A HEIGHT FUNCTION ON THE PICARD GROUP OF SINGULAR ARAKELOV VARIETIES

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For line bundles on possibly singular arithmetic varieties we construct height functions using arithmetic intersection theory. In the case of a model of an algebraic curve of genus g over a number field, for line bundles of degree g equivalence is shown to the height on the Jacobian defined by the Θ-divisor. The behaviour of this equivalence under change of the base field is investigated. In arbitrary dimension a finiteness property is proven.

1 Introduction

1.1 In a previous paper[17] there was given a construction for a height function on the Picard group of a regular arithmetic variety following the philosophy of Bost, Gillet and Soulé[3] that heights should be objects in arithmetic geometry analogous to degrees in algebraic geometry. Let us shortly recall this construction and fix notation.

Let K be a number field, $\mathcal{O}_K$ its ring of integers and $\mathcal{X}/\mathcal{O}_K$ an arithmetic variety, by which we mean a reduced scheme, projective and flat over $\mathcal{O}_K$, whose generic fiber X is geometrically connected. In the main body of the paper we will additionally assume X to be regular in order to have the theory of Quillen metrics available. Denote the dimension of X by d.

Further we choose a Kähler metric ω on the complex manifold $X(\mathbb{C})$ associated to X being invariant under complex conjugation F_∞. This transforms $\mathcal{X}$ into a so-called Arakelov variety $\overline{\mathcal{X}}$. Finally choose some metrized line bundle $\overline{\mathcal{T}} = (\mathcal{T}, \|.\|_\mathcal{T}) \in \widehat{\mathrm{Pic}}\,(\mathcal{X})$.

1.2 Definition. *Let $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X})$. We call a hermitian metric $\|.\|$ on the associated line bundle $\mathcal{L}_\mathbb{C}$ on the complex manifold $X(\mathbb{C})$* distinguished *if its Chern form $c_1(\mathcal{L}_\mathbb{C}, \|.\|)$ is harmonic with respect to ω and*

$$\widehat{\deg}\,(\det R\pi_*\mathcal{L}, \|.\|_Q) = 0. \qquad (1)$$

Here $\pi : \mathcal{X} \longrightarrow \mathrm{Spec}\ \mathcal{O}_K$ is the structural morphism and $\|.\|_Q$ denotes the

Quillen metric[2] *at the infinite places of* K.

1.3 Lemma. *Suppose the Euler characteristic* $\chi(\mathcal{L}_K)$ *is different from zero.*
a) *Then there exists a distinguished metric on* $\mathcal{L}$.
b) *If* $\|.\|$ *and* $\|.\|'$ *are distinguished metrics on* $\mathcal{L}$, *then* $\widehat{c}_1(\mathcal{L}, \|.\|)$ *and* $\widehat{c}_1(\mathcal{L}, \|.\|')$ *are numerically equivalent to each other.*

1.4 Definition *(The height). Let* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ *be a line bundle and assume* $\chi(\mathcal{L}_K) \neq 0$. *Then the* height *of* $\mathcal{L}$ *with respect to* ω *and* $\overline{\mathcal{T}}$ *is the arithmetic intersection number*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) := \widehat{\deg}\, \gamma\left((\mathcal{T}, \|.\|_{\mathcal{T}}),\ \ldots\ , (\mathcal{T}, \|.\|_{\mathcal{T}}), (\mathcal{L}, \|.\|)\right), \tag{2}$$

where $\|.\|$ *is a distinguished metric on* $\mathcal{L}$. *Here*

$$\gamma : \underbrace{\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \times\ \ldots\ \times \widehat{\mathrm{Pic}}(\mathcal{X})}_{d+1 \text{ times}} \to \widehat{\mathrm{CH}}^1(\mathrm{Spec}\ \mathcal{O}_K)_{\mathbb{Q}} \tag{3}$$

is the multi-linear map defined in Lemma A.4. When $\mathcal{X}$ *is regular, it is given by*

$$(\overline{\mathcal{U}_1},\ \ldots\ , \overline{\mathcal{U}_{d+1}}) \mapsto \pi_*\left(\widehat{c}_1(\overline{\mathcal{U}_1}) \cdot\ \ldots\ \cdot \widehat{c}_1(\overline{\mathcal{U}_{d+1}})\right). \tag{4}$$

1.5 We note that the concept of an arithmetic variety considered here is compatible with finite field extensions L/K in that sense that the base change of an arithmetic variety with $\mathrm{Spec}\, \mathcal{O}_L \to \mathrm{Spec}\, \mathcal{O}_K$ is again an arithmetic variety, but over $\mathcal{O}_L$. So one easily mimics the classical approach to heights for L-valued points on varieties defined over K.

Definition *(The normalized height). Let* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X} \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\, \mathcal{O}_L)$ *be some line bundle and assume* $\chi(\mathcal{L}_L) \neq 0$. *Then the* normalized height *of* $\mathcal{L}$ *with respect to* ω *and* $\overline{\mathcal{T}}$ *is given by*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) := \frac{1}{[L:K]}\, h_{\pi_1^*\overline{\mathcal{T}},\pi_1^*\omega}(\mathcal{L}), \tag{5}$$

where $\pi_1 : \mathcal{X} \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\, \mathcal{O}_L \to \mathcal{X}$ *denotes the projection to the first factor.*

1.6 Remark. The constructions above use the first Chern class of a hermitian line bundle and the intersection product on an arithmetic variety. These are the work of Gillet and Soulé[7,8]. They make extensive use of algebraic K-Theory. On an arithmetic variety there is no moving lemma available and thus the classical method to define the intersection product of cycles fails. Instead Gillet and Soulé analyze the Brown-Gersten-Quillen spectral sequence

$$E_1^{pq} := \bigoplus_{x \in \mathcal{X}^{(p)}} \mathrm{K}_{-p-q}(k(x)) \Rightarrow \mathrm{K}_{-p-q}(\mathcal{X}). \tag{6}$$

They show by investigation of the Adams operations that, after tensoring with $\mathbb{Q}$, it degenerates in the E_2-term, at least on the K_0-diagonal. Thus $\mathrm{K}_0(\mathcal{X})_{\mathbb{Q}} \cong \bigoplus_{i\geq 0} \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}}$ and this coincides with the splicing of $\mathrm{K}_0(\mathcal{X})_{\mathbb{Q}}$ into eigenspaces under the Adams operations. Therefore the product structure on $\mathrm{K}_0(\mathcal{X})$ gives rise to a product, transforming $\bigoplus_{i\geq 0} \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}}$ into a graded ring with unit.

In the appendix we will be going to discuss to some extent one point of view on the possible intersection products when $\mathcal{X}$ is singular. We will mainly follow the ideas of Fulton[5] as well as Remark 2.3.1.ii of Bost, Gillet and Soulé[3].

2 Elementary Properties

2.1 Lemma *(Field extensions). Let L'/L be an extension of number fields containing K and let $p : \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_{L'} \to \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L$ be the natural morphism. Then, for line bundles $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ with $\chi(\mathcal{L}_L) \neq 0$ one has*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(p^*\mathcal{L}). \tag{7}$$

Proof. First we note that $p^*\|.\|_{\mathrm{dis}}$ is a distinguished metric again. Indeed the canonical isomorphism $\det R\pi_{L*}p^*\mathcal{L} \xrightarrow{\cong} p^* \det R\pi_*\mathcal{L}$ is isometric since the formation of the Quillen metric commutes with arbitrary base changes. Therefore

$$\widehat{\deg}\,(\det R\pi_{L*}p^*\mathcal{L}, \|.\|_Q) = [L:K]\,\widehat{\deg}\,(\det R\pi_*\mathcal{L}, \|.\|_Q) = 0. \tag{8}$$

Consequently, one has

$$\begin{aligned} h_{\overline{\mathcal{T}},\omega}(p^*\mathcal{L}) &= \frac{1}{[L:K]}\,\widehat{\deg}\,\gamma\,(p^*(\mathcal{L}, \|.\|), p^*(\mathcal{T}, \|.\|), \dots) \\ &= \widehat{\deg}\,\gamma\,((\mathcal{L}, \|.\|), (\mathcal{T}, \|.\|), \dots) \\ &= h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) \end{aligned} \tag{9}$$

by Lemma A.4.ii). □

2.2 Proposition *(Special fibers).* a) *Let D be an effective Cartier divisor on $\mathcal{X}$ being non-trivial only in the special fiber over $\mathfrak{p} \in \mathrm{Specm}\,\mathcal{O}_K$. Then, for $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$, where L/K is a finite field extension and $\chi(\mathcal{L}_L) \neq 0$,*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}(D)) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \log(\sharp\mathcal{O}_K/\mathfrak{p}) \cdot \left[\deg_{\mathcal{T}} D - c_1(\mathcal{T}_K)^d \cdot \frac{\chi(\mathcal{L}(D)/\mathcal{L})}{\chi(\mathcal{L}_L)}\right]. \tag{10}$$

Note that $\mathcal{L}(D)/\mathcal{L} \cong \mathcal{L}(D)|_D$ if on the right hand side we consider D as a Weil divisor.
b) *In particular $h_{\overline{\mathcal{T}},\omega}(\mathcal{L}(D)) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L})$ if D is a complete fiber.*
Proof. This is essentially contained in Proposition 2.4 of a previous paper[16]. □

2.3 Remarks. i) The finiteness and comparison results we show below assume that the line bundles under consideration have bounded degrees in the components of the special fibers. To the contrary, the proposition above shows the behaviour of the height under large perturbations in special fibers.

ii) In the forthcoming propositions we will study $h_{.,.}(\mathcal{L})$ under changes of the initial data. The correction terms turn out to be of algebro-geometric and complex-analytic nature, i.e. they consist of degrees and Euler characteristics. The proofs were essentially given in a previous paper[16].

2.4 Proposition *(Change of defining bundle).* a) *Let F be a Cartier divisor supported over $\mathfrak{p}$. Then for any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ with $\chi(\mathcal{L}_L) \neq 0$*

$$h_{\overline{\mathcal{T}(F)},\omega}(\mathcal{L}) \tag{11}$$

$$= h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + d\log(\sharp\mathcal{O}_K/\mathfrak{p}) \cdot \deg_{\mathcal{T}} \mathcal{L}|_{\overline{F^1}} + \sum_{k=2}^{d} \binom{d}{k} \log(\sharp\mathcal{O}_K/\mathfrak{p}) \cdot \deg_{\mathcal{T}} \mathcal{L}|_{\overline{F^k}},$$

where $\overline{F^k}$ denotes the pull back to the geometric fiber $\overline{\mathcal{X}_\mathfrak{p}}$ of a cycle representing $F^k \in \mathrm{CH}^k_{\mathcal{X}_\mathfrak{p}}(\mathcal{X})_\mathbb{Q}$. Note that the right summand disappears as $F = [\mathcal{X}_\mathfrak{p}]$ or for an arithmetic surface.
b) *On the line bundle $\mathcal{T}_\mathbb{C}$ on $X(\mathbb{C})$ let $\|.\|' = e^\varphi \cdot \|.\|$ be another hermitian metric. Then for any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ with $\chi(\mathcal{L}_L) \neq 0$*

$$h_{\overline{\mathcal{T}}',\omega}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \frac{1}{2\,[L:K]} \sum_{i+j=d-1} \int_{X_L(\mathbb{C})} \varphi\, c_1(\overline{\mathcal{T}})^i c_1(\overline{\mathcal{T}}')^j \mathcal{H}\left(c_1(\mathcal{L}_\mathbb{C})\right), \tag{12}$$

where $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}'$ mean the pull-backs of these hermitian line bundles under the natural projection $X_L(\mathbb{C}) \to X(\mathbb{C})$ and $\mathcal{H}$ denotes the harmonic projection. In particular, when φ is constant

$$h_{\overline{\mathcal{T}}',\omega}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \frac{d}{2}\,\varphi\,[K:\mathbb{Q}]\deg_{\mathcal{T}} \mathcal{L}_L. \tag{13}$$

□

2.5 Proposition *(Change of Kähler metric). Let ω, ω' be Kähler metrics on $X(\mathbb{C})$. Then for every $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ with $\chi(\mathcal{L}_L) \neq 0$ there exists a smooth function $g_\mathcal{L}$ on $X(\mathbb{C})$, depending only on the homological* *equivalence class of* $\mathcal{L}_{\mathbb{C}}$*, such that*

$$h_{\overline{\mathcal{T}},\omega'}(\mathcal{L}) = h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) + \frac{1}{2}\int\limits_{X(\mathbb{C})} g_{\mathcal{L}}\, c_1(\overline{\mathcal{T}}_{\mathbb{C}})^d. \tag{14}$$

□

2.6 Proposition *(Birational morphisms). Let* $p : \mathcal{X}' \longrightarrow \mathcal{X}$ *be a morphism of arithmetic varieties inducing an isomorphism between the generic fibers. Then, for any line bundle* $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ *with* $\chi(\mathcal{L}_L) \neq 0$

$$\begin{aligned} h_{p^*\overline{\mathcal{T}},\omega}(p^*\mathcal{L}) &= h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) \\ &- \frac{c_1(\mathcal{T}_K)^d}{[L:K]\,\chi(\mathcal{L}_L)} \sum_{\mathfrak{p}} \log(\sharp\mathcal{O}_K/\mathfrak{p}) \sum_{j\geq 1}(-1)^j \chi\left(\overline{\mathcal{X}_{\mathfrak{p}}}, (R^j p_*\mathcal{O}_{\mathcal{X}'})|_{\overline{\mathcal{X}_{\mathfrak{p}}}} \otimes \mathcal{L}|_{\overline{\mathcal{X}_{\mathfrak{p}}}}\right). \end{aligned} \tag{15}$$

Note that, if $R^j p_*\mathcal{O}_{X'} = 0$ *for all* $j \geq 1$*, then the correction term vanishes.* □

2.7 As line bundles can be defined by divisors it is natural to expect a relation of the height of a line bundle with that of a corresponding divisor. In general, linearly equivalent divisors will have different heights, such that the relation can not be too simple.

Proposition *(Comparison with the height of divisor). Let* $\mathcal{P}/\mathcal{O}_K$ *be a scheme of finite type whose generic fiber is proper. On* $\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathcal{P}$ *consider a line bundle* $\mathcal{U}$ *such that* $\det R\pi_{2*}\mathcal{U}|_{\mathcal{P}_K} \cong \mathcal{O}_{\mathcal{P}_K}$ *and* $\chi(\mathcal{U}|_{\mathcal{X}\times\overline{y}}) \neq 0$ *for each* $\overline{y} \in \mathcal{P}_K(\overline{K})$*. Assume that for some* $\mathcal{H} \in \mathrm{Pic}\,(\mathcal{P})$ *and* $n \in \mathbb{N}$ *the tensor power* $(\mathcal{U} \otimes \pi_2^*\mathcal{H})^{\otimes n}$ *has a suitable section* s*, i.e.* $s|_{X\times\{\overline{y}\}} \neq 0$ *for each* $\overline{y} \in \mathcal{P}_K(\overline{K})$*. Then for every number field* L *and any* $p \in \mathcal{P}(\mathcal{O}_L)$

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{U}|_{\mathcal{X}\times p}) = \frac{1}{n[L:K]}\, h_{\overline{\mathcal{T}}}(\mathrm{div}(s|_{\mathcal{X}\times p})) - \frac{1}{[L:K]}\, h_{\overline{\mathcal{H}}}(p)\cdot c_1(\mathcal{T}_L)^d + O(1), \tag{16}$$

where $\mathcal{H}$ *is equipped with any hermitian metric, while* $h_{\overline{\mathcal{T}}}$ *and* $h_{\overline{\mathcal{H}}}$ *denote heights for cycles*[3]*.* $O(1)$ *is a bounded function on*

$$\coprod_{L/K \text{ finite}} \mathcal{P}(\mathcal{O}_L), \tag{17}$$

but the bound depends on $\overline{\mathcal{H}}$ *and* s *(and, of course, on the original data* $\mathcal{X}, \omega, \overline{\mathcal{T}}, \mathcal{P}, \mathcal{U}$*).*

Proof. Equip the line bundle $\mathcal{U}_{\mathbb{C}}$ on $\coprod_{\sigma:K\hookrightarrow\mathbb{C}} \mathcal{X}(\mathbb{C}) \times \mathcal{P}(\mathbb{C})$ with a hermitian metric $\|.\|$ being invariant under F_∞ and admitting a harmonic Chern form $c_1(\mathcal{U}_{\mathbb{C}}, \|.\|)$ fiber-by-fiber. One easily sees that the function

$$\begin{array}{rcl} \coprod_{L/K \text{ finite}} \mathcal{P}(\mathcal{O}_L) & \longrightarrow & \mathbb{R} \\ p & \mapsto & \frac{1}{[L:K]}\,\widehat{\deg}\,(\det R\pi_{2*}\mathcal{U}|_{\mathcal{X}\times p}, \|.\|_Q) \end{array} \tag{18}$$

is bounded. Hence there is a distinguished metric $\|.\|_{\mathrm{dis}} = e^{C(p)} \cdot \|.\|$ on $\mathcal{U}|_{\mathcal{X}\times p}$, where C is a bounded function on $\bigsqcup_{L/K \text{ finite}} \mathcal{P}(\mathcal{O}_L)$. Therefore

$$\begin{aligned} h_{\overline{\mathcal{T}},\omega}(\mathcal{U}|_{\mathcal{X}\times p}) &= \frac{1}{[L:K]} \widehat{\deg}\, \gamma\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, (\mathcal{U}|_{\mathcal{X}\times p}, \|.\|_{\mathrm{dis}})\right) \\ &= \frac{1}{[L:K]} \widehat{\deg}\, \gamma\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, (\mathcal{U}|_{\mathcal{X}\times p}, \|.\|)\right) + O(1) \qquad (19) \\ &= \frac{1}{[L:K]} \widehat{\deg}\, \gamma\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, ((\mathcal{U}\otimes \pi_2^*\mathcal{H})|_{\mathcal{X}\times p}, \|.\|)\right) \\ &\quad - \frac{1}{[L:K]} \widehat{\deg}\, \gamma\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, \pi_2^*\overline{\mathcal{H}}|_{\mathcal{X}\times p}\right) + O(1). \end{aligned}$$

Here the second summand is the same as

$$\begin{aligned} & -\frac{1}{[L:K]} \widehat{\deg}\, \gamma\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, \pi^*\overline{\mathcal{H}}|_p\right) \\ &= -\frac{1}{[L:K]} \widehat{\deg}\, \beta\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, \pi^*\overline{(0, \tfrac{2}{[L:\mathbb{Q}]} h_{\overline{\mathcal{H}}}(p), \ \ldots, \tfrac{2}{[L:\mathbb{Q}]} h_{\overline{\mathcal{H}}}(p))}\right) \qquad (20) \\ &= -\frac{1}{[L:K][L:\mathbb{Q}]} \int_{X_L(\mathbb{C})} h_{\overline{\mathcal{H}}}(p) \cdot \omega_{\overline{\mathcal{T}}}^{\wedge d} \\ &= -\frac{1}{[L:K]} h_{\overline{\mathcal{H}}}(p) \cdot c_1(\mathcal{T}_L)^d, \end{aligned}$$

where we used the properties of the various intersection products shown in the appendix. Note that the factor $[L:\mathbb{Q}]$ in the denominator disappears since $X_L(\mathbb{C})$ has $[L:\mathbb{Q}]$ mutually isomorphic connected components. For the first summand in the formula above we obtain

$$\begin{aligned} & \frac{1}{n[L:K]} \widehat{\deg}\, \beta\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, (\mathrm{div}(s|_{\mathcal{X}\times p}), -\log \|s|_{\mathcal{X}\times p}\|^2)\right) \\ &= \frac{1}{n[L:K]} \widehat{\deg}\, \alpha\left(\overline{\mathcal{T}}, \ \ldots, \overline{\mathcal{T}}, \mathrm{div}(s|_{\mathcal{X}\times p})\right) \qquad (21) \\ &\quad - \frac{1}{2n[L:K]} \int_{X_L(\mathbb{C})} \log \|s|_{\mathcal{X}\times p}\|^2 \cdot \omega_{\overline{\mathcal{T}}}^{\wedge d} \\ &= \frac{1}{n[L:K]} h_{\overline{\mathcal{T}}}(\mathrm{div}(s|_{X\times p})) - \frac{1}{2n[L:K]} \sum_{\sigma: L\hookrightarrow \mathbb{C}} \int_{X\otimes_K \mathbb{C}(\mathbb{C})} \log \|s|_{\mathcal{X}\times \sigma(p)}\|^2 \cdot \omega_{\overline{\mathcal{T}}}^{\wedge d}. \end{aligned}$$

Using the work of Stoll[23] one shows easily that the integral

$$\int_{X\otimes_K \mathbb{C}(\mathbb{C})} \log \|s|_{\mathcal{X}\times q}\|^2 \cdot \omega_{\overline{\mathcal{T}}}^{\wedge d} \qquad (22)$$

is a continuous function on $q \in \mathcal{P}(\mathbb{C})$, in particular the second part is bounded. $\Box$

3 A Finiteness Theorem

3.1 Proposition *(Models of line bundles). Let $\pi : \mathcal{X} \to \mathcal{O}_K$ be an arithmetic variety and L/K an extension of number fields. Then any line bundle $\mathcal{K} \in \mathrm{Pic}(\mathcal{X})$ that becomes trivial under base change by* $\mathrm{Spec}\ \mathcal{O}_L \to \mathrm{Spec}\ \mathcal{O}_K$ *is of the form $\mathcal{K} = \pi^*\mathcal{I}$ where $\mathcal{I} \in \mathrm{Pic}(\mathrm{Spec}\ \mathcal{O}_K)$.*

Proof. It is sufficient to assume L/K to be Galois.

$$\{\mathcal{X}_{\mathcal{O}_L} := \mathcal{X} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_L \to \mathcal{X}\} \tag{23}$$

is a covering for the fppf-site on $\mathcal{X}$. We consider the Čech spectral sequence[20]

$$E_2^{pq} := \check{\mathrm{H}}^p(\{\mathcal{X}_{\mathcal{O}_L} \to \mathcal{X}\}, \underline{\mathrm{H}}^q_{\mathrm{fppf}}(\mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{fppf}}(\mathcal{X}, \mathbb{G}_m) \tag{24}$$

associated to this covering. The beginning of its lower term exact sequence looks like this.

$$\begin{array}{ccccc} 0 \to \check{\mathrm{H}}^1(\{\mathcal{X}_{\mathcal{O}_L} \to \mathcal{X}\}, \mathbb{G}_m) \to & \mathrm{H}^1_{\mathrm{fppf}}(\mathcal{X}, \mathbb{G}_m) & \to & \check{\mathrm{H}}^0(\{\mathcal{X}_{\mathcal{O}_L} \to \mathcal{X}\}, \underline{\mathrm{H}}^1_{\mathrm{fppf}}(\mathbb{G}_m)) \\ & \| & & \cap\!| \\ & \mathrm{Pic}\,(\mathcal{X}) & & \mathrm{Pic}\,(\mathcal{X}_{\mathcal{O}_L}) \end{array} \tag{25}$$

Therefore the line bundles under consideration are described by the Čech cohomology group $\check{\mathrm{H}}^1(\{\mathcal{X}_{\mathcal{O}_L} \to \mathcal{X}\}, \mathbb{G}_m)$.

We remark that our assumptions make sure that $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_K$, i.e. every fiber of $\mathcal{X}$ is geometrically connected. So we simply deal with the first cohomology group of the complex

$$0 \longrightarrow \mathcal{O}_L^* \longrightarrow (\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L)^* \longrightarrow (\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L)^* \longrightarrow \dots\ , \tag{26}$$

which means nothing but that π induces an isomorphism

$$\check{\mathrm{H}}^1(\{\mathrm{Spec}\ \mathcal{O}_L \to \mathrm{Spec}\ \mathcal{O}_K\}, \mathbb{G}_m) \xrightarrow{\cong} \check{\mathrm{H}}^1(\{\mathcal{X}_{\mathcal{O}_L} \to \mathcal{X}\}, \mathbb{G}_m). \tag{27}$$

On the other hand, repeating the spectral sequence argument above with $\mathcal{X}$ replaced by $\mathrm{Spec}\ \mathcal{O}_K$ one sees that the group on the left hand side just describes the line bundles on $\mathrm{Spec}\ \mathcal{O}_K$ becoming trivial under base change to $\mathrm{Spec}\ \mathcal{O}_L$. $\Box$

3.2 As for effective cycles one can expect finiteness results only when working in a fixed equivalence class under the relation of algebraic equivalence.

Definition. *Let K be a field and X/K a proper variety. Let $\mathfrak{L} \in \mathrm{NS}(X)$ be an element of the Néron-Severi group, i.e. an equivalence class of line bundles* *modulo algebraic equivalence. We will call* $\mathfrak{L}$ appropriate, *if the following two conditions hold.*

(*) *For some finite field extension* K'/K *there is a* K'*-valued point* $x \in X(K') = X_{K'}(K')$ *with the property below: On* $X_{K'} \times \mathbf{Pic}^{\mathfrak{L}_{K'}}(X_{K'})$ *let* $\mathcal{P}$ *be the tautological line bundle with* $\mathcal{P}|_{\{x\}\times \mathbf{Pic}^{\mathfrak{L}_{K'}}(X_{K'})} \cong \mathcal{O}_{\mathbf{Pic}^{\mathfrak{L}_{K'}}}$. *Then* $(\det R\pi_{2*}\mathcal{P})^{-1}$ *is ample on* $\mathbf{Pic}^{\mathfrak{L}_{K'}}(X_{K'})$.

(**) *If* $\mathcal{L} \in \mathfrak{L}$, *then* $\chi(\mathcal{L}) > 0$.

3.3 Theorem *(Finiteness).* A. *Let* $(\mathcal{X}/\mathcal{O}_K, \omega)$ *be an Arakelov variety,* $\mathfrak{L} \in \mathrm{NS}(\mathcal{X}_K)$ *an appropriate class and assume that the line bundle* $\mathcal{T} \in \mathrm{Pic}\,(\mathcal{X})$*, defining the height, is ample. Then for every* $H \in \mathbb{R}$ *there are only finitely many* $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X})$ *such that*

i) $\mathcal{L}|_{\mathcal{X}_K} \in \mathfrak{L}$,

ii) *for each* $\mathfrak{p} \in \mathrm{Specm}\ \mathcal{O}_K$ *the degree on every irreducible component* $\overline{\mathcal{X}^i_{\mathfrak{p}}}$ *of the geometric fiber* $\overline{\mathcal{X}_{\mathfrak{p}}}$ *is bounded,*

$$|\deg_{\mathcal{T}} \mathcal{L}|_{\overline{\mathcal{X}^i_{\mathfrak{p}}}}| < H, \tag{28}$$

and

iii) *the height is bounded above,*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) < H. \tag{29}$$

B. *Assume in addition* $\mathcal{X}/\mathcal{O}_K$ *to be normal and suppose that stably, i.e. after any finite field extension* L/K*, for any special fiber* $[\mathcal{X}_{\mathcal{O}_L,\mathfrak{q}}] \in \mathrm{Div}(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_L)$ *the coefficients in its decomposition into irreducible Weil divisors are relatively prime. Then for every* $H \in \mathbb{R}$ *there are only finitely many* $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ *such that*

i) $\mathcal{L}|_{\mathcal{X}_L} \in \mathfrak{L}_L$,

ii) *for each* $\mathfrak{q} \in \mathrm{Specm}\ \mathcal{O}_L$ *the degree on every irreducible component* $\overline{\mathcal{X}^i_{\mathfrak{q}}}$ *of the geometric fiber* $\overline{\mathcal{X}_{\mathfrak{q}}}$ *is bounded,*

$$|\deg_{\mathcal{T}} \mathcal{L}|_{\overline{\mathcal{X}^i_{\mathfrak{q}}}}| < H, \tag{30}$$

iii) *if* $\mathfrak{p} \in \mathrm{Specm}\ \mathcal{O}_K$ *splits in* $\mathcal{O}_L$*, say* $\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{\alpha_1} \cdot \ldots \cdot \mathfrak{q}_l^{\alpha_l}$*, then* $\mathcal{L}|_{\overline{\mathcal{X}_{\mathfrak{q}_1}}}, \ldots, \mathcal{L}|_{\overline{\mathcal{X}_{\mathfrak{q}_l}}}$ *become mutually numerically equivalent under the canonical isomorphisms* $\overline{\mathcal{X}_{\mathfrak{q}_1}} \cong \ldots \cong \overline{\mathcal{X}_{\mathfrak{q}_l}}$ *and*

iv) *the normalized height is bounded above,*

$$h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) < H, \tag{31}$$

where

v) L/K *is an arbitrary finite field extension with* $[L:K] < H$,

if no distinction is made between $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ *and its pull back to* $\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_{L'}$ *when* L' *is a finite field extension of* L.

3.4 Remarks. i) The convention in B on the pull back to a field extension is in obvious analogy to the situation of cycles. Note that it implies automatically that no distinction is made between $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$ and $\mathcal{L} \otimes \pi_L^* \mathcal{I}$, where $\mathcal{I} \in \mathrm{Pic}\,(\mathrm{Spec}\,\mathcal{O}_L)$ is an element of the class group. That identification is necessary for finiteness for trivial reasons. Indeed, for $K = \mathbb{Q}$ and $\mathcal{X} = \mathrm{Spec}\,\mathbb{Z}$ there are infinitely many quadratic fields with non-trivial class group.

ii) The assumption on the special fibers of the stabilization of $\mathcal{X}$ is fulfilled, in particular, if every special fiber of $\mathcal{X}$ is geometrically reduced. Unfortunately, as Example 3.5 will show, we can not get rid of an assumption like this. In fact, the minimal assumption being necessary in this place is that stably there exist no torsion line bundles being defined by a Cartier divisor supported in a special fiber except those coming from the class group. Lemma 3.9 below shows this property, being the one actually used in the proof, under our assumption.

iii) Condition B.iii) seems to be a relatively restrictive one. But note that this restriction is only concerned with the behaviour of $\mathcal{L}$ in the non-smooth fibers $\mathcal{X}_\mathfrak{p}$, where $\mathfrak{p}$ splits in $\mathcal{O}_L$. At least if $\mathcal{X}$ is regular and the irreducible components of its fibers are geometrically irreducible, then, after a finite field extension, every line bundle can be brought into the form required by adding a suitably chosen divisor being supported in special fibers. See Proposition 3.10 for this.

iv) $\mathbf{Pic}^{\mathfrak{L}_{K'}}(X_{K'})$ denotes the scheme representing the open subfunctor of the Picard functor assigning to each K'-scheme T the the set of all line bundles on $X_{K'} \times T$, which are rigidified at x and belong to $\mathfrak{L}_{K'}$, the class of the base changes of the line bundles of $\mathfrak{L}$, fiber-by-fiber. In order to work with the tautological line bundle one has to extend the ground field such that there is a K'-valued point on X. In general there exists only a Picard scheme representing the fppf-sheafification of the naive Picard functor and the relation with line bundles becomes less direct.

v) Condition $(*)$ above is fairly independent of choices. It does not depend on the choice of the K'-valued point x. Indeed, the tautological line bundle $\mathcal{P}'$ defined by another K'-valued point x' differs from $\mathcal{P}$ by some $\pi_2^*\mathcal{Z}$, where $\mathcal{Z}$ is algebraically equivalent to zero on $\mathbf{Pic}^{\mathfrak{L}_{K'}}(X_{K'})$. Consequently, $(\det R\pi_{2*}\mathcal{P}')^{-1}$ is algebraically equivalent to $(\det R\pi_{2*}\mathcal{P})^{-1}$ and therefore it is ample, too, by the Nakai-Moishezon criterion. Furthermore, $(*)$ does obviously still hold when K' is replaced by a finite field extension.

vi) Some cases where $(*)$ is actually fulfilled are listed in Theorem 3.6 below.

3.5 Example. Let $K = \mathbb{Q}$ and consider the quadric

$$\mathcal{X} = \mathrm{Proj}\ \mathbb{Z}[X,Y,Z]/(X^2+Y^2-2Z^2). \tag{32}$$

One shows without difficulty that this is a regular arithmetic variety. Its fiber over $\mathfrak{p} = (2)$ is equal to Proj $\mathbb{F}_2[X,Y,Z]/(X+Y)^2$. Therefore $\mathcal{O}([\mathcal{X}_{(2)}]_{\mathrm{red}})$ is a non-trivial torsion bundle. In particular, it is numerically equivalent to zero in any fiber.

Let $\overline{x} := (1\!:\!1\!:\!1) \in \mathcal{X}(\mathbb{Z})$. By Théorème 2.3.1 of Raynaud[22] the rigidified Picard functor

$$\begin{aligned} \mathrm{Pic}_{\mathcal{X}/\mathbb{Z}} : (\mathrm{Sch}/\mathbb{Z}) &\longrightarrow (\mathrm{Sets}) \\ T &\mapsto \{(\mathcal{L}, i) \mid \mathcal{L} \in \mathrm{Pic}\,(T \times_{\mathrm{Spec}\,\mathbb{Z}} \mathcal{X}),\ i : \mathcal{L}|_{T\times\overline{x}} \xrightarrow{\cong} \mathcal{O}_{T\times\overline{x}}\} \end{aligned} \tag{33}$$

is representable as an algebraic space. Further, its subfunctor $\mathrm{Pic}^{\tau}_{\mathcal{X}/\mathbb{Z}}$ collecting the line bundles, which are numerically equivalent to zero in each geometric fiber of $T \times_{\mathrm{Spec}\,\mathbb{Z}} \mathcal{X} \to T$, is representable by an open group subspace $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathbb{Z}}$ being of finite type over $\mathrm{Spec}\,\mathbb{Z}$ as shown in SGA6, Exp. XIII, Thm. 4.7.

It is easy to see that $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathbb{Z}}$ can not be separated. In fact there are two different maps i_1, i_2 : Spec $\mathbb{Z} \to \mathbf{Pic}^{\tau}_{\mathcal{X}/\mathbb{Z}}$ that are both extensions of i : Spec $\mathbb{Q} \to \mathbf{Pic}^{\tau}_{\mathcal{X}/\mathbb{Z}}$ corresponding to the trivial line bundle on X. This effect has serious consequences already on quadratic extensions of $\mathbb{Q}$ split over (2). Namely, if $(2)\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$, then in order to extend i_L : Spec $L \to \mathbf{Pic}^{\tau}_{\mathcal{X}/\mathbb{Z}}$ to Spec $\mathcal{O}_L$ we have two possibilities for the image of $\mathfrak{p}_1$ and two possibilities for the image of $\mathfrak{p}_2$, thus producing two new line bundles in the sense of our statement above. For any other choice of L there will be different ones and we end up with infinitely many of them. Note that in order to produce this pathology we needed extensions of the ground field.

At this point it becomes clear, what condition B.iii) is good for. $\mathbf{Pic}_{\mathcal{X}/\mathcal{O}_K}$ is not separated in general, even when $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{O}_K}$ is. A typical case is a regular arithmetic variety $\mathcal{X}$ having reducible fibers. We could therefore produce infinitely many line bundles not being of much interest by the same procedure.

3.6 Theorem. *Let K be a field and X/K be a regular, proper and geometrically connected scheme. Further, let $\mathfrak{L} \in \mathrm{NS}(X)$ be an equivalence class of line bundles modulo algebraic equivalence. Then $\mathfrak{L}$ is appropriate, if*

a) *X is a curve of genus g and $\mathfrak{L}$ consists of line bundles of degree at least g,*

b) *X is torsor over an abelian variety and $\mathfrak{L}$ is an ample equivalence class,*

c) *(X is arbitrary), $\mathfrak{L}' \in \mathrm{NS}(X)$ is any equivalence class and $\mathfrak{A}$ is an ample equivalence class, for $\mathfrak{L} = \mathfrak{L}' + n\mathfrak{A}$ when $n \gg 0$.*

Proof. This follows directly from Theorem 1.7 in a previous paper[16] and Remark ii) above. □

3.7 Proof of Theorem 3.3. B. 1st step. *We may assume without restriction that* $X(K) \neq \emptyset$. This is just Lemma 3.1.

2nd step. *The Picard functor.* Denote by $\overline{x}$ an $\mathcal{O}_K$-valued point of $\mathcal{X}$ and consider the rigidified Picard functor as in Example 3.5. We claim that the algebraic space $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{O}_K}$ is separated here. In fact, by the valuative criterion we have to consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\ F & \to & \mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{O}_K} \\ \downarrow & \nearrow\!\!\nearrow & \downarrow \\ \mathrm{Spec}\ S & \to & \mathrm{Spec}\ \mathcal{O}_K, \end{array} \tag{34}$$

where S is a discrete valuation ring and F is fraction field. By the modular interpretation we have two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\ (\mathcal{X} \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\ S)$ becoming isomorphic under restriction to $\mathcal{X} \times_{\mathrm{Spec}\, \mathcal{O}_K} \mathrm{Spec}\ F$. Lemma 3.9 shows that under our assumptions the two must be isomorphic. The claim is proven. By Théorème 3.3.1 of Raynaud[22] it follows that $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{O}_K}$ is a scheme already. We define $\mathbf{Pic}^{\sigma}_{\mathcal{X}/\mathcal{O}_K}$ to be the closure of $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{O}_K,K}$ in $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{O}_K}$.

A little bit extending the collection of line bundles under consideration we look at the subfunctor $\mathrm{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K}$ of $\mathrm{Pic}_{\mathcal{X}/\mathcal{O}_K}$, which collects the line bundles satisfying ii) in each characteristic p geometric point of T and, instead of i), being of the same *numerical* type as the bundles in $\mathfrak{L}_C$ for each C-valued point of T, when C is a field in characteristic zero. This subfunctor is given by numerical conditions only, therefore it is represented by an open subspace $\mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K}$ of $\mathbf{Pic}_{\mathcal{X}/\mathcal{O}_K}$. We will not be interested in line bundles on special fibers that can not be generalized. Therefore we consider the closure $\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$ of $\mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K,K}$ in $\mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K}$. Proposition 3.8 shows now that we are dealing with finitely many numerical types only, so, after a finite field extension, $\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K,\mathcal{O}_{L_0}}$ is the union of finitely many cosets of $\mathbf{Pic}^{\sigma}_{\mathcal{X}/\mathcal{O}_K,\mathcal{O}_{L_0}}$. In particular, it is of finite type over Spec $\mathcal{O}_K$. Condition B.iii) translates into the requirement, that we consider $\mathcal{O}_L$-valued points lying in one of the cosets already.

We note that by FGA, Exp. 232, Théorème 3.1 and Exp. 236, Théorème 2.1 the scheme $\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$ is projective outside the fibers of finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l \in \mathrm{Specm}\, \mathcal{O}_K$. By enlarging the exceptional set if necessary we may assume smoothness, i.e. $\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K} \times_{\mathrm{Spec}\, \mathcal{O}_K \backslash \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}} \mathrm{Spec}\, \mathcal{O}_K$ is a disjoint union of finitely many abelian schemes.

3rd step. *For every* $n \in \mathbb{N}$ *there exist* $C(n) \in \mathbb{N}$ *and a morphism* $p' : P' \to \mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K}$ *satisfying the conditions below.*

i) *P' is a separated scheme of finite type over $\mathcal{O}_K$.*

ii) *p' is quasi-finite.*

iii) *p' is finite outside the fibers over the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$.*

iv) *For every number field L and any $i \in \mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K}(\mathcal{O}_L)$ lying after lift to $\mathcal{O}_{LL_0}$ in one of the cosets under $\mathbf{Pic}^{\sigma}_{\mathcal{X}/\mathcal{O}_K,\mathcal{O}_{L_0}}$ there exist a field extension L'/L with $[L':L] \leq C(n)$ and $i' \in P'(\mathcal{O}_{L'})$ lifting i.*

v) *For every $\mathcal{F} \in \mathrm{Pic}\left(\mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K,\overline{\mathbb{Q}}}\right)$ the pull-back $p'^{*}_{\overline{\mathbb{Q}}}\mathcal{F}$ is divisible by n.*

We may choose finitely many sections $i_\alpha : \mathrm{Spec}\, \mathcal{O}_{L_0} \to \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$ to obtain a surjection $P'_1 := \coprod_{\alpha \in A} \mathbf{Pic}^{\sigma}_{X/\mathcal{O}_K,\mathcal{O}_{L_0}} \to \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K,\mathcal{O}_{L_0}} \to \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$, where the morphisms $\mathbf{Pic}^{\sigma}_{X/\mathcal{O}_K,\mathcal{O}_{L_0}} \to \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K,\mathcal{O}_{L_0}}$ are just the multiplications with the $\mathcal{O}_{L_0}$-valued points i_α. This satisfies i), ii) and iii).

For v) there is a cohomological argument. The obstruction against $\mathcal{F} \in \mathrm{Pic}\,(\,.\,)$ to be divisible by n is its Chern class $c_1(\mathcal{F}) \in \mathrm{H}^2_{\text{ét}}(\,.\,, \mu_n)$. Since each connected component of $\mathbf{Pic}^{\delta}_{X/\mathcal{O}_K,\overline{\mathbb{Q}}}$ is an abelian variety, its second cohomology group is generated by the first one. So we only have to annihilate (some of) the classes in $\mathrm{H}^1_{\text{ét}}(P'_{1,\overline{\mathbb{Q}}}, \mu_n)$ and this is done by an étale cover. But this one will be given by finitely many data and therefore be nothing but the base change of some cover $\widetilde{P} \to P'_{1,L}$ defined over a finite field extension L/K already. Consider the composition $\widetilde{P} \to P'_1 \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L \to P'_1$ being a finite morphism in the generic fibers and define an extension P' of $\widetilde{P}$ being finite over P'_1 as the normalization of P'_1 in $K(\widetilde{P})$. By construction, P fulfills v), while i), ii) and iii) remain valid by the finiteness over P'_1.

Finally, let us show iv). It is sufficient to prove this for P'_1. In fact, P' is finite over P'_1 of some degree C. By SGA1, Exp. I, Théorème 10.11 over any L-valued point of P'_1 there is an L'-valued point with $[L':L] \leq C$. Extend it to an $\mathcal{O}_{L'}$-valued point. But for lifts to P'_1 the only field extension we need is the composite with L_0. Indeed, any $i : \mathrm{Spec}\,\mathcal{O}_L \to \mathbf{Pic}^{\delta}_{\mathcal{X}/\mathcal{O}_K}$ corresponds to some line bundle $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L)$. But the sections i_α chosen represent all the possible numerical types in the special fibers. Therefore, as soon as $L \supseteq L_0$, we may find $\alpha \in A$ such that $i - i_\alpha$ is a section in $\mathbf{Pic}^{\sigma}_{X/\mathcal{O}_K}$. This is just the assertion.

4th step. *The tautological line bundle.* Let $\mathcal{P}$ be the rigidified tautological line bundle on $\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$. A direct application of Riemann-Roch shows that for any $\overline{\mathbb{Q}}$-valued point $\overline{x} \hookrightarrow \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$ the Euler characteristic $\chi := \chi(\mathcal{P}|_{\mathcal{X}\times\overline{x}})$ is the same. We put $n := \chi$ and consider the morphism $P' \xrightarrow{p'} \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$ constructed in Step 3. We claim:

There is a projective morphism $q : P \to P'$ satisfying the following conditions.

i) *q is an étale cover outside the fibers over $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$.*

ii) *There exists $C \in \mathbb{N}$ such that for any number field L and any $i \in P'(\mathcal{O}_L)$ there exist a field extension L'/L with $[L':L] \leq C$ and a lift $i' \in P(\mathcal{O}_{L'})$ of i.*

iii) *There exists* $\mathcal{E} \in \mathrm{Pic}(P)$ *such that the line bundle*

$$\det R\pi_{2*}\left((\mathrm{id} \times p)^*\mathcal{P} \otimes \pi_2^*\mathcal{E}\right) \tag{35}$$

is trivial on the generic fiber P_K. *Here* p *denotes the composition* $p = p'q : P \to \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$.

By the projection formula

$$\det R\pi_{2*}\left((\mathrm{id} \times p)^*\mathcal{P} \otimes \pi_2^*\mathcal{E}\right) = p^* \det R\pi_{2*}\mathcal{P} \otimes \mathcal{E}^{\otimes \chi}. \tag{36}$$

So we are looking for a morphism, the pull-back under which makes the line bundle $\det R\pi_{2*}\mathcal{P}$ divisible by χ. p' yields that on the geometric generic fiber by the last step. Therefore it is clear that there exists a number field L/K such that the pull-back of $\det R\pi_{2*}\mathcal{P}$ to $P_1 := P' \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L$ will be divisible by χ on the generic fiber. It remains to extend the quotient $\mathcal{D}$ from the generic fiber $P_{1,K}$ to a model. For this take a Weil divisor D with $\mathcal{D} = \mathcal{O}(D)$ and consider its Zariski closure $\overline{D}$ in P_1. In general it will not define a line bundle there, but the blow-up $P := \mathrm{Bl}_{\overline{D}}(P_1)$ fulfills properties i), ii) and iii). Note that the restriction of $\mathcal{E}$ to P_K is an ample line bundle by assumption $(*)$ and $(**)$.

5th step. *Comparison with height for cycles.* Put $\mathcal{U} := (\mathrm{id} \times p)^*\mathcal{P} \otimes \pi_2^*\mathcal{E}$. For the $x \in X(K)$ used in the rigidification the line bundle $\mathcal{U}|_{\{x\}\times P_K} \cong \mathcal{E}_K$ is ample. Thus, $\mathcal{U}_K$ is ample fiber-by-fiber since all the $\mathcal{U}|_{\{.\}\times P}$ are mutually numerically equivalent (up to extension of ground field). By EGA III, Théorème 4.7.1 it is relatively ample. But then EGA II, Proposition 4.6.13.ii shows that there exists some $\mathcal{H} \in \mathrm{Pic}(\mathcal{X})$ such that $\mathcal{U} \otimes \pi_1^*\mathcal{H}$ induces an ample line bundle on $X \times P_K$. This implies that there is some $n \in \mathbb{N}$ such that $(\mathcal{U} \otimes \pi_1^*\mathcal{H})^{\otimes n}|_{X \times P_K}$ admits a suitable section s, i.e. $s|_{X \times \{\overline{y}\}} \neq 0$ for every geometric point $\overline{y} \hookrightarrow P_K$, as worked out in a previous paper[16]. Multiplying with elements of $\mathcal{O}_K$ if necessary, we may assume that s can be extended to $\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} P$. By Lemma 2.7 we know for any number field L and any $y \in P(\mathcal{O}_L)$

$$\begin{aligned} h_{\overline{\mathcal{T}},\omega}(\mathcal{U}|_{\mathcal{X}\times y}) &= \frac{1}{n[L:K]}\, h_{\overline{\mathcal{T}}}(\mathrm{div}(s|_{\mathcal{X}\times y})) + O(1) \\ &= \frac{1}{n[L:K]}\, h_{\overline{\mathcal{T}}}\Big(\pi_{L*}\,\mathrm{div}(s|_{\mathcal{X}\times y})\Big) + O(1), \end{aligned} \tag{37}$$

where $\pi_L : \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} P \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\,\mathcal{O}_L \to \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} P$ is the natural projection. But the cycles $\mathrm{div}(s|_{\mathcal{X}\times y})$ are pairwise different, even when restricted to the geometric generic fiber, since they represent different line bundles there. One easily sees that each cycle on $\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} P$ has only finitely many possibilities to be a push-forward under one of the π_L. So the proof is finished by the finiteness result for the height for effective cycles[3].

For A we have $\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$ and $\mathbf{Pic}^{\sigma}_{X/\mathcal{O}_K}$ as well, but we neither know $\mathbf{Pic}^{\sigma}_{X/\mathcal{O}_K}$ to be separated nor to be a scheme. So we have to deal carefully with algebraic spaces. We claim:

For every $n \in \mathbb{N}$ there exist $C(n) \in \mathbb{N}$ and a morphism $p' : P' \to \mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$ satisfying the conditions below.

i) *P' is a separated scheme of finite type over $\mathcal{O}_K$.*

ii) *Each component of P' is finite over its image in $\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$.*

iii) *p' is finite outside the fibers over the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$.*

iv) *For any $i \in \mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}(\mathcal{O}_K)$ there are a field extension K'/K with $[K' : K] \leq C(n)$ and $i' \in P'(\mathcal{O}_{K'})$ lifting i.*

v) *For every $\mathcal{F} \in \mathrm{Pic}\left(\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K,\overline{\mathbb{Q}}}\right)$ the pull-back $p'^{*}_{\overline{\mathbb{Q}}}\mathcal{F}$ is divisible by n.*

By the definition of an algebraic space there are a scheme U and an étale covering $U \to \mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$. As $\mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$ is the closure of its generic fiber, we may, not losing the property of being étale, decompose U into its components and consider $\coprod_{\alpha \in A} U_\alpha \to \mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$, where A is a finite index set and U_α are irreducible affine schemes. Without restriction assume $A = A_0 \cup \ldots \cup A_l$, where for $\alpha \in A_i$ with $i \in \{1, \ldots, l\}$ the component U_α meets the generic fiber and among the fibers over $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ exactly that over $\mathfrak{p}_i$, while it does not meet the fibers over $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ for $\alpha \in A_0$. Define $\overline{U_\alpha}$ to be the normalization of $U_{\alpha,\mathrm{red}}$ in the normal closure $K(U_{\alpha,\mathrm{red}})^{\mathrm{n}}$ of its function field, considered as a finite extension of $K(\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K})$. Corresponding to each $\sigma \in G = \mathrm{Gal}(K(U_{\alpha,\mathrm{red}})^{\mathrm{n}}/K(\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}))$ there is a birational map $j_\sigma : \overline{U_\alpha} \dashrightarrow \overline{U_\alpha}$ making the obvious diagram of maps to $\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$ commutative. We consider these copies of $\overline{U_\alpha}$ as different schemes connected by a birational map $j_{\mathrm{id},\sigma} : \overline{U^{\mathrm{id}}_\alpha} \dashrightarrow \overline{U^{\sigma}_\alpha}$, put $j_{\tau,\sigma} := j_\sigma j_\tau^{-1} : \overline{U^{\tau}_\alpha} \dashrightarrow \overline{U^{\sigma}_\alpha}$ and claim that all these can be glued together to give a scheme $\overline{U^{\mathrm{n}}_\alpha}$. In fact there is a maximal open set $\overline{U^{\tau,\sigma}_\alpha}$, where $j_{\tau,\sigma}$ is a morphism. Since $\overline{U_\alpha}$ is normal and quasi-finite over $\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$, $j_{\tau,\sigma}$ is quasi-finite and therefore an open embedding by Zariski's Main Theorem. The same argument is true for the inverse map and the cocycle relations are trivial.

$\overline{U^{\mathrm{n}}_\alpha}$ clearly fulfills i) and is quasi-finite over $\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$. In order to show it is finite over its image we need that $\overline{p}^{T}_\alpha : \overline{U^{\mathrm{n}}_\alpha} \times_{\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}} T \to T$ is finite as soon as it is surjective for a scheme T being étale over $\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$. As by EGA II, Corollaire 5.4.3.ii) properness descends under proper surjective base changes we may assume T to be normal (but no more étale) and $K(T) \supseteq K(\overline{U^{\mathrm{n}}_\alpha}) = K(U_{\alpha,\mathrm{red}})^{\mathrm{n}}$. But in that case p^{T}_α splits completely into a disjoint union of birational maps, i.e. open embeddings. As the full Galois group G acts on $\overline{U^{\mathrm{n}}_\alpha} \times_{\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}} T$ surjectivity implies that all of them must be isomorphisms.

Now, for every l-tuple $(\alpha_1, \ldots, \alpha_l) \in A_1 \times \ldots \times A_l$ let

$K_{(\alpha_1, \dots, \alpha_l)} := K(\overline{U^{\mathrm{n}}_{\alpha_1}}) \cdot \dots \cdot K(\overline{U^{\mathrm{n}}_{\alpha_l}})$ be the composite of the corresponding function fields. We put $U^i_{(\alpha_1, \dots, \alpha_l)}$ for $i \in \{1, \dots, l\}$ to be the normalization of $\overline{U^{\mathrm{n}}_{\alpha_i}}$ in the field extension $K_{(\alpha_1, \dots, \alpha_l)}$ and $U^0_{(\alpha_1, \dots, \alpha_l)}$ to be the normalization of $O := \mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K} \times_{\mathrm{Spec}\, \mathcal{O}_K} (\mathrm{Spec}\, \mathcal{O}_K \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_l\})$ in $K_{(\alpha_1, \dots, \alpha_l)}$. The $U^i_{(\alpha_1, \dots, \alpha_l)}$ can be glued together giving an integral scheme $U_{(\alpha_1, \dots, \alpha_l)}$ being equipped with a map to $\mathbf{Pic}^{\gamma}_{X/\mathcal{O}_K}$, which is finite onto its image. Indeed, by our construction there is something to be glued only outside the the exceptional fibers. But there $\overline{U^{\mathrm{n}}_{\alpha_i}}$ is just an open subscheme of the normalization of O in $K(U_{\alpha_i, \mathrm{red}})^{\mathrm{n}}$. Consequently, outside the exceptional fibers each $U^i_{(\alpha_1, \dots, \alpha_l)}$ is nothing but an open subscheme of the normalization of O in $K_{(\alpha_1, \dots, \alpha_l)}$. So there are natural identifications and the cocycle relations are trivial. We put

$$P'' := \coprod_{(\alpha_1, \dots, \alpha_l) \in A_1 \times \dots \times A_l} U_{(\alpha_1, \dots, \alpha_l)}. \tag{38}$$

By construction P'' admits properties i), ii) and iii). For v) there is exactly the same argument as in step B.3 giving the scheme P' desired. Again to show iv) it is sufficient to consider P''. By the definition of an algebraic space a map $i : \mathrm{Spec}\ \mathcal{O}_K \to \mathbf{Pic}^{\gamma}_{\mathcal{X}/\mathcal{O}_K}$ gives rise to a morphism of schemes $\coprod_{\alpha \in A} \overline{\mathcal{V}_\alpha} \to \coprod_{\alpha \in A} \overline{U^{\mathrm{n}}_\alpha}$. By our construction, $\overline{\mathcal{V}_\alpha}$ is finite over its image in Spec $\mathcal{O}_K$. As we started with a covering there is necessarily one combination $(\alpha_1, \dots, \alpha_l) \in A_1 \times \dots A_l$ giving a base change map $\mathcal{V}_{(\alpha_1, \dots, \alpha_l)} \to \mathcal{U}_{(\alpha_1, \dots, \alpha_l)}$, where $\mathcal{V}_{(\alpha_1, \dots, \alpha_l)}$ is surjective, hence necessarily finite, over Spec $\mathcal{O}_K$. That implies the assertion.

The proof is finished along the lines of steps B.4 and B.5. $\square$

3.8 Proposition. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Further, let $\mathcal{X}$ be an integral scheme, projective and flat over R whose generic fiber is geometrically connected. Equip it with some ample line bundle $\mathcal{T}$. Assume $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ satisfies the following conditions.*

i) *Its restriction $\mathcal{L}_K \in \mathrm{Pic}(\mathcal{X}_K)$ to the generic fiber is trivial.*

ii) *On every component $\overline{\mathcal{X}^i_{\mathfrak{m}}}$ of the geometric fiber $\overline{\mathcal{X}_{\mathfrak{m}}}$*

$$\deg_{\mathcal{T}} \mathcal{L}|_{\overline{\mathcal{X}^i_{\mathfrak{m}}}} = 0. \tag{39}$$

Then $\mathcal{L}|_{\mathcal{X}_{\mathfrak{m}}}$ is numerically equivalent to zero.

Proof. We may assume $\dim \mathcal{X} \geq 2$. Let $p : \mathcal{X}' \to \mathcal{X}$ be the normalization. Note, since p is finite, $p^*\mathcal{T}$ is again an ample line bundle. Thus $p^*\mathcal{L}$ also satisfies conditions i) and ii). In fact this is trivial for i), while for ii) let $\overline{\mathcal{X}'^i_{\mathfrak{m}}}$ be a component of the geometric fiber $\overline{\mathcal{X}'_{\mathfrak{m}}}$. Then $p_* \left[\overline{\mathcal{X}'^i_{\mathfrak{m}}}\right] = n \left[\overline{\mathcal{X}^j_{\mathfrak{m}}}\right] \in \mathrm{CH}^0(\overline{\mathcal{X}_{\mathfrak{m}}})$ for some component $\overline{\mathcal{X}^j_{\mathfrak{m}}}$ and a non-negative integer n. Therefore the assumption $\deg_{\mathcal{T}} \mathcal{L}|_{\overline{\mathcal{X}^j_\mathfrak{m}}} = \deg c_1(\mathcal{L}) \cdot c_1(\mathcal{T})^{\dim \mathcal{X}-2}|_{\overline{\mathcal{X}^j_\mathfrak{m}}} = 0$ implies $\deg_{p^*\mathcal{T}} p^*\mathcal{L}|_{\overline{\mathcal{X}'^i_\mathfrak{m}}} = \deg c_1(p^*\mathcal{L}) \cdot c_1(p^*\mathcal{T})^{\dim \mathcal{X}-2}|_{\overline{\mathcal{X}'^i_\mathfrak{m}}} = 0$. By Lemma 3.9 below $p^*\mathcal{L}$ is torsion, at least after a finite unramified extension of R. But then $\mathcal{L}|_{\mathcal{X}_\mathfrak{m}}$ is numerically equivalent to zero as for this by SGA6, Exp. XIII, Théorème 4.6 one only has to verify that certain intersection numbers of line bundles are 0 and this can be checked after pull-back under a finite morphism (cp. Lemma A.4). □

3.9 Lemma. *Let $\mathcal{X}$ be as in Proposition 3.8 and assume further it is normal and* $\dim \mathcal{X} \geq 2$.

a) *Let $D_\mathfrak{m}$ be the vector space of Weil $\mathbb{Q}$-divisors supported over $\mathfrak{m}$. Then there is a symmetric bilinear form $\langle\, .\, ,\, .\,\rangle : D_\mathfrak{m} \times D_\mathfrak{m} \longrightarrow \mathbb{Q}$ with the properties below.*

i) $\langle D, [\mathcal{X}_\mathfrak{m}]\rangle = 0$ *for every $D \in D_\mathfrak{m}$.*

ii) *If D is effective with coefficient 0 at the component $[\mathcal{X}^i_\mathfrak{m}]$, then $\langle D, [\mathcal{X}^i_\mathfrak{m}]\rangle \geq 0$.*

ii') *If, in addition, D is Cartier and $|D| \cap |[\mathcal{X}^i_\mathfrak{m}]| \neq \emptyset$, then $\langle D, [\mathcal{X}^i_\mathfrak{m}]\rangle > 0$.*

iii) *If D is a Cartier divisor, then $\langle D, D'\rangle = \deg\, D \cdot c_1(\mathcal{T})^{\dim \mathcal{X}-2}|_{D'}$.*

b) *On the vector space $C_\mathfrak{m}$ of Cartier $\mathbb{Q}$-divisors supported over $\mathfrak{m}$, the form $\langle\, .\, ,\, .\,\rangle$ is negative semi-definite, whereas only rational multiples of the fiber $[\mathcal{X}_\mathfrak{m}]$ have square zero. In particular, any line bundle $\mathcal{L} = \mathcal{O}(D)$, where D is supported over $\mathfrak{m}$, whose degree vanishes at every component of $\mathcal{X}_\mathfrak{m}$, is torsion. One even has $\mathcal{L} \cong \mathcal{O}_\mathcal{X}$ if the coefficients of the divisor $[\mathcal{X}_\mathfrak{m}]$ are relatively prime.*

Proof. a) Let $\mathcal{X}^i_\mathfrak{m}$ and $\mathcal{X}^j_\mathfrak{m}$ be two different components of $\mathcal{X}_\mathfrak{m}$. We let

$$\mathcal{X}^i_\mathfrak{m} \cdot \mathcal{X}^j_\mathfrak{m} := \sum_{x \in \mathcal{X}^{(2)}} l\left(\mathcal{O}_{\mathcal{X},x}/(\mathcal{J}_{\mathcal{X}^i_\mathfrak{m}} + \mathcal{J}_{\mathcal{X}^j_\mathfrak{m}})\mathcal{O}_{\mathcal{X},x}\right) \cdot x \qquad (40)$$

be their *naive* intersection product and put

$$\langle [\mathcal{X}^i_\mathfrak{m}], [\mathcal{X}^j_\mathfrak{m}]\rangle := \deg c_1(\mathcal{T}^{\dim \mathcal{X}-2})|_{\mathcal{X}^i_\mathfrak{m} \cdot \mathcal{X}^j_\mathfrak{m}}. \qquad (41)$$

This can easily be extended to a symmetric bilinear form satisfying i). But then iii) is clear, since there are no higher Tor's occurring when intersecting with a Cartier divisor. ii) is trivial from the construction. For ii') note that by dimension theory $\dim |D| \cap |[\mathcal{X}^i_\mathfrak{m}]| \geq \dim \mathcal{X} - 2$ as soon as they meet. For two arbitrary Weil divisors this would be wrong in general.

b) Let $[\mathcal{X}_\mathfrak{m}] = \sum_i C_i[\mathcal{X}^i_\mathfrak{m}]$. Then

$$0 = \langle [\mathcal{X}^i_\mathfrak{m}], [\mathcal{X}_\mathfrak{m}]\rangle = C_i\langle [\mathcal{X}^i_\mathfrak{m}], [\mathcal{X}^i_\mathfrak{m}]\rangle + \sum_{k \neq i} C_k\langle [\mathcal{X}^i_\mathfrak{m}], [\mathcal{X}^k_\mathfrak{m}]\rangle, \qquad (42)$$

hence $\langle C_i[\mathcal{X}^i_{\mathfrak{m}}], C_i[\mathcal{X}^i_{\mathfrak{m}}]\rangle = -\sum_{k\neq i}\langle C_i[\mathcal{X}^i_{\mathfrak{m}}], C_k[\mathcal{X}^k_{\mathfrak{m}}]\rangle$. With rational coefficients b_i one obtains

$$\left\langle \sum_i b_i C_i[\mathcal{X}^i_{\mathfrak{m}}], \sum_i b_i C_i[\mathcal{X}^i_{\mathfrak{m}}] \right\rangle = \sum_i b_i^2 \langle C_i[\mathcal{X}^i_{\mathfrak{m}}], C_i[\mathcal{X}^i_{\mathfrak{m}}]\rangle + \sum_{j\neq k} b_j b_k \langle C_j[\mathcal{X}^j_{\mathfrak{m}}], C_k[\mathcal{X}^k_{\mathfrak{m}}]\rangle = -\frac{1}{2}\sum_{j\neq k}(b_j - b_k)^2 \langle C_j[\mathcal{X}^j_{\mathfrak{m}}], C_k[\mathcal{X}^k_{\mathfrak{m}}]\rangle. \quad (43)$$

Therefore $\langle\, .\, ,\, .\, \rangle$ is negative semi-definite. Now assume $D = \sum_i b_i C_i[\mathcal{X}^i_{\mathfrak{m}}]$ to be $\mathbb{Q}$-Cartier with square zero, while not all the b_i are equal. Without restriction we may suppose some of the b_i are zero, $b_{i_1} = \ldots = b_{i_l} = 0$, and the others are not, $b_{j_1} \neq 0, \ldots, b_{j_m} \neq 0$. But this implies $\langle C_{i_n}[\mathcal{X}^{i_n}_{\mathfrak{m}}], C_{j_o}[\mathcal{X}^{j_o}_{\mathfrak{m}}]\rangle = 0$ for any n and o, therefore $\langle [\mathcal{X}^{i_n}_{\mathfrak{m}}], D\rangle = 0$ for any n being a contradiction to a.ii') since $\mathcal{X}_{\mathfrak{m}}$ is connected. □

3.10 Proposition. *Let L/K be an extension of number fields, $\mathcal{X}/\mathcal{O}_K$ a regular arithmetic variety, the irreducible components of all whose fibers are geometrically irreducible, and $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_L)$. Then there exist some finite field extension L'/L and $\mathcal{H} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_{L'})$ being non-trivial only in non-smooth fibers such that*

$$\mathcal{L}' := \mathcal{L}_{\mathcal{O}_{L'}} \otimes \mathcal{H} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_{L'}) \quad (44)$$

admits the property that for each $\mathfrak{p} \in \mathrm{Specm}\ \mathcal{O}_K$ splitting in $\mathcal{O}_L$, say $\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{\alpha_1} \cdot \ldots \cdot \mathfrak{q}_l^{\alpha_l}$, the restrictions $\mathcal{L}'|_{\mathcal{X}_{\overline{\mathfrak{q}_i}}}$ are mutually numerically equivalent.

Proof. The assumption makes sure that on $\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\ \mathcal{O}_L$ every vertical Weil divisor admits a non-zero multiple which is Cartier. Therefore, on the vector spaces $D_{\mathfrak{q}_i}/[\mathcal{X}_{\mathfrak{q}_i}]$ of Weil $\mathbb{Q}$-divisors supported over $\mathfrak{q}_i$, there is a negative definite intersection pairing. We would be finished, if we were allowed to use a vertical $\mathbb{Q}$-divisor for $\mathcal{H}$. Lemma 3.11 below completes the proof. □

3.11 Lemma. *Let n be a positive integer, S be a Dedekind ring and $\mathcal{X}/S$ a proper scheme whose generic fiber is smooth. Denote its open subscheme consisting of all the smooth fibers by $\mathcal{X}^{(\mathrm{s})}$. Consider $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X})$ such that $\mathcal{L}|_{\mathcal{X}^{(\mathrm{s})}} \cong \mathcal{O}_{\mathcal{X}^{(\mathrm{s})}}$. Then there exists a finite flat morphism $p : \mathrm{Spec}\ T \longrightarrow \mathrm{Spec}\ S$ of spectra of Dedekind rings such that $p^*_{\mathcal{X}}\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,S} \mathrm{Spec}\ T)$ is divisible by n in that sense that there exists $\mathcal{Q} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,R} \mathrm{Spec}\ T)$ with $\mathcal{Q}|_{\mathcal{X}^{(\mathrm{s})}} \cong \mathcal{O}_{\mathcal{X}^{(\mathrm{s})}}$ and $\mathcal{Q}^{\otimes n} \cong p^*_{\mathcal{X}}\mathcal{L}$.*

Proof. The question is local in the base. We may even assume S to be a strictly henselian discrete valuation ring. We note that, if the residue characteristic is $l > 0$, then pull-back under Frobenius of any line bundle is divisible by l. So assume from now on n to be prime to the residue characteristic.

Let $(S, s) \to (T, t)$ be a totally ramified homomorphism of degree n between strictly henselian discrete valuation rings and denote the associated morphism of affine schmes by p.. The only obstruction against $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X})$ to be divisible by n in the sense above is

$$c_1(\mathcal{L}) \in \mathrm{H}^2_{\text{ét},\mathcal{X}_s}(\mathcal{X}, \mu_n) = \mathrm{H}^0_{\text{ét},s}(\mathrm{Spec}\; S,\; R^2\pi_*\mu_n). \tag{45}$$

On the other hand

$$\begin{aligned} \mathrm{H}^2_{\text{ét},(\mathcal{X}_T)_t}(\mathcal{X} \times_{\mathrm{Spec}\, S} \mathrm{Spec}\; T, \mu_n) &= \mathrm{H}^0_{\text{ét},t}(\mathrm{Spec}\; T, R^2\pi_{T\,*}p_{\mathcal{X}}^*\mu_n) \\ &= \mathrm{H}^0_{\text{ét},s}(\mathrm{Spec}\; S, R^2\pi_*(p_{\mathcal{X}\,*}p_{\mathcal{X}}^*\mu_n)) \end{aligned} \tag{46}$$

as $p_{\mathcal{X}} : \mathcal{X} \times_{\mathrm{Spec}\, S} \mathrm{Spec}\; T \to \mathcal{X}$ is acyclic being a finite morphism. We have to study the canonical map between these two spaces. i.e. the map between the stalks of the direct image sheaves occurring. By proper base change this is a map $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, p_{\mathcal{X}_{\overline{s}}*}p^*_{\mathcal{X}_{\overline{s}}}\mu_n)$. But one easily sees $p_{\mathcal{X}_{\overline{s}}*}p^*_{\mathcal{X}_{\overline{s}}}\mu_n \cong \mu_n$. We claim the map is multiplication by $\deg p_{\mathcal{X}_{\overline{s}}}$, hence zero. It comes from the pull-back $p^*_{\mathcal{X}_{\overline{s}}} : \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{s}} \times_{\mathrm{Spec}\, S} \mathrm{Spec}\; T, p^*_{\mathcal{X}_{\overline{s}}}\mu_n)$. But there is an isomorphism of topoi $(p_{\mathcal{X}_{\overline{s}}*}, p^*_{\mathcal{X}_{\overline{s}}}) : (\mathcal{X}_{\overline{s}} \times_{\mathrm{Spec}\, S} \mathrm{Spec}\; T)_{\text{ét}} \to (\mathcal{X}_{\overline{s}})_{\text{ét}}$. Finally, note that $p_{\mathcal{X}_{\overline{s}}*}p^*_{\mathcal{X}_{\overline{s}}}s = \deg p_{\mathcal{X}_{\overline{s}}} \cdot s$ for any class in étale cohomology. $\square$

4 The Case of an Arithmetic Surface

4.1 Conventions. An *arithmetic surface* is an arithmetic variety of dimension 2. Note that this is exactly the case $d = 1$ in the notation introduced in the beginning of the paper. An *Arakelov surface* is an Arakelov variety of dimension 2, i.e. an arithmetic surface $\mathcal{X}$ equipped with a Kähler metric ω being invariant under complex conjugation.

4.2 Proposition. *Let $\mathcal{X}/\mathcal{O}_K$ be an arithmetic surface. Then there exists $C \in \mathbb{R}$ with the property below: Let $L'/L/K$ be finite field extensions and*

$$p : \widetilde{\mathcal{X}} \to \mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\; \mathcal{O}_{L'} \to \mathcal{X} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}\; \mathcal{O}_L \tag{47}$$

the composition of a proper birational morphism with a base extension. Then for each $\overline{\mathcal{T}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$, each Kähler metric ω on $X(\mathbb{C})$ being invariant under F_∞ and every $\mathcal{L} \in \mathrm{Pic}\,(\mathcal{X} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\; \mathcal{O}_L)$ with $\chi(\mathcal{L}_L) \neq 0$

$$|h_{p^*\overline{\mathcal{T}},p^*\omega}(p^*\mathcal{L}) - h_{\overline{\mathcal{T}},\omega}(\mathcal{L})| \leq C \cdot \deg(\mathcal{T}_K). \tag{48}$$

Proof. We put $C := c_1(\mathcal{T}_K)^d \cdot \sum_{\mathfrak{p}} \log(\sharp\mathcal{O}_K/\mathfrak{p})\; C(\mathfrak{p})$, where the $C(\mathfrak{p})$ come from Lemma 4.3 below. In the case that $L = K$ Proposition 2.6 gives the assertion directly, since $R^j p_* \mathcal{O}_{\mathcal{X}'} = 0$ for $j \geq 2$ and the Euler characteristic breaks down to the length of the space of global sections. For arbitrary L note that $\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_L \to \mathcal{X}$ is acyclic for coherent sheaves. $\square$

4.3 Lemma. *Let $\mathcal{X}/\mathcal{O}_K$ be an arithmetic surface. Then, there exists some function $C : \operatorname{Specm} \mathcal{O}_K \to \mathbb{R}$ such that*

i) Only finitely many values of C are different from zero.

ii) For any composition $p : \widetilde{\mathcal{X}} \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_L \to \mathcal{X}$ of a birational morphism and a base extension one has

$$\mathrm{l}(R^1 p_* \mathcal{O}_{\widetilde{\mathcal{X}}}|_{\mathcal{X}_\mathfrak{p}}) \leq C(\mathfrak{p}) \cdot [L : K]. \tag{49}$$

Proof. First note, if $\mathcal{X}_1 \xrightarrow{p_1} \mathcal{X}_2 \xrightarrow{p_2} \mathcal{X}_3$ is any composition of proper birational maps of surfaces, then the lower term sequence associated to the Leray spectral sequence for $\mathcal{O}_{\mathcal{X}_1}$ reads $0 \to R^1 p_{2*} \mathcal{O}_{\mathcal{X}_2} \to R^1 (p_2 p_1)_* \mathcal{O}_{\mathcal{X}_1} \to p_{2*} R^1 p_{1*} \mathcal{O}_{\mathcal{X}_1} \to 0$. So in order to show an estimate for p_2 it is sufficient to prove it for $p_2 p_1$.

For arithmetic surfaces there exists a semi-stable reduction, i.e. there is some field extension L_0 and a birational desingularization $p' : \mathcal{X}' \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_{L_0} \to \mathcal{X}$ such that $\mathcal{X}'$ is semi-stable[19]. We put $C(\mathfrak{p}) := \mathrm{l}(R^1 p'_* \mathcal{O}_{\mathcal{X}'}|_{\mathcal{X}_\mathfrak{p}})/[L_0 : K]$. Note that $R^1 p'_* \mathcal{O}_{\mathcal{X}'}$ is a coherent sheaf supported in finitely many points.

To prove the assertion for $\widetilde{\mathcal{X}} \to \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_L \to \mathcal{X}$ we may assume by flat base change that $L \supseteq L_0$. Further, it is sufficient to consider $\mathcal{X}' \times_{(\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_{L_0})} \widetilde{\mathcal{X}}$ instead of $\widetilde{\mathcal{X}}$ and we may even regard its desingularization $\widetilde{\mathcal{X}'}$. So everything we need is that $\widetilde{\mathcal{X}'} \to \mathcal{X}' \times_{\operatorname{Spec} \mathcal{O}_{L_0}} \operatorname{Spec} \mathcal{O}_L$ is acyclic for the structure sheaf, i.e. that $\mathcal{X}' \times_{\operatorname{Spec} \mathcal{O}_{L_0}} \operatorname{Spec} \mathcal{O}_L$ has only rational singularities.

But $\mathcal{X}' \times_{\operatorname{Spec} \mathcal{O}_{L_0}} \operatorname{Spec} \mathcal{O}_L$ is regular in codimension 1 as $\mathcal{X}'$ is smooth in codimension 1 and Cohen-Macaulay as it is flat of relative dimension zero over $\mathcal{X}'$. So it is normal. In formal coordinates its singularities are given by $\operatorname{Spf} \widehat{\mathcal{O}}_{L_0,\mathfrak{p}}[[X,Y]]/(XY - \mathfrak{p}^n)$ for some positive integer n. Therefore they are toroidal embeddings in the sense of Section IV.3 of Kempf, Knudsen, Mumford and Saint-Donat[19], in particular they are rational. $\square$

4.4 Theorem. *Let $(\mathcal{X}/\mathcal{O}_K, \omega)$ be an Arakelov surface whose generic fiber is of genus g. Assume $x \in \mathcal{X}(K)$ is a K-valued point such that $\mathcal{O}(x)$ can be extended to some $\mathcal{T} \in \operatorname{Pic}(\mathcal{X})$, which we suppose to be equipped with a hermitian metric. Transfer the Θ-divisor to $\mathbf{Pic}^g(X_K)$ via the map $\mathbf{Pic}^{g-1}(X_K) \to \mathbf{Pic}^g(X_K)$ defined by x. Then there exists a function $C = C_{\mathcal{X},\omega,\overline{\mathcal{T}}} : \mathbb{R} \to \mathbb{R}$ with the property below: For each finite field extension L/K and each $\mathcal{L} \in \operatorname{Pic}(\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_L)$ of degree g on $\mathcal{X}_L$ satisfying*

$$|\deg \mathcal{L}|_{\overline{\mathcal{X}^i_\mathfrak{p}}}| < H \tag{50}$$

for every component $\overline{\mathcal{X}_{\mathfrak{p}}^i}$ of the geometric fibers $\overline{\mathcal{X}_{\mathfrak{p}}}$, the inequality

$$\left| h_{\overline{\mathcal{T}},\omega}(\mathcal{L}) - \frac{1}{[L:K]}\, h_{\Theta}([\mathcal{L}_K]) \right| < C(H) \tag{51}$$

is true, where h_Θ is the height on $\mathbf{Pic}^g(\mathcal{X}_K)$ defined by Θ.

Proof. By the existence of semi-stable reduction and Proposition 4.2 we may assume $\mathcal{X}$ to be semi-stable. In that situation the statement of Lemma 3.9 is known for a long time. But then, using Proposition 3.10 and Proposition 2.2 one easily sees that it is sufficient to assume that over any $\mathfrak{p} \in \operatorname{Specm} \mathcal{O}_K$ that splits in $\mathcal{O}_L$, the restrictions of $\mathcal{L}$ to the corresponding geometric fibers are mutually numerically equivalent. By Proposition 3.8 we deal with finitely numerical types only. We may consider them separately. Exactly the same argument as in 3.7, step B.ii), yields a separated scheme $\mathbf{Pic}^{\sigma}_{\mathcal{X}/\mathcal{O}_K}$ being of finite type over $\mathcal{O}_K$. We are forced to work with a coset $\mathcal{P} \hookrightarrow \mathbf{Pic}_{\mathcal{X}/\mathcal{O}_K}$ of it, whose generic fiber is $\mathbf{Pic}^g_{X/K}$.

Consider the line bundle

$$\mathcal{O}((\pi_1 \times \pi_{21})^*\Delta + \ \ldots \ + (\pi_1 \times \pi_{2g})^*\Delta) \in \operatorname{Pic}(X \times X^g), \tag{52}$$

where Δ denotes the diagonal in $X \times X$. It defines a morphism

$$c : X^g \to \mathbf{Pic}^g_{X/K}. \tag{53}$$

Let $\mathcal{P}'$ denote the normalization of $\mathcal{P}$ in $K(X^g)$. It is equipped with a proper morphism $p : \mathcal{P}' \to \mathcal{P}$ such that $\mathcal{P}' \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} K \cong X^g$ and p goes over into c under this isomorphism. By functoriality there is a line bundle $\mathcal{V} \in \operatorname{Pic}(\mathcal{X} \times \mathcal{P}')$ such that $\mathcal{V}|_{X \times X^g} \cong \mathcal{O}((\pi_1 \times \pi_{21})^*\Delta + \ \ldots \ + (\pi_1 \times \pi_{2g})^*\Delta)$. We claim that $\mathcal{D} := \det R\pi_{2*}\mathcal{V}|_{X^g} \cong c^*\mathcal{O}(-\Theta) \otimes \bigotimes_{i=1}^g \pi_i^*\mathcal{O}(x)$. In fact, there is an exact sequence

$$0 \to \mathcal{V}'|_{X \times X^g} \to \mathcal{V}|_{X \times X^g} \to \mathcal{V}|_{\{x\} \times X^g} \to 0, \tag{54}$$

where $\mathcal{V}' := \mathcal{V} \otimes \pi_1^*\mathcal{T}^{-1} \in \operatorname{Pic}(\mathcal{X} \times \mathcal{P}')$. Consequently,

$$\begin{aligned} \det R\pi_{2*}\mathcal{V}|_{X^g} &\cong \det R\pi_{2*}(\mathcal{V}|_{X \times X^g}) \\ &\cong \det R\pi_{2*}(\mathcal{V}'|_{X \times X^g}) \otimes \mathcal{O}(\pi_1^*(x) + \ \ldots \ + \pi_g^*(x)). \end{aligned} \tag{55}$$

But by Faltings[4] or Moret-Bailly[21] one has $\det R\pi_{2*}(\mathcal{V}'|_{X \times X^g}) \cong c^*\mathcal{O}(-\Theta)$.

We are in the situation of Proposition 2.7 with $n = 1$, $\mathcal{U} := \mathcal{V} \otimes \pi_2^*\mathcal{D}^{-1}$ and $\mathcal{H} := \mathcal{D}$. Therefore

$$\begin{aligned} & h_{\overline{\mathcal{T}},\omega}\left(\mathcal{V}|_{\mathcal{X} \times \overline{(p_1, \ldots, p_g)}}\right) \\ &= \frac{1}{[L:K]}\, h_{\overline{\mathcal{T}}}\left(\overline{(p_1)} + \ \ldots \ + \overline{(p_g)}\right) - \frac{1}{[L:K]}\, h_{\mathcal{D}}\left(\overline{(p_1, \ \ldots \ , p_g)}\right) + O(1). \end{aligned} \tag{56}$$

The statement follows easily from Lemma 4.5 below. $\square$

4.5 Lemma. *Let g be a positive integer and $\mathcal{X}/\mathcal{O}_K$ be an arithmetic variety equipped with $\overline{\mathcal{T}} \in \widehat{\operatorname{Pic}}(\mathcal{X})$. Let $\mathcal{P}/\mathcal{O}_K$ be a separated scheme of finite type whose generic fiber is X^g and let $\overline{\mathcal{G}} \in \widehat{\operatorname{Pic}}(\mathcal{P})$ be such that $\mathcal{G}|_{X^g} \cong \bigotimes_{i=1}^g \pi_i^* \mathcal{T}|_{X^g}$. Then, for any number field L/K and any $p_1, \ldots, p_g \in X(L)$*

$$h_{\overline{\mathcal{T}}}((\overline{p_1}) + \ldots + (\overline{p_g})) = h_{\overline{\mathcal{G}}}\left(\overline{(p_1, \ldots, p_g)}\right) + [L:K] \cdot O(1) \tag{57}$$

as soon as $(p_1, \ldots, p_g) \in X^g(L)$ can be extended to some $\overline{(p_1, \ldots, p_g)} \in \mathcal{P}(\mathcal{O}_L)$. Here $O(1)$ is a bounded function on $\bigsqcup_{L/K \text{ finite}} X^g(L)$, but the bound depends on $\overline{\mathcal{T}}$ and $\overline{\mathcal{G}}$.

Proof. Consider $\mathcal{X}^g$ instead of $\mathcal{P}$ first. By Lemma A.2 one has

$$h_{\overline{\mathcal{T}}}((\overline{p_1}) + \ldots + (\overline{p_g})) = \sum_{i=1}^g h_{\pi_i^* \overline{\mathcal{T}}}\left(\overline{(p_1, \ldots, p_g)}\right) = h_{\bigotimes_{i=1}^g \pi_i^* \overline{\mathcal{T}}}\left(\overline{(p_1, \ldots, p_g)}\right). \tag{58}$$

But there is a scheme $\mathcal{P}'$ equipped with birational morphisms $\mathcal{X}^g \xleftarrow{q} \mathcal{P}' \xrightarrow{p} \mathcal{P}$, where p is proper. Indeed let $\mathcal{P}'$ be the closure of the diagonal in $\mathcal{X}^g \times_{\operatorname{Spec} \mathcal{O}_K} \mathcal{P}$. Then $h_{\overline{\mathcal{G}}}\left(\overline{(p_1, \ldots, p_g)}\right) = h_{p^* \overline{\mathcal{G}}}\left(\overline{(p_1, \ldots, p_g)}\right)$ and

$$h_{\bigotimes_{i=1}^g \pi_i^* \overline{\mathcal{T}}}\left(\overline{(p_1, \ldots, p_g)}\right) = h_{q^* \bigotimes_{i=1}^g \pi_i^* \overline{\mathcal{T}}}\left(\overline{(p_1, \ldots, p_g)}\right). \tag{59}$$

The difference of the two terms on the right hand side is bounded as there are changes only in the special fibers (and the metric). $\square$

4.6 Remark (Asymptotic behaviour)**.** If $\mathcal{E}, \mathcal{M} \in \operatorname{Pic}(\mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_L)$ are equipped with distinguished metrics, then there is a formula for $h_{\overline{\mathcal{T}},\omega}(\mathcal{E} \otimes \mathcal{M}^n)$ in terms of arithmetic intersection numbers[17]. In particular, if $\deg(\mathcal{E}_L) = g$, $\deg(\mathcal{M}_L) = 0$ and $\deg(\mathcal{T}_L) = 1$, then the dominating term for $n \to \infty$ is

$$-\frac{1}{2} \widehat{\deg}\, \gamma\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right) \cdot n^2. \tag{60}$$

By Theorem 4.4 we recover the formulas $h_{\mathrm{NT},\Theta}([\mathcal{M}_K]) = -\frac{1}{2} \widehat{\deg}\, \gamma\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right)$ and

$$h_{\mathrm{NT},\Theta+\Theta^-}([\mathcal{M}_K]) = -\widehat{\deg}\, \gamma\left(\overline{\mathcal{M}}, \overline{\mathcal{M}}\right) \tag{61}$$

for the Néron-Tate, which are due to Faltings and Hriljac.

Appendix

A Some Facts Concerning Intersection Theory on Singular Arithmetic Varieties

A.1 Remark. It is a well-known phenomenon that there is no good intersection product available for arbitrary cycles on singular varieties. Nevertheless it is possible to intersect cycles with Chern classes of vector bundles. The same situation occurs in arithmetic intersection theory. It seems that the right context to describe possible intersection products is arithmetic K-Theory[9] more than arithmetic Chow theory.

A.2 Lemma. A. *For any projective arithmetic variety $\mathcal{X}/\mathcal{O}_K$, whose generic fiber X is regular, any natural number r and any partition $(r_1, \ldots, r_s)$ of r there is a unique multi-linear map*

$$\alpha_{r_1, \ldots, r_s} : \underbrace{\widehat{\mathrm{K}}_0(\mathcal{X}) \times \ \ldots \ \times \widehat{\mathrm{K}}_0(\mathcal{X})}_{s \text{ times}} \times \mathrm{Z}_r(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathrm{Spec}\ \mathcal{O}_K)_{\mathbb{Q}}, \tag{62}$$

called restriction product*, satisfying the following conditions.*

i) *If $\mathcal{X}$ is regular, then $\alpha_{r_1, \ldots, r_s}$ coincides with the product*

$$\left(\widehat{ch}_{r_1}(.) \cdot \ \ldots \ \cdot \widehat{ch}_{r_s}(.) \,\Big|\, . \right) \tag{63}$$

defined by Bost, Gillet and Soulé[3].

ii) *For any morphism $f : \mathcal{X} \to \mathcal{X}'$ there is a projection formula*

$$\alpha_{r_1, \ldots, r_s}(v_1, \ldots, v_s, f_*(Z)) = \alpha_{r_1, \ldots, r_s}(f^*(v_1), \ldots, f^*(v_s), Z). \tag{64}$$

B. *If $(r_1, \ldots, r_s) = (1, \ldots, 1)$, the specialized product*

$$\alpha : \underbrace{\widehat{\mathrm{Pic}}(\mathcal{X}) \times \ \ldots \ \times \widehat{\mathrm{Pic}}(\mathcal{X})}_{r \text{ times}} \times \mathrm{Z}_r(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathrm{Spec}\ \mathcal{O}_K)_{\mathbb{Q}} \tag{65}$$

exists without the assumption on X to be regular and fulfills i) and ii) below.

i) *If $\mathcal{X}$ is regular, then α coincides with the product $(\widehat{c}_1(.) \cdot \ \ldots \ \cdot \widehat{c}_1(.) \mid .)$.*

ii) *For any morphism $f : \mathcal{X} \to \mathcal{X}'$ there is a projection formula*

$$\alpha\left(\overline{\mathcal{U}_1}, \ldots, \overline{\mathcal{U}_r}, f_*(Z)\right) = \alpha\left(f^*(\overline{\mathcal{U}_1}), \ldots, f^*(\overline{\mathcal{U}_r}), Z\right). \tag{66}$$

Proof. We will show A only. If $\overline{\mathcal{V}_1}, \ldots, \overline{\mathcal{V}_s}$ are hermitian vector bundles on $\mathcal{X}$, then there are a morphism $\iota : \mathcal{X} \to P$, where P is projective and smooth over Spec $\mathcal{O}_K$, and vector bundles $\mathcal{V}_{1,P}, \ldots, \mathcal{V}_{r,P}$ on P such that $\iota^*(\mathcal{V}_{j,P}) = \mathcal{V}_j$. One may even choose ι such that the hermitian metric on $\overline{\mathcal{V}_j}$ is a pullback of one on $\mathcal{V}_{j,P}$ and any differential form on $X(\mathbb{C})$ extends to $P(\mathbb{C})$.

$$\iota^*(v_{j,P}) = v_j \tag{67}$$

Then for an r-dimensional cycle Z in $\mathcal{X}$ define

$$\alpha_{r_1,\,\dots\,,r_s}(v_1,\,\dots\,,v_s,Z) := \Big(\widehat{ch}_{r_1}(v_{1,P})\cdot\,\dots\,\cdot\widehat{ch}_{r_s}(v_{s,P})\,\Big|\,\iota_*(Z)\Big)\,, \qquad (68)$$

where $(.|.)$ denotes the restriction product $\widehat{\mathrm{CH}}^r(P)\times \mathrm{Z}_r(P)\to\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathcal{O}_K)_{\mathbb{Q}}$. In their remark after Proposition 3.2.1 Bost, Gillet and Soulé[3] have shown independence of the ι chosen. Uniqueness of $\alpha_{r_1,\,\dots\,,r_s}$ is clear. □

A.3 Lemma. a) *For any projective arithmetic variety $\pi:\mathcal{X}\to\mathcal{O}_K$, whose generic fiber X is regular, any natural number r and any partition $(r_1,\,\dots\,,r_s)$ of r, there is a unique multi-linear map*

$$\beta_{r_1,\,\dots\,,r_s} : \underbrace{\widehat{\mathrm{K}}_0(\mathcal{X})\times\,\dots\,\times\widehat{\mathrm{K}}_0(\mathcal{X})}_{s\text{ times}}\times\widehat{\mathrm{CH}}_r(\mathcal{X})\longrightarrow\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathcal{O}_K)_{\mathbb{Q}}, \qquad (69)$$

called intersection product, *satisfying the following condition.*

If $y\in\widehat{\mathrm{CH}}_r(\mathcal{X})$ is represented by a cycle (Y,g_Y), where g_Y is a differential form with logarithmic singularities along $|Y|$, then

$$\beta_{r_1,\,\dots\,,r_s}(v_1,\,\dots\,,v_s,y) := \alpha_{r_1,\,\dots\,,r_s}(v_1,\,\dots\,,v_s,Y) + \overline{\Big(0,\,\Big(\int_{\mathcal{X}(\mathbb{C})} g_Y\,\omega_{\widehat{ch}_{r_1}(v_1)}\cdot\,\dots\,\cdot\omega_{\widehat{ch}_{r_s}(v_s)}\Big)_{\sigma:K\hookrightarrow\mathbb{C}}\Big)}. \qquad (70)$$

b) *In particular, if $(r_1,\,\dots\,,r_s)=(1,\,\dots\,,1)$, there is a specialized product*

$$\beta : \underbrace{\widehat{\mathrm{Pic}}(\mathcal{X})\times\,\dots\,\times\widehat{\mathrm{Pic}}(\mathcal{X})}_{r\text{ times}}\times\widehat{\mathrm{CH}}_r(\mathcal{X})\longrightarrow\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathcal{O}_K)_{\mathbb{Q}} \qquad (71)$$

c) *If $\mathcal{X}$ is regular, then $\beta_{r_1,\,\dots\,,r_s}$ coincides with $\pi_*\Big[\widehat{ch}_{r_1}(.)\cdot\,\dots\,\cdot\widehat{ch}_{r_s}(.)\cdot\,.\Big]$.*
d) *For any morphism $f:\mathcal{X}\to\mathcal{X}'$ of projective arithmetic varieties having regular generic fibers such that f_K is smooth, there is the projection formula*

$$\beta_{r_1,\,\dots\,,r_s}(f^*(v_1),\,\dots\,,f^*(v_s),y) = \beta_{r_1,\,\dots\,,r_s}(v_1,\,\dots\,,v_s,f_*(y))\,. \qquad (72)$$

e) *If $f:\mathcal{X}\to\mathcal{X}'$ is a proper and flat morphism such that f_K is étale, then*

$$\beta_{r_1,\,\dots\,,r_s}(f^*(v_1),\,\dots\,,f^*(v_s),f^*(z)) = \deg f\cdot\beta_{r_1,\,\dots\,,r_s}(v_1,\,\dots\,,v_s,z)\,. \qquad (73)$$

Proof. a) Take the property given as a definition. The integral over $\mathcal{X}(\mathbb{C})$ converges because of the logarithmic singularities of g_Y. Independence of the cycle chosen immediately carries over from the regular case. Indeed, assume (Y,g_Y) is an arithmetic r-cycle, rationally equivalent to zero. If $\iota:\mathcal{X}\to P$ is a closed embedding into a regular arithmetic variety, then (ι_*Y,ι_*g_Y) is also an arithmetic r-cycle, rationally equivalent to zero.

b) and c) are trivial now and d) follows from the projection formula for $\alpha_{r_1,\ldots,r_s}$ and the same argument for the integrals as in the regular case.

e) If $f : \mathcal{X} \to \mathcal{X}'$ is flat inducing an étale covering on generic fibers, then there is a pull-back homomorphism $f^* : \widehat{\mathrm{CH}}_d(\mathcal{X}') \to \widehat{\mathrm{CH}}_d(\mathcal{X})$ satisfying $f_* f^* z = \deg f \cdot z$. $\square$

A.4 Lemma. A. a) *For any projective arithmetic variety $\pi : \mathcal{X} \to \mathcal{O}_K$, whose generic fiber X is regular and equidimensional of dimension d, and any partition $(d_1, \ldots, d_s)$ of $d+1$ there is a unique multi-linear map*

$$\gamma_{d_1,\ldots,d_s} : \underbrace{\widehat{\mathrm{K}}_0(\mathcal{X}) \times \widehat{\mathrm{K}}_0(\mathcal{X}) \times \ \ldots \ \times \widehat{\mathrm{K}}_0(\mathcal{X})}_{s \text{ times}} \longrightarrow \widehat{\mathrm{CH}}^1(\mathrm{Spec}\ \mathcal{O}_K)_{\mathbb{Q}} \tag{74}$$

satisfying the following conditions.

i) *If $\mathcal{X}$ is regular, then*

$$\gamma_{d_1,\ldots,d_s}(v_1, \ldots, v_s) = \pi_* \left[\widehat{ch}_{d_1}(v_1) \cdot \ \ldots \ \cdot \widehat{ch}_{d_s}(v_s)\right]. \tag{75}$$

ii) *If $f : \mathcal{X}' \to \mathcal{X}$ is an alteration, i.e. a surjective and generically finite morphism, then*

$$\gamma_{d_1,\ldots,d_s}(f^*(v_1), \ldots, f^*(v_s)) = \deg f \cdot \gamma_{d_1,\ldots,d_s}(v_1, \ldots, v_s). \tag{76}$$

b) *$\gamma_{.}(\,.\,)$ is compatible with the natural action of the symmetric group.*
c) *One has*

$$\gamma_{d_1,\ldots,d_s}(v_1, \ldots, v_s) = \alpha_{d_1,\ldots,d_s}(v_1, \ldots, v_s, [\mathcal{X}]), \tag{77}$$

where $[\mathcal{X}]$ denotes the fundamental class of $\mathcal{X}$.
d) *If $d_s = 1$ and X is regular, then*

$$\gamma_{d_1,\ldots,d_{s-1},1}(v_1, \ldots, v_{s-1}, v_s) = \beta_{d_1,\ldots,d_{s-1}}(v_1, \ldots, v_{s-1}, \widehat{c}_1(v_s)). \tag{78}$$

B. *If $(d_1, \ldots, d_s) = (1, \ldots, 1)$, then the specialized product*

$$\gamma : \underbrace{\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \times \ \ldots \ \times \widehat{\mathrm{Pic}}(\mathcal{X})}_{d+1 \text{ times}} \longrightarrow \widehat{\mathrm{CH}}^1(\mathrm{Spec}\ \mathcal{O}_K)_{\mathbb{Q}} \tag{79}$$

exists without assuming X to be regular. It has the properties below.

i) *If $\mathcal{X}$ is regular, then*

$$\gamma\left(\overline{\mathcal{U}_1}, \ldots, \overline{\mathcal{U}_{d+1}}\right) = \pi_* \left[\widehat{c}_1(\overline{\mathcal{U}_1}) \cdot \ \ldots \ \cdot \widehat{c}_1(\overline{\mathcal{U}_{d+1}})\right]. \tag{80}$$

ii) *If $f : \mathcal{X}' \to \mathcal{X}$ is an alteration, i.e. a surjective and generically finite morphism, then*

$$\gamma\left(f^*(\overline{\mathcal{U}_1}), \ldots, f^*(\overline{\mathcal{U}_{d+1}})\right) = \deg f \cdot \gamma\left(\overline{\mathcal{U}_1}, \ldots, \overline{\mathcal{U}_{d+1}}\right). \tag{81}$$

iii) γ *is symmetric.*

iv) *One has*

$$\gamma\left(\overline{\mathcal{U}_1},\,\ldots\,,\overline{\mathcal{U}_{d+1}}\right)=\alpha\left(\overline{\mathcal{U}_1},\,\ldots\,,\overline{\mathcal{U}_{d+1}},\,[\mathcal{X}]\right), \tag{82}$$

where $[\mathcal{X}]$ *denotes the fundamental class of* $\mathcal{X}$*.*

v) *If the generic fiber* X *of* $\mathcal{X}$ *is regular, then*

$$\gamma\Big(\overline{\mathcal{U}_1},\,\ldots\,,\overline{\mathcal{U}_d},\overline{\mathcal{U}_{d+1}}\Big)=\beta\Big(\overline{\mathcal{U}_1},\,\ldots\,,\overline{\mathcal{U}_d},\,\widehat{c}_1(\overline{\mathcal{U}_{d+1}})\Big). \tag{83}$$

Proof. Again we show A only. By de Jong[18] for every $\mathcal{X}$ there is a resolution of singularities by alterations, so let $p:\mathcal{X}'\to\mathcal{X}$ be surjective and generically finite with $\mathcal{X}'$ regular and put

$$\gamma_{d_1,\ldots,d_s}(v_1,\,\ldots\,,v_s):=\frac{1}{\deg p}\,(\pi p)_*\,\Big[\widehat{ch}_{d_1}(p^*v_1)\cdot\,\ldots\,\cdot\widehat{ch}_{d_s}(p^*v_s)\Big]. \tag{84}$$

As ii) is true for $\mathcal{X}$ regular, this definition is independent of the choice of p. Property i) is trivial, ii) and b) immediately carry over from the regular case.

c) Lemma A.2.A.ii) implies

$$\begin{aligned}\alpha_{d_1,\ldots,d_s}(v_1,\,\ldots\,,v_s,[\mathcal{X}])&=\frac{1}{\deg p}\,\alpha_{d_1,\ldots,d_s}(v_1,\,\ldots\,,v_s,p_*[\mathcal{X}'])\\&=\frac{1}{\deg p}\,\Big(\widehat{ch}_{d_1}(p^*v_1)\cdot\,\ldots\,\cdot\widehat{ch}_{d_s}(p^*v_s)\Big|\,\mathcal{X}'\Big)\end{aligned} \tag{85}$$

being equal to our formula for $\gamma_{d_1,\ldots,d_s}(v_1,\,\ldots\,,v_s)$ above.

d) Assume without loss of generality that v_s is given by a hermitian line bundle $\overline{\mathcal{U}}$ and there is some section $0\neq s\in\Gamma(\mathcal{X},\mathcal{U}_{d+1})$. Then, by definition,

$$\begin{aligned}&\beta_{d_1,\ldots,d_{s-1},1}\left(v_1,\,\ldots\,,v_{s-1},\widehat{c}_1(\overline{\mathcal{U}})\right)\\=\;&\alpha_{d_1,\ldots,d_{s-1},1}\left(v_1,\,\ldots\,,v_{s-1},\mathrm{div}(s)\right)\\&\overline{+\Big(0,\Big(\int_{\mathcal{X}(\mathbb{C})}-\log\|s\|^2\,\omega_{\widehat{ch}_{d_1}(v_1)}\cdot\,\ldots\,\cdot\omega_{\widehat{ch}_{d_{s-1}}(v_{s-1})}\Big)_{\sigma:K\hookrightarrow\mathbb{C}}\Big)}.\end{aligned} \tag{86}$$

On the other hand, we have $p_*(\mathrm{div}(p^*s))=\deg p\cdot\mathrm{div}(s)$. This is clear in the case when $\mathcal{X}$ is normal, as then p is flat over some $\mathcal{X}\setminus\mathcal{Y}$ with codim $\mathcal{Y}\geq 2$. If p is the normalization of $\mathcal{X}$, then it follows from Example 1.2.3 in Fulton's book[6]. Consequently,

$$\begin{aligned}&\alpha_{d_1,\ldots,d_{s-1},1}\left(v_1,\,\ldots\,,v_{s-1},\mathrm{div}(s)\right)\\=\;&\frac{1}{\deg p}\,\alpha_{d_1,\ldots,d_{s-1},1}\left(p^*(v_1),\,\ldots\,,p^*(v_{s-1}),\mathrm{div}(p^*s)\right).\end{aligned} \tag{87}$$

Finally, it is obvious that

$$\int\limits_{\mathcal{X}(\mathbb{C})} -\log \|s\|^2 \, \omega_{\widehat{ch}_{d_1}(v_1)} \cdot \ \ldots \ \cdot \omega_{\widehat{ch}_{d_{s-1}}(v_{s-1})} \tag{88}$$

$$= \frac{1}{\deg p} \int\limits_{\mathcal{X}'(\mathbb{C})} -\log \|p^* s\|^2 \, \omega_{\widehat{ch}_{d_1}(p^* v_1)} \cdot \ \ldots \ \cdot \omega_{\widehat{ch}_{d_{s-1}}(p^* v_{s-1})}.$$

□

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