Heights for line bundles on arithmetic varieties

Jörg Jahnel*

Abstract

Let X be an arithmetic variety and L be an element of the Néron-Severi group of its generic fiber X_K. Then there are only finitely many line bundles $\mathcal{L}$ on X, generically belonging to L, such that the degrees of $\mathcal{L}$ on the irreducible components of the special fibers of X and the height of $\mathcal{L}$ are bounded. The concept of a height used here is recalled. Several elementary properties of this height are proven.

1 Introduction

Let K be a number field, $\mathcal{O}_K$ its ring of integers and $\pi : X \to \operatorname{Spec} \mathcal{O}_K$ an arithmetic variety, i.e. a regular scheme, projective and flat over $\mathcal{O}_K$, whose generic fiber X_K we assume to be geometrically connected of dimension d. In [J96] there was suggested a definition for a concept of a height for line bundles on X, following the philosophy of [BoGS] that heights should be objects in arithmetic geometry, analogous to degrees in algebraic geometry. These heights depend on

i) a Kähler metric ω on $X(\mathbb{C})$ being invariant under complex conjugation F_∞ and

ii) a hermitian line bundle $\overline{\mathcal{T}} = (\mathcal{T}, \|.\|_\mathcal{T}) \in \widehat{\operatorname{Pic}}\,(X)$ or, equivalently, its first Chern class

$$\widehat{c}_1(\mathcal{T}, \|.\|_\mathcal{T}) = \overline{(T, g_T)} \in \widehat{\operatorname{CH}}^1(X).$$

1.1 Definition. *A hermitian metric $\|.\|_{\rm dis}$ on a line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ being invariant under F_∞ is called* distinguished, *if*

i) *its Chern form $c_1(\mathcal{L}_\mathbb{C}, \|.\|_{\rm dis})$ is harmonic and*

ii) $\widehat{\deg}\,(\det R\pi_*\mathcal{L}, \|.\|_Q) = 0$.

Here $\|.\|_Q$ is Quillen's metric [Qu], [BGS] *induced by $\|.\|_{\rm dis}$ on the determinant of cohomology* $\det R\pi_*\mathcal{L} \in \operatorname{Pic}(\operatorname{Spec} \mathcal{O}_K)$.

1.2 Lemma. *Assume the Euler characteristic $\chi(\mathcal{L}_K)$ is different from zero. Then*

a) *there exists a distinguished metric $\|.\|_{\rm dis}$ on $\mathcal{L}$.*

b) *$\widehat{c}_1(\mathcal{L}, \|.\|_{\rm dis}) \in \widehat{\operatorname{CH}}^1(X)$ is uniquely determined up to a summand $\overline{(0, C)}$, where C is a locally constant function on $X(\mathbb{C})$ being invariant under F_∞ and satisfying*

$$\int_{X(\mathbb{C})} C\omega^{\wedge d} = 0.$$

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c) *Such* $\overline{(0,C)} \in \widehat{\mathrm{CH}}^1(X)$ *are numerically equivalent to zero.*
The **Proof** is an application of the $\partial\overline{\partial}$-lemma of Hodge theory and elementary calculations. See [J96, section 1]. $\square$

1.3 Definition. *The* height $h_{\overline{\mathcal{T}},\omega}(\mathcal{L})$ *of a line bundle* $\mathcal{L} \in \mathrm{Pic}\,(X)$ *is given by*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) := \widehat{\deg}\,\pi_*\left[\widehat{c}_1(\mathcal{L}, \|.\|_{\mathrm{dis}}) \cdot \widehat{c}_1(\mathcal{T}, \|.\|_{\mathcal{T}})^d\right],$$

where $\|.\|_{\mathrm{dis}}$ *is a distinguished metric on* $\mathcal{L}$.

1.4 In [J96] we have analyzed the case of an arithmetic surface.
Theorem (Equivalence)**.** *Let* $C/\mathcal{O}_K$ *be a regular projective variety of dimension 2, flat over* $\mathcal{O}_K$ *and generically connected of genus* g, $x \in C_K(K)$ *be a* K*-valued point and* Θ *be the Theta divisor on* $J = \mathbf{Pic}^g(C_K)$ *(defined using* x*). On* $C(\mathbb{C})$ *let* ω *be a Kähler form invariant under* F_∞*. Fix, finally, a real number* H.

Then, for line bundles $\mathcal{L} \in \mathrm{Pic}\,(C)$*, fiber-by-fiber of degree* g *and of degree of absolute value less than* H *on every irreducible component of the special fibers of* C,

$$h_{\overline{\mathcal{O}(x)},\omega}(\mathcal{L}) = h_\Theta(\mathcal{L}_K) + \mathrm{O}(1),$$

where h_Θ *is the height on* J *defined using the ample divisor* Θ *and* $\overline{\mathcal{O}(x)} \in \widehat{\mathrm{Pic}}\,(C)$ *is any hermitian line bundle extending* $\mathcal{O}(x) \in \mathrm{Pic}\,(C_K)$.

1.5 In the higher dimensional case there is no analogue of that theorem to be expected, since there is no canonical polarization on the Picard scheme. That is why we are going directly to investigate the fundamental finiteness property with respect to $h_{\overline{\mathcal{T}},\omega}$. For arithmetic surfaces C and line bundles fiber-by-fiber of degree $g = g(C_K)$ this is a direct consequence of the theorem above.
Theorem (Finiteness)**.** *Let* $X/\mathcal{O}_K$ *be a regular, projective and flat scheme with* $X(K) \neq \emptyset$*, whose generic fiber* X_K *we assume to be connected. Equip* $X(\mathbb{C})$ *with a Kähler form* ω *being invariant under* F_∞*. Let* $L \in \mathrm{NS}(X_K)$ *be an equivalence class of line bundles satisfying the following condition.*

Fix $x \in X_K(K)$ *and let* $\mathcal{P} \in \mathrm{Pic}\,(X_K \times \mathbf{Pic}^L(X_K))$ *be tautological with* $\mathcal{P}|_{\{x\}\times\mathbf{Pic}^L(X_K)} \cong \mathcal{O}_{\mathbf{Pic}^L(X_K)}$*. Then* $(\det R\pi_{2*}\mathcal{P})^{-1}$ *is ample.* (1)

Further assume $\chi(\mathcal{L}_K) > 0$ *for* $\mathcal{L} \in L$*. Let, finally,* $\mathcal{T} \in \mathrm{Pic}\,(X)$*, underlying the hermitian line bundle* $\overline{\mathcal{T}} \in \widehat{\mathrm{Pic}}\,(X)$*, defining the heights, be ample. Then for every* $H \in \mathbb{R}$ *there are only finitely many* $\mathcal{L} \in \mathrm{Pic}\,(X)$ *with*

i) $\mathcal{L}_K \in L$,

ii) $|\deg_{\mathcal{T}}\mathcal{L}|_{X_{\mathfrak{p},i}}| < H$ *for every irreducible component* $X_{\mathfrak{p},i}$ *of the special fibers,*

iii) $h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) < H$.

1.6 Remarks. i) We note that $\mathbf{Pic}^0(X_K)$ is an abelian variety by [FGA, Exposé 236, Theorem 2.1]. It represents the functor

$$\begin{array}{rcl} \underline{\mathrm{Pic}}^0(X_K) : (\mathrm{Sch}/K) & \to & (\mathrm{Sets}) \\ T & \mapsto & \{\mathcal{F} \in \mathrm{Pic}\,(X_K \times T), \text{fiber-by-fiber algebraically} \\ & & \text{equivalent to zero}\}\,/\,\pi_2^*\,\mathrm{Pic}\,(T). \end{array}$$

$\mathbf{Pic}^L(X_K)$ is a torsor over $\mathbf{Pic}^0(X_K)$.

ii) Condition (1) seems to be a little bit dubious. Nevertheless it turns out to be fulfilled as the class L is sufficiently large.

1.7 Theorem. *Let K be a field and R/K be a smooth, proper and connected scheme with $R(K) \neq \emptyset$ and fix an $x \in R(K)$. Further, let $L \in \mathrm{NS}(R)$. Consider the tautological $\mathcal{P} \in \mathrm{Pic}\,(R \times \mathbf{Pic}^L(R))$ with $\mathcal{P}|_{\{x\}\times\mathbf{Pic}^L(R)} \cong \mathcal{O}_{\mathbf{Pic}^L(R)}$. Then*

$$(\det R\pi_{2*}\mathcal{P})^{-1}$$

is ample on $\mathbf{Pic}^L(R)$, if

a) *R is a curve,*

b) *R is an abelian variety and L is an ample class,*

c) *($R \in \mathrm{NS}(R)$ is arbitrary), L' is any equivalence class and A an ample equivalence class, for $L = L' + nA$ when $n \gg 0$.*

1.8 Remark. By the Nakai-Moishezon criterion [Kl, Chapter III, §1] the property of a line bundle to be ample depends only on its class modulo numerical equivalence. In particular we would be allowed to weaken our assumptions to $\mathcal{P}|_{\{x\}\times\mathbf{Pic}^L(R)}$ having to be numerically equivalent to zero.

1.9 Remarks. i) The remainder of this paper will mainly be devoted to the proof of the two theorems above. We will organize it as follows. In the next section we prove some elementary properties of the heights $h_{\overline{\mathcal{T}},\omega}(\mathcal{L})$. Section 3 will be devoted to the proof of Theorem 1.7. Then, in section 4, the finiteness statement is shown ignoring one technical detail concerning the existence of "suitable" sections of sufficiently high powers of an ample line bundle on the total space of a family. That point we put at the end of the paper to section 5.

ii) In this paper we do not deal with the questions related to extensions of the ground field K. The behaviour of the equivalence given in Theorem 1.4 under field extensions and a corresponding generalization of the Finiteness Theorem will be discussed in a forthcoming paper [J97]. Note that these questions are technically more difficult as the base changes $X \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L$ are not regular in general.

2 Elementary Properties

2.1 Remark. In the finiteness statement we restrict ourselves to line bundles (of given equivalence class modulo algebraic equivalence), whose degrees in the components of the special fibers are limited. To the contrary, Proposition 2.2 will investigate the behaviour of the height, when $\mathcal{L}$ is changed by a vertical divisor.

2.2 Proposition. *Let $D \in \mathrm{Div}\,(X)$ be supported over the prime $\mathfrak{p}$. Then, for $\mathcal{L} \in \mathrm{Pic}\,(X)$ with $\chi(\mathcal{L}_K) \neq 0$,*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}(D)) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \log(\sharp\mathcal{O}_K/\mathfrak{p}) \cdot \left[\deg_{\mathcal{T}} D - (\mathcal{T}_K^d) \cdot \frac{\chi\,(\mathcal{L}(D)|_D)}{\chi(\mathcal{L}_K)}\right].$$

In particular $h_{\overline{\mathcal{T}},\omega}(\mathcal{L}(D)) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L})$ if D is a complete fiber.

Proof. The short exact sequence $0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{L}(D)|_D \longrightarrow 0$ induces $\det R\pi_*\mathcal{L}(D) = \det R\pi_*\mathcal{L} \otimes \det R\pi_*\mathcal{L}(D)|_D$, which implies by Lemma 2.3 below, if we assume $\|.\|_{\mathcal{L}(D)} = C \cdot \|.\|_{\mathcal{L}}$,

$$\begin{aligned} \widehat{\deg}\,(\det R\pi_*\mathcal{L}(D), \|.\|_Q) &= \widehat{\deg}\,(\det R\pi_*\mathcal{L}, \|.\|_Q) \\ &+ \log(\sharp\mathcal{O}_K/\mathfrak{p}) \cdot \chi(\mathcal{L}(D)|_D) - \chi(\mathcal{L}_K)\,[K:\mathbb{Q}]\,\log C. \end{aligned}$$

For the distinguished metrics it follows that one has to put $C = (\sharp\mathcal{O}_K/\mathfrak{p})^{\frac{\chi(\mathcal{L}(D)|_D)}{\chi(\mathcal{L}_K)\cdot[K:\mathbb{Q}]}}$. Intersecting $\widehat{c}_1(\overline{\mathcal{T}})^d$ with $\widehat{c}_1(\mathcal{L}(D), \|.\|_{\mathrm{dis}}) - \widehat{c}_1(\mathcal{L}, \|.\|_{\mathrm{dis}})$ gives the claim. If D is a complete fiber the term in the brackets breaks down to the difference of the two self-intersection numbers $(\mathcal{T}_{\mathfrak{p}}^d) - (\mathcal{T}_K^d)$. $\square$

2.3 Lemma. *Let $Y/\mathcal{O}_K$ be an arithmetic variety and $\mathcal{L} \in \mathrm{Pic}\,(Y)$. If h and Ch with $C > 0$ are two hermitian metrics on $\mathcal{L}$, then*

$$\widehat{\deg}\,(\det R\pi_*\mathcal{L}, h_{Q,Ch}) = \widehat{\deg}\,(\det R\pi_*\mathcal{L}, h_{Q,h}) + [K:\mathbb{Q}]\,\chi(\mathcal{L}_K)\log C.$$

Proof. This is [J96, Lemma 1.3] and the definition of arithmetic degree. $\square$

2.4 Remark. We will study the behaviour of $h_{..}(\mathcal{L})$ under changes of the initial data. The differences will turn out to be of algebro-geometric or complex-analytic nature, i.e. they consist of degrees and Euler characteristics, data being bounded in the considerations concerning finiteness.

2.5 Proposition (Change of hermitian line bundle by vertical divisors and metric). *Let $\mathcal{L} \in \mathrm{Pic}\,(X)$ with $\chi(\mathcal{L}_K) \neq 0$.*

a) *Let $F \in \mathrm{Div}\,(X)$ be an (effective) divisor supported over $\mathfrak{p}$. Then*

$$h_{\overline{\mathcal{T}(F)},\omega}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + d\log(\sharp\mathcal{O}_K/\mathfrak{p})\cdot\deg_{\mathcal{T}}\mathcal{L}|_F + \sum_{k=2}^{d}\binom{d}{k}\log(\sharp\mathcal{O}_K/\mathfrak{p})\cdot\deg_{\mathcal{T}}\mathcal{L}|_{F^k},$$

where F^k denotes a representative of $F^k \in \mathrm{CH}^k_{X_{\mathfrak{p}}}(X)_{\mathbb{Q}}$. The right summand disappears as $F = [X_{\mathfrak{p}}]$ or $d = 1$.

b) *On $\mathcal{T}_{\mathbb{C}} \in \mathrm{Pic}\,(X(\mathbb{C}))$ let $\|.\|' = e^{\varphi}\cdot\|.\|$ be another hermitian metric. Then*

$$h_{\overline{\mathcal{T}}',\omega}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \frac{1}{2}\sum_{i+j=d-1}\int_{X(\mathbb{C})}\varphi\, c_1(\overline{\mathcal{T}})^i\, c_1(\overline{\mathcal{T}}')^j\,\mathcal{H}(c_1(\mathcal{L})),$$

where $\mathcal{H}$ denotes the harmonic representative of a cohomology class. In particular, when φ is constant,

$$h_{\overline{\mathcal{T}}',\omega}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \frac{d}{2}\,\varphi\,[K:\mathbb{Q}]\deg_{\mathcal{T}}\mathcal{L}_K.$$

Proof. a) As the change considered is of no effect on the distinguished metrics, we can calculate explicitly

$$\begin{aligned} h_{\overline{\mathcal{T}(F)},\omega}(\mathcal{L}) - h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) &= \widehat{\deg}\,\pi_*\Big(\widehat{c}_1\,(\mathcal{L}, \|.\|_{\mathrm{dis}})\cdot\Big[\Big(\widehat{c}_1(\overline{\mathcal{T}}) + \overline{(F,0)}\Big)^d - \widehat{c}_1(\overline{\mathcal{T}})^d\Big]\Big) \\ &= \sum_{k=1}^{d}\binom{d}{k}\widehat{\deg}\,\pi_*\Big(\widehat{c}_1\,(\mathcal{L}, \|.\|_{\mathrm{dis}})\cdot\widehat{c}_1(\overline{\mathcal{T}})^{d-k}\cdot\overline{(F,0)}^k\Big). \end{aligned}$$

b) Here a direct calculation shows

$$\begin{aligned} & h_{\overline{\mathcal{T}}',\omega}(\mathcal{L}) - h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) \\ = \; & \widehat{\deg}\,\pi_*\Big(\widehat{c}_1\,(\overline{\mathcal{L}})\cdot\Big[\widehat{c}_1(\overline{\mathcal{T}}')^d - \widehat{c}_1\,(\overline{\mathcal{T}})^d\Big]\Big) \end{aligned}$$

$$\begin{aligned} &= \widehat{\deg}\,\pi_*\Big(\widehat{c}_1\,(\overline{\mathcal{L}})\cdot\sum_{i+j=d-1}\widehat{c}_1(\overline{\mathcal{T}}')^j\,\widehat{c}_1(\overline{\mathcal{T}})^i\,\widehat{c}_1(\overline{\mathcal{T}}'\otimes\overline{\mathcal{T}}^{-1})\Big)\\ &= \widehat{\deg}\,\pi_*\Big(\widehat{c}_1\,(\overline{\mathcal{L}})\cdot\sum_{i+j=d-1}\widehat{c}_1(\overline{\mathcal{T}}')^j\,\widehat{c}_1(\overline{\mathcal{T}})^i\,\overline{(0,\varphi)}\Big)\\ &= \frac{1}{2}\sum_{i+j=d-1}\int_{X(\mathbb{C})}\mathcal{H}(c_1(\mathcal{L}))\,c_1(\overline{\mathcal{T}}')^j\,c_1\,(\overline{\mathcal{T}})^i\,\varphi. \qquad\square\end{aligned}$$

2.6 Proposition (Change of Kähler metric). *Let ω, ω' be Kähler metrics on $X(\mathbb{C})$. Then for every $l \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X(\mathbb{C}))$ there exists a smooth function g_l on $X(\mathbb{C})$, such that for every $\overline{\mathcal{T}} \in \widehat{\mathrm{Pic}}(X)$ and $\mathcal{L} \in \mathrm{Pic}\,(X)$ with $c_1(\mathcal{L}_\mathbb{C}) = l$*

$$h_{\overline{\mathcal{T}},\omega'}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \int_{X(\mathbb{C})} g_l\, c_1(\overline{\mathcal{T}})^d.$$

Proof. Let $\mathcal{L}_0 \in \mathrm{Pic}\,(X)$ with $c_1(\mathcal{L}_{0\mathbb{C}}) = l$ and $\|.\|$ and $\|.\|'$ be distinguished metrics on $\mathcal{L}_0$ with respect to ω and ω'. We obtain $g_l = \log\frac{\|.\|'^2}{\|.\|^2}$ and have to show that every $\mathcal{L} \in \mathrm{Pic}\,(X)$ with $c_1(\mathcal{L}_\mathbb{C}) = l$ yields exactly the same function. Hence it has to be shown that, if $\|.\|_\mathcal{L}$ is distinguished for ω, then $\|.\|'_\mathcal{L} := \frac{\|.\|'}{\|.\|}\cdot\|.\|_\mathcal{L}$ is for ω'. First,

$$c_1(\mathcal{L}_\mathbb{C}, \|.\|'_\mathcal{L}) = c_1(\mathcal{L}_{0\mathbb{C}}, \|.\|') - c_1(\mathcal{L}_{0\mathbb{C}}, \|.\|) + c_1(\mathcal{L}_\mathbb{C}, \|.\|_\mathcal{L})$$

is harmonic with respect to ω' since the last two summands are equal as they are harmonic with respect to ω and in the same cohomology class. Furthermore, by [BGS, Theorem 0.2 and Theorem 0.3]

$$\begin{aligned} 0 &= \widehat{\deg}\,(\det R\pi_*\mathcal{L}_0, \|.\|_{Q,\omega}) - \widehat{\deg}\,(\det R\pi_*\mathcal{L}_0, \|.\|'_{Q,\omega'})\\ &= \frac{1}{2}\int_{X(\mathbb{C})}\widetilde{\mathrm{ch}}(\mathcal{L}_{0\mathbb{C}}, \|.\|, \|.\|')\,\mathrm{Td}(\pi,\omega) + \frac{1}{2}\int_{X(\mathbb{C})}\mathrm{ch}(\mathcal{L}_{0\mathbb{C}}, \|.\|')\,\widetilde{\mathrm{Td}}(\pi,\omega,\omega'). \end{aligned}$$

But when we turn over to $\mathcal{L}$ in the formula above all the data remain unchanged, since this is tensoring by $(\mathcal{L}, \|.\|_\mathcal{L}) \otimes (\mathcal{L}_0, \|.\|)^{-1}$, the Chern form of whose restriction to $X(\mathbb{C})$, and therefore also the Chern character form, vanish. Note formula (1.3.5.2) in [GS90]. $\square$

2.7 Proposition (Birational morphisms). *Let $p : X' \to X$ be a morphism of arithmetic varieties inducing an isomorphism between the generic fibers. Then, for $\mathcal{L} \in \mathrm{Pic}\,(X)$,*

$$\begin{aligned} &h_{p^*\overline{\mathcal{T}},\omega}(p^*\mathcal{L})\\ &= h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) - \frac{(\mathcal{T}_K^d)}{\chi(\mathcal{L}_K)}\sum_\mathfrak{p}\log(\sharp\mathcal{O}_K/\mathfrak{p})\sum_{j\geq 1}(-1)^j\chi\left(X_\mathfrak{p}, (R^jp_*\mathcal{O}_{X'})|_{X_\mathfrak{p}}\otimes\mathcal{L}\right). \end{aligned}$$

If $R^jp_*\mathcal{O}_{X'} = 0$ for all $j \geq 1$, then the correction term vanishes. This is the case if X' is the blow-up of a regular embedding.

Proof. Let $\|.\|_\mathcal{L}$ be distinguished and $p^*\mathcal{L}$ be equipped with the same metric. We easily obtain

$$\begin{aligned} &\widehat{\deg}\,(\det R(\pi p)_*p^*\mathcal{L}, \|.\|_Q) - \widehat{\deg}\,(\det R\pi_*\mathcal{L}, \|.\|_Q)\\ &= \sum_{q\geq 0}(-1)^q\,(\log\sharp\mathrm{coker}h_q - \log\sharp\ker h_q)\,, \end{aligned}$$

where $h_q : H^q(X, \mathcal{L}) \to H^q(X', p^*\mathcal{L})$ is the natural map. Indeed, the Quillen metrics are the same on both sides and what remains is elementary linear algebra.

The Leray spectral sequence $H^i(X, R^j p_* p^* \mathcal{L}) \Longrightarrow H^{i+j}(X', p^*\mathcal{L})$ gives, when we note that $p_* p^* \mathcal{L} = \mathcal{L}$,

$$\begin{aligned}
& \sum_{q \geq 0} (-1)^q \left(\log \sharp \mathrm{coker} h_q - \log \sharp \ker h_q \right) \\
= \quad & \sum_{j \geq 1, i \geq 0} (-1)^{i+j} \log \sharp H^i(X, R^j p_* p^* \mathcal{L}) \\
= \quad & \sum_{j \geq 1} \sum_{\mathfrak{p}} (-1)^j \chi\big(X, (R^j p_* \mathcal{O}_{X'})|_{X_{\mathfrak{p}}} \otimes \mathcal{L}\big) \log(\sharp \mathcal{O}_K / \mathfrak{p}) \\
=: \quad & \varphi.
\end{aligned}$$

So the metric on $p^*\mathcal{L}$ has to be changed by the factor $e^{-\frac{\varphi}{\chi(\mathcal{L}_K)[K:\mathbb{Q}]}}$. It follows that

$$\widehat{c}_1 \left(p^* \mathcal{L}, \|.\|_{\mathrm{dis}} \right) = p^* \, \widehat{c}_1 \left(\mathcal{L}, \|.\|_{\mathrm{dis}} \right) - \frac{2\varphi}{\chi\left(\mathcal{L}_K\right) \, [K : \mathbb{Q}]} \, \overline{(0;1)}$$

and, thus

$$\begin{aligned}
h_{p^*\overline{\mathcal{T}}, \omega} \left(p^* \mathcal{L} \right) \quad &= \quad \widehat{\deg} \, \pi_* p_* \left[\left(p^* \widehat{c}_1(\mathcal{L}, \|.\|_{\mathrm{dis}}) - \frac{2\varphi}{\chi\left(\mathcal{L}_K\right) \, [K : \mathbb{Q}]} \overline{(0;1)} \right) \cdot p^* \widehat{c}_1(\overline{\mathcal{T}})^d \right] \\
&= \quad h_{\overline{\mathcal{T}}, \omega}(\mathcal{L}) - \frac{(\mathcal{T}_K^d)}{\chi(\mathcal{L}_K)} \, \varphi.
\end{aligned}$$

The last statement is [SGA6, Exposé VII, Lemma 3.5]. $\square$

2.8 Remark (Change of model). Let X and X' be arithmetic varieties connected by an isomorphism $\iota : X_K \to X'_K$ between their generic fibers. If $\overline{\mathcal{T}} \in \widehat{\mathrm{Pic}}\,(X)$ and $\overline{\mathcal{T}'} \in \widehat{\mathrm{Pic}}\,(X')$ are such that there is an isomorphism $\iota^* \overline{\mathcal{T}'_K} \xrightarrow{\cong} \overline{\mathcal{T}_K}$, then the results above control the difference $h_{\overline{\mathcal{T}}, \omega}(\mathcal{L}) - h_{\overline{\mathcal{T}'}, \omega}(\mathcal{L}')$ for $\mathcal{L} \in \mathrm{Pic}\,(X)$ and $\mathcal{L}' \in \mathrm{Pic}\,(X')$ with $\iota^* \mathcal{L}'_K \cong \mathcal{L}_K$. To make this precise one needs birational morphisms, inducing isomorphisms between the generic fibers, as follows:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "p"'] \arrow[dr, "p'"] & \\
X & & X'
\end{tikzcd}$$

For $\widetilde{X}$ one could choose the Zariski closure of the diagonal in

$$X_K \times_K X'_K \hookrightarrow X \times_{\mathcal{O}_K} X',$$

but, unfortunately, there is no regular model available. Nevertheless the results above work well, since on a singular scheme we can use the intersection product from [GS92, Theorem 4]. Alternatively one can resolve singularities by an alteration as proven in [Jo, Theorem 8.2], intersect the Chern classes of the pull-backs, push-forward again and divide by the degree. See [J97, Appendix] for details.

3 Negativity of the Determinant Line Bundle

3.1 Remark. In this section we will prove Theorem 1.7. First we recall the case of curves where the statement is well known. Then we deal with the higher dimensional case by some induction argument which is mainly based on the weak Lefschetz Theorem. Afterwards the situation of an abelian variety will be considered separately by a direct computation using the Grothendieck-Riemann-Roch theorem.

For this section we fix the following notations. K is a field and R/K a connected smooth projective scheme with $R(K) \neq \emptyset$. We choose one point $x \in R(K)$. $\mathbf{Pic}^0(R)$ denotes the neutral connected component of the Picard scheme of R. As R is regular, this is an abelian variety. Let $\mathcal{P} \in \mathrm{Pic}\big(R \times \mathbf{Pic}^0(R)\big)$ be the tautological line bundle, i.e. assume

i) $\mathcal{P}|_{\{x\} \times \mathbf{Pic}^0(R)} \cong \mathcal{O}_{\mathbf{Pic}^0(R)}$,

ii) If P/K is an arbitrary scheme and $\mathcal{E} \in \mathrm{Pic}\,(R \times P)$, fiber-by-fiber algebraically equivalent to zero, then there exist exactly one morphism $i : P \to \mathbf{Pic}^0(R)$ and $\mathcal{E}_0 \in \mathrm{Pic}\,(P)$ such that $\mathcal{E} \cong (\mathrm{id} \times i)^* \mathcal{P} \otimes \pi_1^* \mathcal{E}_0$.

Further, let $\mathcal{L}_0 \in \mathrm{Pic}\,(R)$. Our question is whether $(\det R\pi_{2*}(\mathcal{P} \otimes \pi_1^*\mathcal{L}_0))^{-1}$ is ample on $\mathbf{Pic}^0(R)$.

3.2 Lemma (Curves). *If R is a curve and $\mathcal{L}_0 \in \mathrm{Pic}\,(R)$ is arbitrary, then*

$$(\det R\pi_{2*}(\mathcal{P} \otimes \pi_1^*\mathcal{L}_0))^{-1}$$

is ample on the Jacobian $J := \mathbf{Pic}^0(R)$. It is, independently of $\mathcal{L}_0$, algebraically equivalent to $\mathcal{O}(\Theta)$.

Proof. The question depends only on the class of $\mathcal{L}_0$ modulo algebraic equivalence. So assume $\mathcal{L}_0 \cong \mathcal{O}(dx)$ for some $d \in \mathbb{Z}$. Consider two integers $d_1 < d_2$. The exact sequence $0 \to \mathcal{O}(d_1 x) \to \mathcal{O}(d_2 x) \to \mathcal{O}_{(d_2 - d_1)x} \to 0$ induces

$$\det R\pi_{2*}\,(\mathcal{P} \otimes \pi_1^*\mathcal{O}(d_1)) \cong \det R\pi_{2*}\,(\mathcal{P} \otimes \pi_1^*\mathcal{O}(d_2))\,.$$

So we are reduced to the case $d = g - 1$. But then by [Fa84, p. 396] or [MB, Proposition 2.4.2]

$$(\det R\pi_{2*}(\mathcal{P} \otimes \pi_1^*\mathcal{L}_0))^{-1} \cong \mathcal{O}(\Theta)$$

and this is ample on J. $\square$

3.3 Proposition (Higher dimensional varieties). *Let R/K be a connected proper smooth variety, $\mathcal{P} \in \mathrm{Pic}\,(R \times \mathbf{Pic}^0(R))$ the tautological line bundle and $\mathcal{A} \in \mathrm{Pic}\,(R)$ be ample. Then for every $\mathcal{L}_0 \in \mathrm{Pic}\,(R)$ there is an $n_0 \in \mathbb{N}$ such that*

$$\big(\det R\pi_{2*}(\mathcal{P} \otimes \pi_1^*(\mathcal{L}_0 \otimes \mathcal{A}^{\otimes n}))\big)^{-1}$$

is ample for $n \geq n_0$.

Proof. The statement is trivial for $\dim R = 0$ and we know it is true for $\dim R = 1$. So using induction we assume the theorem is clear for varieties of dimension $d-1$ and want to prove it for dimension $d \geq 2$.

$\mathcal{A}$ can be replaced by an arbitrary tensor power. So assume it is very ample. Then, by Bertini's theorem, $\mathcal{A} = \mathcal{O}_R(D)$, where D can be assumed to be a smooth connected prime divisor. The short exact sequence

$$0 \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}|_D \longrightarrow 0$$

induces

$$0 \longrightarrow \mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}) \longrightarrow \mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}) \longrightarrow \mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D) \longrightarrow 0$$

and therefore an isomorphism

$$\begin{aligned} & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}\big)\big) \\ \cong\ & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}\big)\big)\otimes\det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D\big)\big)\,. \end{aligned} \tag{2}$$

This formula will lead us to the assertion.

First Case: $\dim X = 2$.

The functor $\det R\pi_{2*}$ commutes with arbitrary base changes and the restriction morphism $\mathbf{Pic}^0(R) \to \mathbf{Pic}^0(D)$ is finite by the weak Lefschetz Theorem [SGA6, Exposé XIII, Lemma 3.11]. Therefore $(\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D)))^{-1}$ is dual to ample and, up to algebraic equivalence, it is independent of n as D is one dimensional. It follows that

$$\begin{aligned} & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}\big)\big) \\ \overset{\text{alg}}{\sim}\ & \det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*\mathcal{L}_0)\otimes(\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*(\mathcal{L}_0|_D)))^{\otimes n} \end{aligned}$$

being ample for $n \gg 0$.

Second Case: $\dim X > 2$.

Here the same argumentation as in the first case gives, combined with the induction assumption, still that $\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D))$, the correction term in (2), is dual to ample for $n \gg 0$. We shall study the dependence of this determinant on n.

The line bundle $\mathcal{O}(D)|_D$ is very ample on D and therefore it can be represented by a smooth connected prime divisor E on D. We get a formula analogous to (2).

$$\begin{aligned} & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D\big)\big) \\ \cong\ & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}|_D\big)\big)\otimes\det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_E\big)\big)\,. \end{aligned}$$

Here $(\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_E)))^{-1}$ is ample for $n \gg 0$ by weak Lefschetz and the induction assumption, meaning that

$$\begin{aligned} & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D\big)\big) \\ \cong\ & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}|_D\big)\big)\otimes(\text{ample})^{-1}. \end{aligned}$$

Formula (2) goes over into

$$\begin{aligned} & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+r}\big)\big) \\ \cong\ & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}\big)\big)\otimes\bigotimes_{i=1}^{r}\det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+i}|_D\big)\big) \\ =\ & \det R\pi_{2*}\big(\mathcal{P}\otimes\pi_1^*\big(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n}\big)\big)\otimes\big(\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*(\mathcal{L}_0\otimes\mathcal{A}^{\otimes n+1}|_D))\big)^{\otimes r} \\ \otimes\ & (\text{ample})^{-1}, \end{aligned}$$

and this is dual to ample as $r \gg 0$. $\square$

3.4 Proposition (Abelian varieties). *Let R/K be an abelian variety and $\mathcal{P} \in \operatorname{Pic}(R \times R^\vee)$ be the tautological line bundle. If $\mathcal{L}_0 \in \operatorname{Pic}(R)$ is ample, then*

$$(\det R\pi_{2*}(\mathcal{P} \otimes \pi_1^*\mathcal{L}_0))^{-1}$$

is ample on $R^\vee$.

Proof. $\mathcal{L}_0$ defines a polarization of the abelian variety R. So we have a finite morphism $c : R \to R^\vee$ satisfying $(\mathrm{id} \times c)^*\mathcal{P} \overset{\text{alg}}{\sim} m^*\mathcal{L}_0 \otimes (\pi_1^*\mathcal{L}_0)^{-1} \otimes (\pi_2^*\mathcal{L}_0)^{-1}$, where $m : R \times R \to R$ denotes the composition and π_1 and π_2 are the natural projections as usual [Mu]. Obviously, we have to show that

$$\big(\det R\pi_{2*}\big(m^*\mathcal{L}_0 \otimes (\pi_2^*\mathcal{L}_0)^{-1}\big)\big)^{-1}$$

is ample on R. For this we will use the Grothendieck-Riemann-Roch theorem

$$\operatorname{ch}(\pi_{2*}\alpha)\operatorname{Td}(T_R) = \pi_{2*}(\operatorname{ch}(\alpha)\operatorname{Td}(T_{R\times R}))\,, \tag{3}$$

which simplifies in our case to the formula $\operatorname{ch}(\pi_{2*}\alpha) = \pi_{2*}(\operatorname{ch}(\alpha))$ as for abelian varieties the Todd character is trivial. Here α denotes an element of $K_0(R \times R)$, the Chern characters map $K_0(?)$ to the Chow ring $\mathrm{CH}^{\cdot}_{\mathbb{Q}}(?)$ and π_{2*} is the push-forward in the sense of K-, respectively Chow-theory. But push-forward by π_2 in K-theory is the $R\pi_{2*}$, we are interested in, such that we can specialize (3) to

$$\begin{aligned} c_1\big(\det R\pi_{2*}\big(m^*\mathcal{L}_0 \otimes (\pi_2^*\mathcal{L}_0)^{-1}\big)\big) &= \big(\pi_{2*}\big(\operatorname{ch}\big(m^*\mathcal{L}_0 \otimes (\pi_2^*\mathcal{L}_0)^{-1}\big)\big)\big)^{(1)} \\ &= \frac{1}{(d+1)!}\,\pi_{2*}\Big(\big(m^*c_1(\mathcal{L}_0) - \pi_2^*c_1(\mathcal{L}_0)\big)^{d+1}\Big), \end{aligned}$$

where $d = \dim R$. Note that for line bundles $\operatorname{ch}(\mathcal{L}) = \sum_{i=0}^{\infty} \frac{1}{i!}(c_1(\mathcal{L}))^i$ but we are interested in the direct summand $\mathrm{CH}^1(R)_{\mathbb{Q}}$ only. Since computations in the Chow groups involving the composition map m are somewhat difficult we introduce the isomorphism

$$\begin{aligned} s : R \times R &\to R \times R \\ (r_1, r_2) &\mapsto (r_1 - r_2, r_2). \end{aligned}$$

We note $\pi_2 s = \pi_2$ and $ms = \pi_1$. It follows that

$$\begin{aligned} & c_1\left(\det R\pi_{2*}\left(m^*\mathcal{L}_0 \otimes (\pi_2^*\mathcal{L}_0)^{-1}\right)\right) \\ = \ & \frac{1}{(d+1)!}\,\pi_{2*}s_*\Big(s^*(m^*c_1(\mathcal{L}_0) - \pi_2^*c_1(\mathcal{L}_0))^{d+1}\Big) \\ = \ & \frac{1}{(d+1)!}\,\pi_{2*}\Big((\pi_1^*c_1(\mathcal{L}_0) - \pi_2^*c_1(\mathcal{L}_0))^{d+1}\Big) \\ = \ & \frac{1}{(d+1)!}\,\pi_{2*}\left[\sum_{i=0}^{d+1}(-1)^{d+1-i}\binom{d+1}{i}(\pi_1^*c_1(\mathcal{L}_0))^i\,(\pi_2^*c_1(\mathcal{L}_0))^{d+1-i}\right]. \end{aligned}$$

But the summand for $i = d$ is the only one not vanishing under π_{2*} and we get

$$\begin{aligned} c_1\big(\det R\pi_{2*}\big(m^*\mathcal{L}_0 \otimes (\pi_2^*\mathcal{L}_0)^{-1}\big)\big) &= -\frac{1}{d!}\,\pi_{2*}\Big[(\pi_1^*c_1(\mathcal{L}_0))^d\,(\pi_2^*c_1(\mathcal{L}_0))\Big] \\ &= -\frac{1}{d!}\,\pi_{2*}\pi_1^*c_1(\mathcal{L}_0)^d \cdot c_1(\mathcal{L}_0) \\ &= -\frac{1}{d!}\,(c_1(\mathcal{L}_0)^d) \cdot c_1(\mathcal{L}_0), \end{aligned}$$

where $c_1(\mathcal{L}_0)$ is an ample divisor and the self-intersection number $\left(c_1(\mathcal{L}_0)^d\right)$ is positive. Note finally that ampleness can be tested in $\mathrm{CH}^1(R)_{\mathbb{Q}}$. $\square$

3.5 Questions. Can one get a more precise result in the general case than that shown in Proposition 3.3? Can, in particular, analytic methods lead to better results than ours? One could use an analytic Riemann-Roch formula [BGS, Theorem 0.1] and investigate whether the determinant of cohomology has a hermitian metric with negative curvature. Proposition 3.4 above can be proven this way by a rather straightforward computation.

4 Finiteness

4.1 Scheme-Theoretic Constructions

4.1.1 Remark. In this section we will prove Theorem 1.5. It is very natural to adopt the point of view of moduli spaces. So fix $x \in X_K(K)$ and let $\mathcal{P} \in \mathrm{Pic}\,(X_K \times \mathbf{Pic}^L(X_K))$ be the tautological line bundle satisfying

$$\mathcal{P}|_{\{x\}\times\mathbf{Pic}^L(X_K)} \cong \mathcal{O}_{\mathbf{Pic}^L(X_K)}.$$

Since our concept of a height depends on the determinant of cohomology, we would like to change $\mathcal{P}$ in such a way that $\det R\pi_{2*}\mathcal{P}'$ becomes trivial. We have $\det R\pi_{2*}(\mathcal{P} \otimes \pi_2^*\mathcal{M}) = \det R\pi_{2*}\mathcal{P} \otimes \mathcal{M}^{\otimes l}$, where $l = \chi(\mathcal{P})$. So we would have to divide the line bundle $\det R\pi_{2*}\mathcal{P} \in \mathrm{Pic}\,(\mathbf{Pic}^L(X_K))$ by l. In general this is impossible. There are obstructions lying in $H^2_{\text{ét}}(\mathbf{Pic}^L(X_K), \mu_l)$. That is why we carry out the idea to consider pull-backs under étale covers.

4.1.2 Convention. *We will use the phrase* commutative proper group scheme (over a field) *for an extension of a commutative finite group scheme by an abelian variety. Note that in characteristic zero finite group schemes are nothing but finite groups* [Mu, §11].

4.1.3 Lemma. *Let K be a field, A/K an abelian variety and l a natural number. Then there is a finite flat morphism $p : A' \to A$ such that*

a) *for every $\mathcal{L} \in \mathrm{Pic}\,(A)$ the pull-back $p^*\mathcal{L}$ is l-divisible,*

b) *A' is an abelian variety,*

c) *if $2, l \nmid \mathrm{char}(K)$, then p is étale,*

d) *if K is a number field, then $A(K)/p_*A'(K)$ is a finite group.*

Proof. We choose the multiplication map $p = [2l] : A \to A$. Then b) is trivial and c) is standard as well as the fact that p is finite flat.

a) By [Mu, §6, Corollary 3] $p^*\mathcal{L} = [2l]^*\mathcal{L} = \mathcal{L}^{\otimes(2l^2+l)} \otimes ([-1]^*\mathcal{L})^{\otimes(2l^2-l)}$ being obviously l-divisible.

d) We have $A(K)/p_*A'(K) = A(K)/2lA(K)$ and this is a finite group by the weak Mordell-Weil theorem. $\square$

4.1.4 Corollary. *Let A/K be an abelian variety over a number field with $A(K) \neq \emptyset$ and $l \in \mathbb{N}$. Then there exist a commutative proper group scheme A'/K and a finite étale morphism $p : A' \to A$ such that*

a) *for every $\mathcal{L} \in \mathrm{Pic}\,(A)$ the pull-back $p^*\mathcal{L}$ is l-divisible.*

b) *The natural map* $p_* : A'(K) \to A(K)$ *is surjective.*
Proof. Let $G \subseteq A(K)$ be a (finitely generated) subgroup such that $G \twoheadrightarrow A(K)/2lA(K)$ becomes surjective and $G' \subseteq G$ be a torsion-free subgroup with $\sharp G/G' < \infty$. We put

$$A' := (A \times G)/G',$$

where $G' \subseteq A \times G$ denotes the subgroup $\{(g, -2l \cdot g) | g \in G'\}$. Obviously, A' is a commutative proper group scheme consisting of $\sharp G/2lG'$ components isomorphic to A.

$$\begin{array}{rcl} A \times G & \to & A \\ (a, g) & \mapsto & 2l \cdot a + g, \end{array}$$

induces the finite étale morphism $p : A' \to A$. Now assertion b) is clear since $A(K) \times G \twoheadrightarrow A(K)$ is surjective by construction. For a) we note that on every component $A'_i \subseteq A'$ one has $p|_{A'_i} = v_{x_i}[2l]$, where v_{x_i} denotes delay by some x_i. Hence $(p|_{A'_i})^*\mathcal{L} = [2l]^* v_{x_i}^* \mathcal{L}$ being obviously l-divisible. □

4.1.5 Remark. By the Mordell-Weil theorem one would be allowed to put $G = A(K)$. Note that we do not use this theorem here.

4.1.6 Proposition. *There exist a commutative proper group scheme* $\underline{P}/K$, *a line bundle* $\underline{\mathcal{U}} \in \mathrm{Pic}\,(X_K \times_K \underline{P})$, *a divisor* $\underline{D} \in \mathrm{Div}\,(X_K)$ *and a natural number* n *such that*

i) $\underline{\mathcal{U}}$ *belongs to* L *fiber-by-fiber.*

ii) *The morphism* $p : \underline{P} \to \mathbf{Pic}^L(X_K)$ *induced by* $\underline{\mathcal{U}}$ *is finite, étale and surjective on* K*-valued points. Further one has*

$$\det R\pi_{2*}\underline{\mathcal{U}} \cong \mathcal{O}_{\underline{P}}.$$

iii) $\underline{\mathcal{U}}(\pi_1^*\underline{D})^{\otimes n}$ *admits a suitable section* s, *i.e.* $s|_{X \times \{y\}} \neq 0$ *for every* $y \in \underline{P}$.
Proof. Consider the tautological $\mathcal{P} \in \mathrm{Pic}\,(X_K \times \mathbf{Pic}^L(X_K))$ and put $A := \mathbf{Pic}^L(X_K)$ and $l := \chi(\mathcal{P}|_{X_K \times \{.\}})$. Then for $\underline{P}$ take the A' given by Corollary 4.1.4 above. Choose $\mathcal{M} \in \mathrm{Pic}\,(\underline{P})$ such that $\mathcal{M}^{\otimes l} = p^*(\det R\pi_{2*}\mathcal{P})^{-1}$. Then put

$$\underline{\mathcal{U}} := (\mathrm{id} \times p)^*\mathcal{P} \otimes {\pi_2}^*\mathcal{M}.$$

This is easily seen to fulfill i) and the first part of ii). Further, one has

$$\begin{array}{rcl} \det R\pi_{2*}\underline{\mathcal{U}} & = & \det R\pi_{2*}((\mathrm{id} \times p)^*\mathcal{P}) \otimes \mathcal{M}^{\otimes l} \\ & = & p^* \det R\pi_{2*}\mathcal{P} \otimes \mathcal{M}^{\otimes l} \\ & = & \mathcal{O}_{\underline{P}}. \end{array}$$

iii) By assumption $(\det R\pi_{2*}\mathcal{P})^{-1}$ is ample, hence $p^*(\det R\pi_{2*}\mathcal{P})^{-1}$ and $\mathcal{M}$ are, too [EGA II, Proposition 4.6.13.ii]. Consequently, the line bundle $\underline{\mathcal{U}}|_{\{x\} \times \underline{P}} = p^*\mathcal{P}|_{\{x\} \times \mathbf{Pic}^L(X_K)} \otimes \mathcal{M} = \mathcal{M}$ is ample. Thus, $\underline{\mathcal{U}}$ is ample fiber-by-fiber since all the $\underline{\mathcal{U}}|_{\{.\} \times \underline{P}}$ are mutually algebraically equivalent (up to extension of ground field). By [EGA III, Théorème 4.7.1] it is relatively ample. Therefore, by [EGA II, Proposition 4.6.13.ii], there is a divisor $\underline{D} \in \mathrm{Div}\,(X_K)$ such that $\underline{\mathcal{U}}(\pi_1^*\underline{D})$ is ample. In the case $\dim X \geq 1$ the assertion is a consequence of Proposition 5.2. The case $\dim X = 0$ is very degenerate; one has necessarily $X = \mathrm{Spec}\,K$ and $\mathbf{Pic}^L(X_K) = \mathrm{Spec}\,K$ such that one can put $l := 1$ and the whole statement becomes trivial. □

4.1.7 Corollary. *There exist a quasi-projective smooth group scheme $P/\mathcal{O}_K$, a line bundle $\mathcal{U} \in \mathrm{Pic}\,(X \times_{\mathcal{O}_K} P)$, a divisor $D \in \mathrm{Div}\,(X)$ and $n \in \mathbb{N}$ such that*

i) $\mathcal{U}|_{X_K \times_K P_K}$ *belongs to* L *fiber-by-fiber.*

ii) *The morphism* $p : P_K \to \mathbf{Pic}^L(X_K)$ *induced by* $\mathcal{U}|_{X_K \times_K P_K}$ *is finite, étale and surjective on* K*-valued points. Further one has*

$$\det R\pi_{2*}\mathcal{U} \cong \mathcal{O}_P(F),$$

where $F = \sum_{\mathfrak{p} \in \mathrm{Spec}\,\mathcal{O}_K, i} a_{\mathfrak{p},i}\,(P_{\mathfrak{p},i})$ *is a divisor supported in the special fibers.*

iii) $\mathcal{U}(\pi_1^* D)^{\otimes n}$ *admits a generically suitable section* s, *i.e.,* $s|_{X_K \times \{y\}} \neq 0$ *for every geometric point* y *in* P_K.

iv) *All* K*-valued points of* P *can be extended to* $\mathcal{O}_K$*-valued points.*

Proof. For P choose the Néron model of the variety $\underline{P}$. As this is smooth, $X \times_{\mathcal{O}_K} P$ is a regular scheme. In particular, a divisor defining $\underline{\mathcal{U}} \in \mathrm{Pic}\,(X \times \underline{P})$ can be closed in the Zariski topology and defines a line bundle $\mathcal{U}$. For D we choose the Zariski closure of $\underline{D}$. Then iv) is the defining property of the Néron model, i) and the first part of ii) are clear and $\det R\pi_{2*}\mathcal{U}$ is trivial on P_K. Thus it allows a rational section defining F. For iii) we know that $\mathcal{U}(\pi_1^* D)^{\otimes n}$ has a suitable section s' being defined on $X_K \times_K P_K$. Its extension is a rational section whose poles are supported in the special fibers. Putting $s := as'$, where $a \in \mathcal{O}_K$ is sufficiently highly divisible, gives the claim. $\square$

4.2 Special Fibers

4.2.1 Proposition. *Let* $X_\mathfrak{p}$ *be a special fiber of* X *and* $X_{\mathfrak{p},i}$ *be an irreducible component of* $X_\mathfrak{p}$. *Then for any scheme* $P/\mathcal{O}_K$ *being of finite type and* $\mathcal{U} \in \mathrm{Pic}\,(X \times_{\mathcal{O}_K} P)$ *there exists* $D \in \mathbb{R}$ *such that* $|\deg_\mathcal{T} \mathcal{U}|_{X_{\mathfrak{p},i} \times y}| < D$ *for each* $y \in P(\mathcal{O}_K)$.

Proof. The question depends only on $y_\mathfrak{p} \in P_\mathfrak{p}(\mathcal{O}_K/\mathfrak{p})$. So we work with $P_\mathfrak{p}$ instead of P. Further, we consider, more generally, geometric points $\overline{y} \in P_\mathfrak{p}(\overline{\mathcal{O}_K/\mathfrak{p}})$. Obviously we may replace $P_\mathfrak{p}$ by an integral scheme T. By the result of de Jong [Jo] we may assume it is regular.

Note that by assumption every local ring of a closed point of $X_\mathfrak{p}$ is Cohen-Macaulay of dimension d. In particular $\dim X_{\mathfrak{p},i} = d$ for every component $X_{\mathfrak{p},i}$. So the degrees can be given as follows. Without restriction assume $\mathcal{T}$ to be very ample. Then let $i : X_{\mathfrak{p},i} \to \mathbf{P}^N_{\mathcal{O}_K/\mathfrak{p}}$ be an embedding such that $\mathcal{T} = i^*\mathcal{O}(1)$. Take the Chern class $c_1(\mathcal{U}|_{X_{\mathfrak{p},i} \times \{\overline{y}\}}) \in \mathrm{CH}^1(\overline{X_{\mathfrak{p},i}})$ and put

$$\begin{aligned} \deg_\mathcal{T} \mathcal{U}|_{X_{\mathfrak{p},i} \times \{\overline{y}\}} &:= \deg \left[\left(i_* c_1(\mathcal{U}|_{X_{\mathfrak{p},i} \times \{\overline{y}\}}) \right) \cdot c_1(\mathcal{O}(1))^{d-1} \right] \\ &= \deg\,(\mathrm{id} \times \{\overline{y}\})^* \left[((i \times \mathrm{id})_* c_1(\mathcal{U}|_{X_{\mathfrak{p},i} \times T})) \cdot c_1(\pi_1^* \mathcal{O}(1))^{d-1} \right] \end{aligned}$$

As we work on $\mathbf{P}^N_{\mathcal{O}_K/\mathfrak{p}} \times T$ now, the claim follows from the next lemma. $\square$

4.2.2 Lemma. *Let* k *be a field,* M/k *be a regular proper scheme of dimension* N *and* T/k *be a regular scheme of finite type. Further, let* $\alpha \in \mathrm{CH}^N(M \times T)$. *Then*

$$\deg \alpha_t = \deg\,(\mathrm{id} \times t)^* \alpha$$

is bounded when t runs over the geometric points of T, $t \in T(\overline{k})$.

Proof. Assume, the statement is wrong. By Noetherian induction there is a minimal closed subscheme $T_0 \subseteq T$ for whose geometric points $\deg \alpha_t$ is still unbounded. Obviously, T_0 must be integral. Let $T_{0,\mathrm{reg}} \subseteq T_0$ be its regular locus. Then choose a cycle $A \in Z^N(M \times T_{0,\mathrm{reg}})$ representing $\alpha|_{M \times T_{0,\mathrm{reg}}}$. By [EGA IV, Théorème 6.9.1] there is a nonempty open subset $U \subseteq T_{0,\mathrm{reg}}$ over which A is flat; consequently $\deg \alpha_t$ is bounded there. Thus it must be unbounded on $T_0 \setminus U$ being a contradiction. $\square$

4.2.3 We are now going to deal with the whole group of divisors concentrated in a special fiber $X_\mathfrak{p}$ of X.

Lemma. *Let $D_\mathfrak{p} \subseteq \mathrm{Div}\,(X)$ be the group of divisors supported over $\mathfrak{p}$. Then*

a) $\langle X_{\mathfrak{p},i}, X_{\mathfrak{p},j}\rangle := \deg_{\mathcal{T}} \mathcal{O}(X_{\mathfrak{p},i})|_{X_{\mathfrak{p},j}}$ *gives rise to a symmetric bilinear pairing.*

b) $\langle X_{\mathfrak{p},i}, X_{\mathfrak{p},j}\rangle$ *is the degree of* $[X_{\mathfrak{p},i}] \cdot [X_{\mathfrak{p},j}] \cdot c_1(\mathcal{T})^{d-1}$ *in* $\mathrm{CH}^{d+1}_{X_\mathfrak{p}}(X)_\mathbb{Q}$.

c) $\langle X_{\mathfrak{p},i}, X_{\mathfrak{p},j}\rangle \geq 0$ *for* $i \neq j$. *Equality holds if and only if* $|X_{\mathfrak{p},i}| \cap |X_{\mathfrak{p},j}| = \emptyset$.

d) $\langle X_{\mathfrak{p},i}, X_\mathfrak{p}\rangle = 0$ *for every* i.

Proof. a), b) and d) are standard. For c) the case $|X_{\mathfrak{p},i}| \cap |X_{\mathfrak{p},j}| = \emptyset$ is trivial. Otherwise $[X_{\mathfrak{p},i}] \cdot [X_{\mathfrak{p},j}]$ is an effective cycle as the intersection multiplicities can be defined using the Tor-formula of Serre and there are no higher Tor's occurring when one intersects properly with a locally main divisor. $\langle X_{\mathfrak{p},i}, X_{\mathfrak{p},j}\rangle$ is the degree of its image in $\mathbf{P}^N_{\mathcal{O}_K/\mathfrak{p}}$. $\square$

4.2.4 Corollary. *The pairing $\langle .,. \rangle$ on $D_\mathfrak{p}$ is negative semi-definite, whereas only multiples of the fiber have square zero.*

Proof. That works as in the case of a surface. Let $[X_\mathfrak{p}] = \sum_i C_i[X_{\mathfrak{p},i}]$. Then $0 = \langle X_{\mathfrak{p},i}, X_\mathfrak{p}\rangle = C_i\langle X_{\mathfrak{p},i}, X_{\mathfrak{p},i}\rangle + \sum\limits_{k\neq i} C_k\langle X_{\mathfrak{p},i}, X_{\mathfrak{p},k}\rangle$, hence

$$\langle C_i X_{\mathfrak{p},i}, C_i X_{\mathfrak{p},i}\rangle = -\sum_{k\neq i}\langle C_i X_{\mathfrak{p},i}, C_k X_{\mathfrak{p},k}\rangle.$$

With $b_1, \ldots, b_i, \ldots \in \mathbb{Q}$ it follows that

$$\begin{aligned}\left\langle \sum_i b_i C_i X_{\mathfrak{p},i}, \sum_i b_i C_i X_{\mathfrak{p},i} \right\rangle &= \sum_i b_i^2 \langle C_i X_{\mathfrak{p},i}, C_i X_{\mathfrak{p},i}\rangle + \sum_{j\neq k} b_j b_k \langle C_j X_{\mathfrak{p},j}, C_k X_{\mathfrak{p},k}\rangle \\ &= -\frac{1}{2}\sum_{j\neq k}(b_j - b_k)^2 \langle C_j X_{\mathfrak{p},j}, C_k X_{\mathfrak{p},k}\rangle.\end{aligned}$$

Therefore $\langle .,. \rangle$ is negative semi-definite; if the square is zero we must have $b_j = b_k$ as soon as $X_{\mathfrak{p},j}$ and $X_{\mathfrak{p},k}$ meet. Since $X_\mathfrak{p}$ is connected all b_i must be mutually equal. $\square$

4.2.5 Corollary. *Let $D \in \mathbb{R}$. Then there are only finitely many $\mathcal{L} \in \mathrm{Pic}\,(X)$ with $\mathcal{L}|_{X_K} \cong \mathcal{O}_{X_K}$ and $\left|\deg \mathcal{L}|_{X_{\mathfrak{p},i}}\right| < D$ for every $X_{\mathfrak{p},i}$.*

Proof. If $X_{\mathfrak{p},i}$ is irreducible, then $\deg \mathcal{L}|_{X_\mathfrak{p}} = 0$. Hence it is enough to show finiteness when $\deg \mathcal{L}|_{X_{\mathfrak{p},i}} = D_{\mathfrak{p},i}$ are fixed. Then by Corollary 4.2.4 $\mathcal{L}$ is uniquely determined up to a pull-back from $\mathrm{Pic}\,(\mathcal{O}_K)$. But the class number is finite. $\square$

4.2.6 Proposition. *Let $P/\mathcal{O}_K$ be a scheme of finite type, $\mathcal{U} \in \mathrm{Pic}\,(X \times_{\mathcal{O}_K} P)$ and $F \in \mathrm{Div}\,(X)$ a divisor being supported over $\mathfrak{p} \in \mathrm{Specm}\,\mathcal{O}_K$. Then*

$$h_{\overline{T},\omega}\,(\mathcal{U}|_{X\times y}(F)) = h_{\overline{T},\omega}\,(\mathcal{U}|_{X\times y}) + \mathrm{O}(1)$$

when y runs over $P(\mathcal{O}_K)$.

Proof. One has to use the formula from Proposition 2.2. It is clear that everything in the correction term remains constant except, may be, the term $\chi\,(\mathcal{L}(D)|_D)$ being here $\chi\,(\mathcal{U}|_{X\times y}(F)|_F) = \chi((\mathcal{U}(\pi_1^*F)|_{F\times P_\mathfrak{p}})_{y_\mathfrak{p}})$. But by flatness the Euler characteristic is locally constant on $P_\mathfrak{p}$ and therefore bounded. $\square$

4.3 Comparison With Heights For Cycles

4.3.1 Remark. In the situation of Corollary 4.1.7 we want to compare $h_{\overline{T},\omega}\,(\mathcal{U}|_{X\times y})$ with the height $h_{\overline{T}}\,(\mathrm{div}(s|_{X\times y}))$ of the cycle $\mathrm{div}(s|_{X\times y})$ for $y \in P(\mathcal{O}_K)$. We are going to use the finiteness statement for this concept of a height [BoGS, Theorem 3.2.5].

4.3.2 Proposition. *Let $P/\mathcal{O}_K$ be a scheme of finite type whose generic fiber P_K is proper and $\mathcal{U} \in$ Pic $(X \times_{\mathcal{O}_K} P)$ such that $\det R\pi_{2*}\mathcal{U} = \mathcal{O}_P(F)$ where F is a divisor supported in special fibers. Assume for some $n \in \mathbb{N}$ there are some divisor $D \in \mathrm{Div}\,(X)$ and a suitable section $s \in \Gamma\big(X \times_{\mathcal{O}_K} P, \mathcal{U}(\pi_1^*D)^{\otimes n}\big)$. Then, when y runs over $P(\mathcal{O}_K)$, one has*

$$h_{\overline{T},\omega}(\mathcal{U}|_{X\times y}) = \frac{1}{n}\,h_{\overline{T}}\,(\mathrm{div}(s|_{X\times y})) + \mathrm{O}(1).$$

Proof. We equip the line bundle $\mathcal{U}_\mathbb{C}$ on $(X \times_{\mathcal{O}_K} P)(\mathbb{C})$ with a hermitian metric, which is invariant under F_∞ and whose Chern form $c_1(\mathcal{U}_\mathbb{C}, \|.\|)$ is harmonic fiber-by-fiber. The hermitian line bundle $(\det R\pi_{2*}\mathcal{U}, \|.\|_Q) \in \widehat{\mathrm{Pic}}\,(P)$ commutes with arbitrary base changes, in particular we may consider its restrictions to $\mathcal{O}_K$-valued points $y \in P(\mathcal{O}_K)$. Then the unit section

$$\mathbf{1} \in \Gamma(P, \det R\pi_{2*}\mathcal{U}) = \Gamma(P, \mathcal{O}_P(F))$$

yields only finitely many different pole-zero-divisors on $\mathrm{Spec}\,\mathcal{O}_K$, while $\|\mathbf{1}\|$ remains bounded as $P(\mathbb{C})$ is compact. Consequently,

$$\widehat{\deg}\,(\det R\pi_{2*}\mathcal{U}|_{X\times y}, \|.\|_Q)$$

is bounded for $y \in P(\mathcal{O}_K)$. By Lemma 2.3 there is a distinguished metric $\|.\|_{\mathrm{dis}} = e^{C_y}\cdot\|.\|$ on $(\mathcal{U}|_{X\times y})_\mathbb{C}$, where C_y remains bounded in its dependence on $y \in P(\mathcal{O}_K)$. Hence,

$$\begin{aligned} h_{\overline{T},\omega}\,(\mathcal{U}|_{X\times y}) &= \widehat{\deg}\,\pi_*\Big[\widehat{c}_1(\mathcal{U}|_{X\times y}, \|.\|_{\mathrm{dis}})\cdot\overline{(T,g_T)}^d\Big] \\ &= \widehat{\deg}\,\pi_*\Big[\widehat{c}_1(\mathcal{U}|_{X\times y}, \|.\|)\cdot\overline{(T,g_T)}^d\Big] + \widehat{\deg}\,\pi_*\Big[(0,-2C_y)\cdot\overline{(T,g_T)}^d\Big] \\ &= \frac{1}{n}\,\widehat{\deg}\,\pi_*\Big[\widehat{c}_1(\mathcal{U}^{\otimes n}|_{X\times y}, \|.\|)\cdot\overline{(T,g_T)}^d\Big] + \mathrm{O}(1). \end{aligned}$$

But $\widehat{c}_1\,(\mathcal{O}(\pi_1^*D)|_{X\times y}, \|.\|) = \widehat{c}_1\,(\mathcal{O}(D), \|.\|)$ is independent of y as soon as we have metrized appropriately, i.e. we can replace $\mathcal{U}^{\otimes n}$ by $\mathcal{U}(\pi_1^*D)^{\otimes n}$, admitting the section s. We get

$$
\begin{aligned}
& h_{\overline{T},\omega}(\mathcal{U}|_{X\times y}) \\
= \; & \frac{1}{n}\,\widehat{\deg}\,\pi_*\Big[\overline{(\operatorname{div}(s|_{X\times y}), -\log\|s|_{X\times y}\|^2)}\cdot\overline{(T,g_T)}^{d}\Big] + \mathrm{O}(1) \\
= \; & \frac{1}{n}\,\widehat{\deg}\,\Big(\overline{(T,g_T)}^{d}\Big|\operatorname{div}(s|_{X\times y})\Big) - \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\int_{X(\mathbb{C})}\log\|s|_{X\times y}\|^2\cdot\omega_{(T,g_T)}^{\wedge d} + \mathrm{O}(1) \\
= \; & \frac{1}{n}\,h_{\overline{T}}\,(\operatorname{div}(s|_{X\times y})) - \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\int_{X(\mathbb{C})}\log\|s|_{X\times y}\|^2\cdot\omega_{(T,g_T)}^{\wedge d} + \mathrm{O}(1),
\end{aligned}
$$

where $(.|.) : \widehat{\mathrm{CH}}^d\,(X)\times \mathrm{Z}_d\,(X) \to \widehat{\mathrm{CH}}^1\,(\operatorname{Spec}\mathcal{O}_K)$ denotes the restriction pairing introduced in [BoGS, section 2.3] and $h_{\overline{T}}$ is the height for cycles being the main subject of that paper. It remains to bound the integrals. This is done in Proposition 4.3.4 below. $\square$

4.3.3 Remark. All the cycles $\operatorname{div}(s|_{X\times y})$ have the the same degree and they are different as soon as the line bundles defined by them are different. The Proposition above and [BoGS, Theorem 3.2.5] imply, among the line bundles $\mathcal{U}|_{X\times y}$ there are only finitely many with bounded height. In order to prove the asserted finiteness it will remain to bring the degree condition into play.

4.3.4 Proposition (fiber integrals). *Let M be a connected compact complex manifold of dimension d and T be a complex space. We consider a hermitian line bundle $\overline{\mathcal{L}}\in\overline{\operatorname{Pic}}\,(M\times T)$ and a section $s\in\Gamma(M\times T,\mathcal{L})$ not vanishing in any fiber $M\times\{t\}$. Let $\alpha\in C^{dd}(M\times T)$ be a continuous (d,d)-form. Then*

$$\int_M \alpha_t\cdot\log\|s_t\|^2$$

depends continuously on $t\in T$.

Proof. Let $t\in T$. For every $x\in M$ there are open euclidian neighbourhoods U_x of x and $U_t^{(x)}$ of t such that $\mathcal{L}$ is trivial on $U_x\times U_t^{(x)}$ and there exists a trivialization $(\mathcal{L}|_{U_x\times U_t^{(x)}}, \|.\|)\to(\mathcal{O}_{U_x\times U_t^{(x)}}, \|.\|_{\mathrm{can}})$ of bounded norm whose inverse is of bounded norm, too. Because of compactness finitely many U_x cover M, say $M=\bigcup_{i=1}^r U_{x_i}$. We may choose a smooth partition of unity embedded into $\{U_{x_i}\}_{i\in\{1,\ldots,r\}}$.

Thus, shrinking M and T we may assume that $\mathcal{L}$ is trivial if we allow M to be no more compact. Instead we still have at least $\operatorname{supp}\alpha\subset G\times T$, where $G\subseteq M$ is open and $\overline{G}$ is compact. The section s is reduced to a holomorphic function $M\times T\to\mathbb{C}$ not vanishing in any fiber. The asserted continuity is [St, Theorem 4.9]. $\square$

4.3.5 Conjecture. *Let M and T as above and $\alpha\in A^{kk}(M\times T)$ be a closed form. Let Z_1 and Z_2 be cycles on $M\times T$ of codimensions p_1, respectively p_2, $p_1>0$. Put $k:=d_1+1-p_1-p_2$. Further, let g be a Green form for Z_1 of* *logarithmic type along $|Z_1|$. Assume, Z_1 and Z_2 meet properly and for every $t \in T$, Z_1, Z_2 and $|Z_1| \cap |Z_2|$ meet $M \times \{t\}$ properly. Then*

$$\int_M \alpha_t g_t \delta_{Z_2,t}$$

depends continuously on t.

4.3.6 Remark. In [BoGS, Proposition 1.5.1] the case $\dim T = 1$ of Conjecture 4.3.5 is shown. The general case, although widely accepted, seems to be surprisingly complicated. Good results for the case $p_1 = 1$ are given in [St] and [Ki]. For our purposes they turned out to be sufficient. Nevertheless it seems that what is known is very much constructed for special cases.

4.3.7 Proof of Theorem 1.5. Let $\mathcal{L} \in \mathrm{Pic}\,(X)$ be as in the theorem. By construction $\mathcal{L}_K = \mathcal{U}|_{X \times \{y_K\}}$ for some $y \in P(\mathcal{O}_K)$, hence $\mathcal{L} = \mathcal{U}|_{X \times y} \otimes \mathcal{F}$, where $\mathcal{F}$ is supported in the special fibers. Now let $D := \max_{X_{\mathfrak{p},i};\, y \in P(\mathcal{O}_K)} \left|\deg_{\mathcal{T}} \mathcal{U}|_{X_{\mathfrak{p},i} \times \{y_{\mathfrak{p}}\}}\right|$ be the constant given by Proposition 4.2.1. Then $\left|\deg_{\mathcal{T}} \mathcal{F}|_{X_{\mathfrak{p},i}}\right| < H + D$ for every $X_{\mathfrak{p},i}$, i.e. there are only finitely many possibilities for $\mathcal{F}$. So it is enough to show finiteness for a fixed $\mathcal{F}$. But this is Proposition 4.2.6 combined with Remark 4.3.3. $\square$

5 Suitable Sections

5.1 Definition. *Let $Y \to T$ be a surjective proper morphism of schemes and $\mathcal{L} \in \mathrm{Pic}\,(Y)$. We say, $s \in \Gamma(Y, \mathcal{L})$ would be* suitable, *if $s|_{Y_t} \neq 0$ for all $t \in T$.*

5.2 It is our goal to prove the following
Proposition. *Let $Y \to T$ be a proper morphism of schemes of finite type over an infinite field k, all whose geometric fibers are at least one-dimensional, and let $\mathcal{L} \in \mathrm{Pic}\,(Y)$ be ample. Then there are $n \in \mathbb{N}$ and a suitable section $s \in \Gamma(Y, \mathcal{L}^{\otimes n})$.*

5.3 Definition. *Let k be an algebraically closed field and $P \subseteq \mathbf{P}_k^N$ be a reduced subscheme. Then the* linear hull $L(P)$ *of P is given by*

$$L(P) := \bigcap_{H \subseteq \mathbf{P}^N \text{ hyperplane},\, H \supseteq P} H.$$

5.4 Lemma. *Let k be algebraically closed, $P \subseteq \mathbf{P}_k^N$ be a closed subscheme and $P \to T$ be a surjective morphism. Assume*

$$\dim L\left((P_t)_{\mathrm{red}}\right) \geq \dim T$$

for every closed point $t \in T$. Then a general section $s \in \Gamma(P, \mathcal{O}(1))$ is suitable.
Proof. We have to consider the linear system $|\mathcal{O}(1)|$ of the hyperplanes H in $\mathbf{P}_k^N$ and we are looking for such satisfying $H \not\supseteq L\left((P_t)_{\mathrm{red}}\right)$ for every closed point $t \in T$. Those with $H \supseteq L\left((P_t)_{\mathrm{red}}\right)$ form a subspace of $(\mathbf{P}_k^N)^\vee$ of dimension $N - 1 - \dim L\left((P_t)_{\mathrm{red}}\right) \leq N - 1 - \dim T$. They form a family in $(\mathbf{P}_k^N)^\vee \times T$, the dimension of whose total space can obviously be at most $N-1$. The same is true for its projection into $(\mathbf{P}_k^N)^\vee$. $\square$

5.5 Corollary. *Let k be infinite, $P \subseteq \mathbf{P}_k^N$ be a closed subscheme and $P \to T$ be a surjective morphism. Assume*

$$\dim L\left((P_t)_{\mathrm{red}}\right) \geq \dim T$$

for every closed geometric point t of T. Then there exists a suitable section $s \in \Gamma(P, \mathcal{O}(1))$.

Proof. We are looking for $p \in \mathbf{P}^N(k)$, which does not belong to a given closed subvariety $V(f_1, \ldots, f_r)$. So it is enough to find $x_0, \ldots, x_N \in k$ such that the algebraic equation $f_1(x_0, \ldots, x_N) = 0$ is not fulfilled. This is possible as $\sharp k = \infty$. $\square$

5.6 Lemma. *Let k be algebraically closed, $P \subseteq \mathbf{P}_k^N$ be a reduced scheme of dimension ≥ 1 and $P' \subseteq \mathbf{P}_k^{N'}$ be its 2-uple embedding. Then*

$$\dim L(P') > \dim L(P).$$

Proof. Let $C(P) \subseteq \mathbf{A}^{N+1}$ and $C(P') \subseteq \mathbf{A}^{N'+1}$ denote the affine cones. Let $\delta := \dim L(P)$. Then there are $\delta + 1$ linearly independent points

$$Q_1, \ldots, Q_{\delta+1} \in C(P) \subseteq \mathbf{A}^{N+1}.$$

Choose coordinates such that Q_i becomes the i-th unit vector. Let $Q_{\delta+2} \in C(P) \subseteq \mathbf{A}^{N+1}$ be another point, whose image in $\mathbf{P}^N$ is different from those of $Q_1, \ldots, Q_{\delta+1}$.

$$\begin{array}{rcl} Q_1 & = & (1,\ 0,\ \ldots,\ 0,\ \ 0,\ \ldots,\ 0) \\ & \cdots & \\ Q_{\delta+1} & = & (0,\ \ \ldots,\ 0, 1,\ \ 0,\ \ldots,\ 0) \\ Q_{\delta+2} & = & (\alpha_1,\ \ldots,\ \alpha_{\delta+1}, 0,\ \ldots,\ 0) \end{array}$$

The 2-uple embedding gives

$$\begin{array}{rcl} Q_1 & \mapsto & (1,\ 0,\ \ldots,\ 0,\ \ 0,\ \ldots,\ 0) \\ & \cdots & \\ Q_{\delta+1} & \mapsto & (0,\ \ \ldots,\ 0, 1,\ \ 0,\ \ldots,\ 0) \\ Q_{\delta+2} & \mapsto & (\alpha_1^2,\ \ldots,\ \alpha_{\delta+1}^2,\ \ldots,\ \alpha_i\alpha_j,\ \ldots\ ). \end{array}$$

Here at least one of the products $\alpha_i\alpha_j$ is different from zero, implying $\dim L(P') \geq \delta + 1$, which is the claim. $\square$

5.7 Proof of Proposition 5.2. Let $\mathcal{L}^{\otimes m}$ define an embedding $Y \hookrightarrow \mathbf{P}^N$. Then Lemma 5.6 implies that $\mathcal{L}^{\otimes m \cdot 2^{\dim T}}$ gives an embedding such that

$$\dim L\left((Y_t)_{\mathrm{red}}\right) \geq \dim T$$

for every geometric fiber of $Y \to T$. Corollary 5.5 yields the assertion. $\square$

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Universität Göttingen
Mathematisches Institut
Bunsenstraße 3-5
37073 Göttingen
Germany

and

Université de Paris-Sud
Laboratoire de Mathématique
Bâtiment 425
91405 Orsay Cedex
France