Q}$. For each of the examples, one has $\mathrm{Pic}\, S_{\overline{\mathbb{Q}}} = \mathrm{Pic}\, S$ and therefore $\Delta_{\mathrm{Pic}}(S) = 1 \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$. Moreover, Algorithm 2.6.1 determines $\Delta_{H^2}(S)$ to, in this order,

$$-5 \cdot 151 \cdot 22490817357414371041 \cdot$$
$$387308497430149337233666358807996260780875056740850984213276970343278935342068889706146733313789\,,$$

$$53 \cdot 2624174618795407 \cdot 5128545618469648171394942020727783 41 \cdot$$
$$12152183700890287690767181021269217443536287 3 \cdot$$
$$68471243971589504569213004351581154456270727349961490419905638575 03\,,$$

and

$$-47 \cdot 3109 \cdot 4969 \cdot 14857095849982608071 \cdot 44541027766092834776258676433187443220258468801614 9 \cdot$$
$$65865270852505269999342419873884248599811521866797956036221419883010165025449071 1\,.$$

Each of the factors listed is reported as being prime by `magma`, version 2.21.8.

Example 2.6.12. Let S be the $K3$ surface over $\mathbb{Q}$, given by the equation

$$X_3^4 + f_2(X_0, X_1, X_2)X_3^2 + f_4(X_0, X_1, X_2) = 0\,, \tag{10}$$

for

$$f_2(X_0, X_1, X_2) := X_0^2 - X_0X_1 - X_0X_2 - X_1X_2 \qquad \text{and}$$
$$f_4(X_0, X_1, X_2) := -X_0^3X_2 + X_0X_1^2X_2 - X_1^4 - X_2^4\,.$$

Then the geometric Picard rank of S is 8 and the jump character of S is trivial.

Proof. First of all, a space quartic S of the form (10) is of geometric Picard rank at least 8. Indeed, the surface S comes equipped with a finite morphism $p\colon S \to S'$, which is generically 2:1, to an underlying degree two del Pezzo surface S'. The induced homomorphism $p^*\colon \mathrm{Pic}\, S'_{\overline{K}} \to \mathrm{Pic}\, S_{\overline{K}}$ doubles all intersection numbers. As, on a degree two del Pezzo surface, there are no non-trivial invertible sheaves that are numerically equivalent to zero, we see that p^* is necessarily injective. The claim follows.

Thus, for the first assertion, it suffices to find a prime p of good reduction such that $\mathrm{rk}\,\mathrm{Pic}\, S_{\overline{\mathbb{F}_p}} = 8$. For example, $p = 19$, 43, 61, 101, 109, 139, 149, 151, 157, and 163 do the job, as is easily shown in the usual way, based on counting points. Cf. [CT] for more details and further references.

On the other hand, a calculation using Gröbner bases shows that the model $\mathscr{S}$ of S given by the same equation in $\mathbf{P}^3_{\mathbb{Z}}$ has bad reduction only at the primes 2, 3, 47, and 431. Using Algorithm 2.6.3, one then proves the triviality of the jump character. In fact, only the first five non-jump primes 19, 43, 61, 101 and 109 are needed in order to do this. □

Remark 2.6.13. This example, and several others of the same kind, were found by a systematic inspection of all space quartics of the form (10), with coefficients from $\{-1, 0, 1\}$. This led to a sample of 183 098 318 non-singular surfaces in total, among which only a few hundred have trivial jump character, together with geometric Picard rank 8.

Example 2.6.14. Let S be the $K3$ surface over $\mathbb{Q}$, given by the equation

$$X_3^4 + f_4(X_0, X_1, X_2) = 0\,, \tag{11}$$

for

$$f_4(X_0, X_1, X_2) := X_0^4 - X_0^3X_1 - 2X_0^3X_2 - X_0^2X_1X_2 + X_0X_1^2X_2 - X_1^4 - X_2^4\,.$$

Then the geometric Picard rank of S is 8 and the jump character of S is $(\frac{-1}{\cdot})$.

Proof. Again, for the first claim, it suffices to find a prime p of good reduction such that $\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_p}} = 8$. For example, $p = 5$, 13, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137, 149, 157, 173, 181, 193, and 197 do the job.

Moreover, a calculation using Gröbner bases shows that the model $\mathscr{S}$ of S given in $\mathbf{P}^3_{\mathbb{Z}}$ by equation (11) has bad reduction only at the primes 2, 7, 6449, and 39 870 353. For the obvious integral model of the underlying degree two del Pezzo surface S', the same is true. Algorithm 2.6.1 then proves that $\Delta_{H^2}(S) = 2 \cdot 7 \cdot 6449 \cdot 39\,870\,353$ and that $\Delta_{H^2}(S') = -2 \cdot 7 \cdot 6449 \cdot 39\,870\,353$. The seven good primes up to 23 are in fact sufficient.

As, on a del Pezzo surface, every cohomology class is algebraic, we conclude that $\Delta_{\mathrm{NS}}(S') = \Delta_{H^2}(S')$. Furthermore, the linear map

$$p^* \otimes_{\mathbb{Z}} \mathbb{Q}\colon\ \mathrm{NS}(S') \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{Pic} S' \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Pic} S \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism, since it is injective and either $\mathbb{Q}$-vector space is of dimension 8. Therefore, $\Delta_{\mathrm{Pic}}(S) = \Delta_{\mathrm{NS}}(S')$, which implies the claim. □

Remark 2.6.15. Note that, for the surface above, Algorithm 2.6.3 would only prove that the jump character is $(\frac{-1}{\cdot})$ or trivial. Moreover, the non-triviality criterion, given in Corollary 2.5.7.b), could not resolve the ambiguity either.

2.7. **Interaction of jumps.**

As is well known, the geometric Picard rank always jumps under reduction when $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ is odd. The same is true, when there is real multiplication by some field E and one has an odd quotient $(22 - \operatorname{rk}\operatorname{Pic} S_{\overline{K}})/[E\colon\mathbb{Q}]$ [Ch14, Theorem 1(2)]. One might speculate in these cases, whether the jump character causes the jumps to be even larger. This does, however, not happen, as is shown by the examples below.

Lemma 2.7.1 ($K3$ surfaces having a non-singular degree two model)**.** *Let K be a number field and S a $K3$ surface over K, given by $w^2 = f_6(X_0, X_1, X_2)$, for f_6 a homogeneous form of degree 6. Write*

$$S_\lambda \colon \lambda w^2 = f_6(X_0, X_1, X_2)$$

for the quadratic twist by $\lambda \in K^*$. Then

$$\Delta_{H^2}(S_\lambda) = \lambda \Delta_{H^2}(S) \quad \textit{and} \quad \Delta_{\mathrm{Pic}}(S_\lambda) = \lambda^{\mathrm{rk}\,\mathrm{Pic}\,S_{\overline{K}}-1}\Delta_{\mathrm{Pic}}(S)\,.$$

Proof. Let $\mathfrak{p}$ be a good prime of S such that λ is a $\mathfrak{p}$-adic unit. Then, for the reductions mod $\mathfrak{p}$, one has that $(S_\lambda)_\mathfrak{p}$ is a non-trivial twist of $S_\mathfrak{p}$ in the case that λ is a non-square modulo $\mathfrak{p}$, and $S_\mathfrak{p} \cong (S_\lambda)_\mathfrak{p}$, otherwise. The assertion therefore follows from [EJ10, Fact 25]. □

Remark 2.7.2 (The odd rank case). Assume that $\mathrm{rk}\,\mathrm{Pic}\,S_{\overline{K}} = 1$. Then, for any prime $\mathfrak{p}$ of good reduction, there exists some $\mathfrak{p}$-adic unit $\lambda \in K^*$ such that $\Delta_{H^2}(S_\lambda)\Delta_{\mathrm{Pic}}(S_\lambda)$ is a non-square modulo $\mathfrak{p}$. If the effect of the odd rank added up with that of the transcendental character, then this would imply

$$\mathrm{rk}\,\mathrm{Pic}\,S_{\overline{\mathbb{F}_\mathfrak{p}}} = \mathrm{rk}\,\mathrm{Pic}(S_\lambda)_{\overline{\mathbb{F}_\mathfrak{p}}} \geqslant 4\,.$$

There are, however, explicit degree 2 $K3$ surfaces known of geometric Picard rank 1 that reduce to geometric Picard rank 2 at certain primes [vL, Theorem 3.1], cf. [EJ11, Example 5.1.1].

Example 2.7.3 (The case of real multiplication). Let S be the minimal desingularisation of the double cover of $\mathbf{P}^2_\mathbb{Q}$, given by $w^2 = X_0X_1X_2 \cdot f_3(X_0, X_1, X_2)$, for

$$f_3(X_0, X_1, X_2) := X_0^3 + 3X_0^2X_1 - 2X_0^2X_2 + 5X_0X_1^2 - X_0X_2^2 + 3X_1^3 \\ - 2X_1^2X_2 - 3X_1X_2^2 + 2X_2^3\,.$$

There is strong evidence that S has real multiplication by $\mathbb{Q}(\sqrt{3})$. Indeed, S is the surface $V^{(3)}_{1,1}$ from [EJ16, Conjectures 5.2]. Its model $\mathscr{S}$ being the double cover of $\mathbf{P}^2_\mathbb{Z}$ given by the same equation has bad reduction only at 2, 3, and 5. Modulo all other primes $p < 1000$, the reduction $\mathscr{S}_{\mathbb{F}_p}$ is of geometric Picard rank 18, except for $p = 263$, where the geometric Picard rank is 22. On the other hand, a sublattice of $\mathrm{Pic}\,S_{\overline{\mathbb{Q}}}$ of rank 16 may be explicitly given. Altogether, taking real multiplication for granted, one concludes that $\mathrm{rk}\,\mathrm{Pic}\,S_{\overline{\mathbb{Q}}} = 16$.

Concerning $\mathrm{Pic}\,S_{\overline{\mathbb{Q}}}$, there are 13 obvious generators, given by the pull-back of a general line on $\mathbf{P}^2_\mathbb{Q}$ and the exceptional curves obtained by blowing up the twelve singular points of the ramification locus. Ten of these singular points are defined over $\mathbb{Q}$, the other two over $\mathbb{Q}(\sqrt{-2})$. Hence, this part of $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_\mathbb{Z} \mathbb{C}$ splits into irreducible components as $\chi^{12}_{\mathrm{triv}} \oplus \chi_{\mathbb{Q}(\sqrt{-2})}$. Further generators are formed by a line and two conics in $\mathbf{P}^2_\mathbb{Q}$, the preimages of which split in S. From these altogether, one calculates that

$$\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_\mathbb{Z} \mathbb{C} = \chi^{12}_{\mathrm{triv}} \oplus \chi_{\mathbb{Q}(\sqrt{-2})} \oplus \chi_{\mathbb{Q}(\sqrt{2})} \oplus \chi_{\mathbb{Q}(\sqrt{6})} \oplus \chi_{\mathbb{Q}(\sqrt{-6})}$$

and, consequently, $\Delta_{\mathrm{Pic}}(S) = 1$. The splitting field of $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})$ is $\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$. On the other hand, Algorithm 2.6.1 yields $\Delta_{H^2}(S) = 3$, so the jump character is given by $(\frac{3}{\cdot})$.

If the effect of real multiplication added up with that of the transcendental character then this would imply $\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}}_p} > 18$ for every prime p such that $(\frac{3}{p}) = -1$, a contradiction.

3. Infinitely many rational curves

It has since long been conjectured that every $K3$ surface S over an algebraically closed field K contains infinitely many rational curves. The problem has been settled only recently by X. Chen, F. Gounelas, and C. Liedtke [CGL]. Many particular cases had been known before, most notably, that of odd Picard rank ([LL], based on the ideas of [BHT], cf. [Ben15]). Other sufficient conditions included those that S has infinitely many automorphisms, that S is elliptic [BT], or that K is of characteristic zero and S cannot be defined over $\overline{\mathbb{Q}}$ [BHT, Theorem 3].

As an application of Theorem 2.4.4, we show that the existence of infinitely many rational curves may be obtained rather easily in the situation that S is defined over $\overline{\mathbb{Q}}$, the jump character is non-trivial and the surface is otherwise generic. Our result is as follows.

Theorem 3.1. *Let K be a number field and S a $K3$ surface over K. Assume that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ is even, that $S_{\overline{K}}$ has neither real nor complex multiplication, and that $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)$ is a non-square in K.*
Then $S_{\overline{K}}$ contains infinitely many rational curves.

Remark 3.2. The transcendental part of $T \subset H^2(S_{\mathbb{C}}, \mathbb{Q})$, considered as a pure weight-2 Hodge structure, has an endomorphism algebra $\mathrm{End}_{\mathrm{Hs}}(T)$ that may only be a totally real field or a CM field [Za, Theorem 1.6.(a) and Theorem 1.5.1]. Our assumption concerning real and complex multiplication just means that $\mathrm{End}_{\mathrm{Hs}}(T) = \mathbb{Q}$, which is fulfilled as long as X is sufficiently general.

Lemma 3.3. *Let K be a number field and S a $K3$ surface over K. Assume that $S_{\overline{K}}$ has neither real nor complex multiplication. Then, for every quadratic field extension L/K, there are infinitely many inert primes $\mathfrak{p} \subset \mathscr{O}_K$ such that the reduction $S_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ is non-supersingular.*

Proof. The case that $\operatorname{rk}\operatorname{Pic} S_{\overline{K}} = 20$ is degenerate of the kind that T contains no $(1,1)$-classes. It is known in this situation that $S_{\overline{K}}$ automatically has complex multiplication [SI, Theorem 4]. We may therefore assume that $r := \dim T \geqslant 3$.

We choose a prime l and put $T_l \subset H^2_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$ to be the transcendental part of l-adic cohomology. Then $\mathrm{GO}(T_l, \langle .\,, . \rangle)_{\overline{\mathbb{Q}}_l}$ is an algebraic group over $\mathbb{Q}_l$. One has $\mathrm{GO}(T_l, \langle .\,, . \rangle)_{\overline{\mathbb{Q}}_l} \cong \mathrm{GO}_{r, \overline{\mathbb{Q}}_l}$, so that there are two connected components when r is even, while the group is connected for odd r.

As $\mathrm{End}_{\mathrm{Hs}}(T) = \mathbb{Q}$, we know that the image of the canonical continuous representation $\varrho_l \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GO}(T_l, \langle .\,, . \rangle)$ is Zariski dense either in $\mathrm{GO}(T_l, \langle .\,, . \rangle)$ or in

the neutral component $\mathrm{GO}^0(T_l, \langle .\,, . \rangle)$. Indeed, this follows from the Mumford–Tate conjecture, proven by S. G. Tankeev [Ta90, Ta95], together with Yu. G. Zarhin's explicit description of the Mumford–Tate group in the case of a $K3$ surface [Za, Theorem 2.2.1].

Now, let us assume, to the contrary, that for all but finitely many inert primes $\mathfrak{p}$, the reduction $S_{\overline{\mathbb{F}_\mathfrak{p}}}$ were supersingular. We put

$$M := \{\mathfrak{p} \subset \mathscr{O}_K \text{ prime ideal } | \\ \mathbb{F}_\mathfrak{p} \text{ is a prime field}, \#\mathbb{F}_\mathfrak{p} \neq l,\ \mathfrak{p} \text{ inert in } L,\ \mathfrak{p} \text{ good for } S,\ S_{\overline{\mathbb{F}_\mathfrak{p}}} \text{ supersingular}\}\,.$$

Then $M \subseteq I$, for I the set of all inert primes, and the difference $I \backslash M$ is of analytic density zero. Indeed, the prime ideals $\mathfrak{p}$ such that $\mathbb{F}_\mathfrak{p}$ is a prime field form a set of density 1.

For every prime ideal $\mathfrak{p} \subset \mathscr{O}_K$, we choose a geometric Frobenius automorphism $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(\overline{K}/K)$. According to the Chebotarev density theorem, the elements $\sigma^{-1}\,\mathrm{Frob}_\mathfrak{p}\,\sigma \in \mathrm{Gal}(\overline{K}/K)$, for $\mathfrak{p} \in M$ and $\sigma \in \mathrm{Gal}(\overline{K}/K)$, are topologically dense in the non-trivial coset of $\mathrm{Gal}(\overline{K}/K)$ modulo $\mathrm{Gal}(\overline{K}/L)$. Thus, there are two elements $\sigma_1, \sigma_2 \in \mathrm{Gal}(\overline{K}/K)$ such that

$$\{\, \sigma_j \sigma^{-1}\,\mathrm{Frob}_\mathfrak{p}\,\sigma \mid j = 1, 2,\ \mathfrak{p} \in M,\ \sigma \in \mathrm{Gal}(\overline{K}/K)\,\}$$

is dense in $\mathrm{Gal}(\overline{K}/K)$.

On the other hand, for $\mathfrak{p} \in M$ one has, due to supersingularity, $p \mid \mathrm{Tr}\,\mathrm{Frob}_{\mathfrak{p},T_l}$, when writing $p := \#\mathbb{F}_\mathfrak{p}$. Moreover, $\det \mathrm{Frob}_{\mathfrak{p},T_l} = \pm p^r$. As $|\,\mathrm{Tr}\,\mathrm{Frob}_{\mathfrak{p},T_l}\,| \leqslant rp$, this shows that

$$(\mathrm{Tr}\,\mathrm{Frob}_{\mathfrak{p},T_l})^r = \pm k^r \det \mathrm{Frob}_{\mathfrak{p},T_l}\,,$$

for $-22 < -r \leqslant k \leqslant r < 22$. Accordingly, let $C_k \subset \mathrm{GO}(T_l, \langle .\,, . \rangle)$ be the closed subscheme, defined by the equation $(\mathrm{Tr}\,A)^r = \pm k^r \det A$, and put $C := \bigcup_{k=-r}^{r} C_k$. Then $C \subset \mathrm{GO}(T_l, \langle .\,, . \rangle)$ is a closed subscheme and invariant under conjugation.

As $\mathrm{GO}(T_l, \langle .\,, . \rangle)_{\overline{\mathbb{Q}}_l} \cong \mathrm{GO}_{r, \overline{\mathbb{Q}}_l}$, for $r \geqslant 3$, it is easily seen that C cannot include a complete component of $\mathrm{GO}(T_l, \langle .\,, . \rangle)_{\overline{\mathbb{Q}}_l}$. I.e., one has $\dim C < \dim \mathrm{GO}(T_l, \langle .\,, . \rangle)$. Thus, the union $\sigma_1 C \cup \sigma_2 C$ cannot be the whole group. Consequently, the image of $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GO}(T_l, \langle .\,, . \rangle)$ is Zariski dense neither in $\mathrm{GO}(T_l, \langle .\,, . \rangle)$, nor in $\mathrm{GO}^0(T_l, \langle .\,, . \rangle)$, a contradiction. □

Proposition 3.4 (Li–Liedtke)**.** *Let K be a number field and $S \subset \mathbf{P}^N_K$ a $K3$ surface. Assume that $\mathrm{Pic}\,S = \mathrm{Pic}\,S_{\overline{K}}$ and that there is an infinite set J of primes such that $\mathrm{rk}\,\mathrm{Pic}\,S_{\overline{\mathbb{F}_\mathfrak{p}}} > \mathrm{rk}\,\mathrm{Pic}\,S$ for $\mathfrak{p} \in J$.*

Then there exist a sequence without repetitions $(\mathfrak{p}_j)_{j \in \mathbb{N}}$ of primes from J and a sequence $(D_{\mathfrak{p}_j})_{j \in \mathbb{N}}$ of rational curves $D_{\mathfrak{p}_j} \subset S_{\overline{\mathbb{F}_{\mathfrak{p}_j}}}$ such that the following two conditions are satisfied.

The class $(D_{\mathfrak{p}_j}) \in \mathrm{Pic}(S_{\overline{\mathbb{F}_{\mathfrak{p}_j}}})$ does not lie in the image of $\mathrm{Pic}\,S_{\overline{K}}$ under specialisation, for any j, and $\lim\limits_{j \to \infty} \deg D_{\mathfrak{p}_j} = \infty$.

Proof. This is [LL, Proposition 4.2]. □

Proposition 3.5 (Li–Liedtke)**.** *Let K be a number field and $S \subset \mathbf{P}^N_K$ a $K3$ surface. Assume that there exist a sequence without repetitions $(\mathfrak{p}_j)_{j\in\mathbb{N}}$ of primes and a sequence $(D_{\mathfrak{p}_j})_{j\in\mathbb{N}}$ of rational curves $D_{\mathfrak{p}_j} \subset S_{\overline{\mathbb{F}_{\mathfrak{p}_j}}}$ satisfying the following conditions.*

i) *Each $S_{\overline{\mathbb{F}_{\mathfrak{p}_j}}}$ is non-supersingular,*

ii) *$(D_{\mathfrak{p}_j})$ does not lie in the image of the Picard group $\operatorname{Pic} S_{\overline{K}}$ of the generic fibre under specialisation, for any j, and*

iii) $\lim\limits_{j\to\infty} \deg D_{\mathfrak{p}_j} = \infty$.

Then, for every $j \gg 0$, there exists a rational curve $D_j \subset S_{\overline{K}}$ such that its specialisation to $S_{\overline{\mathbb{F}_{\mathfrak{p}_j}}}$ is reducible, containing $D_{\mathfrak{p}_j}$ as one of its components. In particular, $\deg D_j > \deg D_{\mathfrak{p}_j}$.

Proof. This is shown in the proof of [LL, Theorem 4.3]. □

3.6. Proof of Theorem 3.1. As $\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)$ is a non-square in K, the field $L := K(\sqrt{\Delta_{H^2}(S)\Delta_{\mathrm{Pic}}(S)})$ is indeed a quadratic extension. By Lemma 3.3, we have an infinite set J of inert primes such that $S_{\overline{\mathbb{F}_{\mathfrak{p}}}}$ is non-supersingular for every $\mathfrak{p} \in N$. Moreover, $\operatorname{rk}\operatorname{Pic} S_{\overline{\mathbb{F}_{\mathfrak{p}}}} > \operatorname{rk}\operatorname{Pic} S_{\overline{K}}$ according to Theorem 2.4.4.b).

Let now $K' \supseteq K$ be the splitting field of $\operatorname{Pic} S_{\overline{K}}$. For each $\mathfrak{p} \in J$, there is at least one prime $\mathfrak{p}' \subset \mathscr{O}_{K'}$ lying above $\mathfrak{p}$. This yields an infinite set J' of primes in $\mathscr{O}_{K'}$, to which Proposition 3.4 applies. It provides a sequence $(\mathfrak{p}_j)_{j\in\mathbb{N}}$ of primes in J' without repetitions and rational curves $D_{\mathfrak{p}_j} \subset S_{\overline{\mathbb{F}_{\mathfrak{p}_j}}}$, not lying in the image of $\operatorname{Pic} S_{\overline{K}}$ under specialisation, such that $\lim_{j\to\infty} \deg D_{\mathfrak{p}_j} = \infty$. Knowing this, Proposition 3.5 yields a sequence $(D_j)_{j\in\mathbb{N}}$ of rational curves on $S_{\overline{K}}$ of degrees tending towards infinity. This completes the proof. □

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