Computations with algebraic surfaces

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Abstract. Computations with algebraic number fields and algebraic curves have been carried out for a long time. They resulted in many interesting examples and the formation of various conjectures.
The aim of this talk is to report on some computations with algebraic surfaces that are currently possible.

Keywords: Algebraic surfaces · Computer algebra · point counting.

1 Introduction

Algebraic geometry is the study of the sets of solutions of systems of algebraic equations. In dimension zero, these sets consist of a finite number of points. From an arithmetic perspective, the points are usually not defined over the base field. Thus, a detailed inspection requires to work with algebraic number fields. Many algorithms for them are described in [9] and [10].

In dimension 1, the solution sets are algebraic curves. Projective curves are classified by the degree, abstract irreducible curves by the genus. A smooth plane curve of degree 1 or 2 has genus 0. Thus, from a geometric perspective, the curve is isomorphic to the projective line. However, the isomorphism is only defined over the base field if the curve has a rational point. The answer to this question can be found by using the famous theorem of Hasse-Minkowski.

Smooth curves of degree 3 are of genus one. They have been studied by many authors from various perspectives. Most notable are the investigations towards the Birch and Swinnerton-Dyer conjecture [1], [2].

Increasing the dimension once more leads us to algebraic surfaces. Here, we have the Enriques-Kodaira classification, which is based on the Kodaira dimension. Prominent surfaces of Kodaira dimensions $-\infty$ are the projective plane, quadratic and cubic surfaces. Ruled surfaces, i.e. surfaces birationally equivalent to $\mathbf{P}^1 \times C$, for a curve C of arbitrary genus, are in this class too.

The most important family of surfaces of Kodaira dimension 0 are K3 surfaces. This family includes all smooth quartic surfaces. Further, abelian surfaces have Kodaira dimension 0, as well.

Finally, there are surfaces of Kodaira-dimensions 1 and 2. They will not be considered in this talk.

The algorithms presented in this article have been implemented by the authors over the past 10 years as a part of their research. The given examples are based on `magma` [4], version 2.25.

2 Computation with cubic surfaces

2.1 Definition

A cubic surface is a smooth algebraic surface in $\mathbf{P}^3$ given as the zero set of a homogeneous cubic form in four variables.

2.2 Properties of cubic surfaces

1. It is well known that every smooth cubic surface contains exactly 27 lines. As the lines generate the Picard group of the surface, many other properties of the surface relate to them [24].
2. The moduli stack of cubic surfaces if of dimension 4.

For a modern presentation of the geometry of cubic surfaces we refer the interested read to [12, Chap. 9].

2.3 Computational questions

1. Given two cubic surfaces, can we test for isomorphy?
2. Given a cubic surface over a finite field, can we count the number of points on the surface efficiently?
3. Given a cubic surface over the rationals, what is known about the number of rational points on the surface? Is there a computational approach to this?

2.4 Invariants and isomorphy testing

As proven by Clebsch, the ring of invariants of even weight of cubic surfaces is generated by five invariants of degrees 8, 16, 24, 32, and 40 [8]. Further, there is an invariant of odd weight an degree 100. These invariants can be computed in `magma`:

```
r4<x,y,z,w> := PolynomialRing(Rationals(),4);
f := x^3 + y^3 + z^3 + w^3 + (3*x+3*z+4*w)^3;
time ClebschSalmonInvariants(f);
[ -2579, -46656, 0, 0, 0 ]
-708235798046072773554016875
Time: 0.010
> time Factorization(708235798046072773554016875);
[ <3, 27>, <5, 4>, <13, 4>, <2281, 2> ]
```

```
Time: 0.000
> time SkewInvariant100(f);
0
Time: 0.010
```

The last return value of `ClebschSalmonInvariants` is the discriminant of the surface. Thus, this surface has bad reduction only at 3, 5, 13 and 2281. Further, the degree 100 invariant vanishes. This shows that the surface has at least one non-trivial automorphism. Multiplication of y by a 3rd root of unity and interchanging x- and z-coordinate are automorphisms of the example.

A detailed description of the algorithm is given in [19]. As an isomorphism of smooth cubic surfaces is always given by a projective linear map, they are isomorphic if and only if the invariants coincide.

2.5 Counting points over finite fields

The number of points on a variety over a finite field relates to the Galois module structure on its etale cohomology [25]. In the case of a cubic surface, the cohomology is generated by the lines on the surface. Using Grbner bases, one can explicitly determine the lines on a cubic surface and compute the Galois module structure. In `magma`, this is available as follows:

```
r4<x,y,z,w> := PolynomialRing(Rationals(),4);
f := x^3 + y^3 + z^3 + w^3 + (x+2*x+3*z+4*w)^3;
p := NextPrime(17^17);
time NumberOfPointsOnCubicSurface(PolynomialRing(GF(p),4)!f);
684326450885775034172205518946819088355253 9
Time: 0.170
```

The second return value is the action of the Frobenius on the lines encoded by its Swinnerton-Dyer number. A detailed description of the lines on a cubic surface and potential Frobenius actions are given in [24].

2.6 Rational points on cubic surfaces

As soon as a smooth cubic surface over $\mathbb{Q}$ has one rational point, one can construct infinitely many other rational points. Further, there are numerous conjectures and questions towards the set of rational points.

If we fix a search bound B, then we can ask for the number of points,

$$n(B) := \#\{(x:y:z:w) \in \mathbf{P}^3(\mathbb{Q}) \mid x,y,z,w \in \mathbb{Z}, |x|,|y|,|z|,|w| < B,\\ f(x,y,z,w) = 0\}\,,$$

on the surface, given by $f = 0$. For cubic surfaces, this question is covered by the Manin conjecture [20]. More precisely, when counting only the points that are not contained in any of the lines on the surface

$$n'(B) := \#\{(x:y:z:w) \in \mathbf{P}^3(\mathbb{Q}) \mid x,y,z,w \in \mathbb{Z}, |x|,|y|,|z|,|w| < B,\\ f(x,y,z,w) = 0, (x:y:z:w) \text{ not on a line of } f = 0\}\,,$$

the conjecture predicts the existence of a constant C such that

$$n' \sim C \cdot B \log^{r-1}(B) \,.$$

Here, r is the rank of the arithmetic Picard group. A conjecture for the value of C is presented in [26]. Today, we have a lot of numerical and theoretical evidence for this conjecture. Some numerical examples of smooth cubics are given in [15]. Examples, such that a more complex set then just a fixed finite collection of lines on the variety needs to be excluded from the count are give in [13] and [14].

Finally, the conjecture is proven for some singular surfaces [5]. The interested reader my also consult [6] for a general introduction to the Manin conjecture.

3 Computations with K3 surfaces

3.1 Definition

A K3 surface is a smooth algebraic surface which is simply connected and has trivial canonical class.

3.2 Examples

As the definition of K3 surfaces is abstract, they arise in various forms.

1. Let $f_6(x, y, z) = 0$ be a smooth plane curve of degree 6. Then the double cover of $\mathbf{P}^2$, given by
$$w^2 = f_6(x, y, z) \,,$$
is a K3 surface of degree 2 in $\mathbf{P}(1, 1, 1, 3)$.
2. A smooth quartic surface in $\mathbf{P}^3$ is a K3 surface.
3. A smooth complete intersection of a quadric and a cubic in $\mathbf{P}^4$ is a K3 surface of degree 6.
4. A smooth complete intersection of three quadrics in $\mathbf{P}^5$ is a K3 surface of degree 8.

If a surface of the shape above has only ADE-singularites, then the minimal resolution of singularities is still a K3 surface.

3.3 Questions towards K3 surfaces

1. Can we test isomorphy of K3 surfaces?
2. What is known about its cohomology?
3. Can we count point over finite fields on K3 surfaces efficiently?
4. What is known about the rational points on a K3 surface defined over $\mathbb{Q}$?

3.4 Invariants and Isomorphy

For none of the above models of K3 surfaces, a complete system of invariants is known. In contrast to cubic surfaces, an isomorphism of K3 surfaces may not be given by a projective linear map. Thus, the isomorphy test is harder in this instance as different embeddings have to be taken into account.

3.5 Cohomology of K3 surfaces

The cohomology $H^2(V, \mathbb{Z})$ of a K3 surface V over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}^{22}$. All algebraic cycles defined over $\overline{\mathbb{Q}}$ generate a sublattice called the geometric Picard group. Its rank $r \in \{1, \ldots, 20\}$ is called the geometric Picard rank.

3.6 Counting points over finite fields

Counting points over finite fields can always be done naively by enumeration. However, there are more efficient methods available. Most noteable are the p-adic methods developed by Kedlaya, Harvey, and others [22], [21], [11]. The following is available in `magma`:

```
r3<x,y,z> := PolynomialRing(Rationals(),3);
f := x^6+y^6+z^6+(x+2*y+3*z)^6;
time WeilPolynomialOfDegree2K3Surface(f,31);
Time: 72.700
t^22 + 58*t^21 + 372*t^20 - 55738*t^19 - 1549132*t^18
    - 12929294*t^17 - 572583020*t^16 - 15975066258*t^15
    + 495227053998*t^14 + 10234692449292*t^13 - 608111309695433*t^12
    + 5843949686173111113*t^10 - 94519534054625971332*t^9
    - 4395158333540107666638*t^8 + 136249908339743333765778*t^7
    + 4693052398368937185990020*t^6 + 101839237044605936935987341*t^5
    + 1172606072256462645291511372*t^4
    + 40545109959944612235271873978*t^3
    - 260047946639644754336571329652*t^2
    - 3896385067150677235809627089285 8*t
    - 6455906981951380730367330401385 61
```

This gives us the characteristic polynomial of the Frobenius on the etale cohomology and the number of points over $\mathbb{F}_{31^d}$ is encoded in this. Some details on this function are given in [18].

3.7 Computing algebraic cycles

As explained above, the algebraic cycles on a K3 surface form a lattice of rank $r = 1, \ldots, 20$. Thus, a first step to determine its rank is a computation of lower and upper bounds. Lower bounds can be generated by enumerating cycles. Upper bounds can be derived by point counting [23]. Applying `WeilPolynomialToRankBound` to the above example gives us the bound $r \leq 10$. Combining this with a modulo 71 computation and `ArtinTateFormula`, one can sharpen this bound to $r \leq 9$. These functions were implemented as part of the research described in [16].

As worked out by Charles [7], primes resulting in sharp upper bounds have positive density, as long as the surface does not have real multiplication. The first explicit family V_t of K3 surfaces such that the approach fails was given in [17]:

$$V_t\colon w^2 = q_1 q_2 q_3$$

with

$$\begin{aligned} q_1 &:= \left(\frac{1}{8}t^2 - \frac{1}{2}t + \frac{1}{4}\right) y^2 + (t^2 - 2t + 2)yz + (t^2 - 4t + 2)z^2 \,, \\ q_2 &:= \left(\frac{1}{8}t^2 + \frac{1}{2}t + \frac{1}{4}\right) x^2 + (t^2 + 2t + 2)xz + (t^2 + 4t + 2)z^2 \,, \\ q_3 &:= 2x^2 + (t^2 + 2)xy + t^2y^2 \,. \end{aligned}$$

3.8 Rational points

For the structure of the set of rational points on K3 surfaces, only conjectures are known. Most notable is a conjecture of Bogomolov [3]: *Every rational point on a K3 surfaces lies on some rational curve on the surface.*

Up to the authors knowledge, there are no computational investigations on this conjecture.

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