On the smallest point on a diagonal quartic threefold

Andreas-Stephan Elsenhans* and Jörg Jahnel*

Abstract

For the family $a_0x^4 = a_1y^4 + a_2z^4 + a_3v^4 + a_4w^4$, $a_0, \ldots, a_4 > 0$, of diagonal quartic threefolds, we study the behaviour of the height of the smallest rational point versus the Tamagawa type number introduced by E. Peyre.

1 Introduction – A computer experiment

1.1. —— Let $V \subseteq \mathbf{P}^n_\mathbb{Q}$ be a Fano variety defined over $\mathbb{Q}$. If $V(\mathbb{Q}_\nu) \neq \emptyset$ for every $\nu \in \mathrm{Val}(\mathbb{Q})$ then it is natural to ask whether $V(\mathbb{Q}) \neq \emptyset$ (Hasse's principle). Further, it would be desirable to have an *a-priori* upper bound for the height of the smallest $\mathbb{Q}$-rational point on V as this would allow to effectively decide whether $V(\mathbb{Q}) \neq \emptyset$ or not.

When V is a conic, Legendre's theorem on zeroes of ternary quadratic forms proves the Hasse principle and, moreover, yields an effective bound for the smallest point. For quadrics of arbitrary dimension, the same is true by an observation due to J. W. S. Cassels [Ca]. Further, there is a theorem of C. L. Siegel [Si, Satz 1] which provides a generalization to hypersurfaces defined by norm equations. For more general Fano varieties, no theoretical upper bound is known for the height of the smallest $\mathbb{Q}$-rational point. Some of these varieties fail the Hasse principle.

In this note, we present some theoretical and experimental results concerning the height of the smallest $\mathbb{Q}$-rational point on quartic hypersurfaces in $\mathbf{P}^4_\mathbb{Q}$.

1.2. —— There is a conjecture, due to Yu. I. Manin, that the number of $\mathbb{Q}$-rational points of anticanonical height $< B$ on a Fano variety V is asymptotically equal to $\tau B \log^{\mathrm{rk}\,\mathrm{Pic}(V)-1} B$, for $B \to \infty$.

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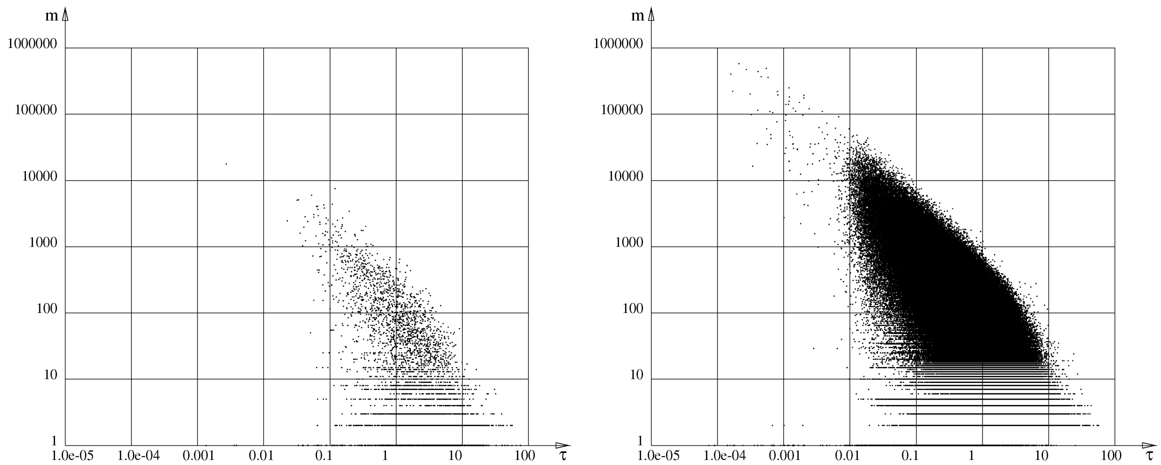
In the particular case of a quartic threefold, the anticanonical height is the same as the naive height. Further, $\operatorname{rk}\operatorname{Pic}(V) = 1$ and, finally, the coefficient $\tau \in \mathbb{R}$ equals the Tamagawa-type number $\tau(V)$ introduced by E. Peyre in [Pe]. Thus, one expects $\sim\tau(V)B$ points of height $<B$. Assuming equidistribution, the height of the smallest point should be $\sim\frac{1}{\tau(V)}$. Being a bit optimistic, this might lead to the expectation that $\mathrm{m}(V)$, the height of the smallest $\mathbb{Q}$-rational point on V, is always less than $\frac{C}{\tau(V)}$ for a certain absolute constant C.

1.3. —— To test this expectation, we computed the Tamagawa number and ascertained the smallest $\mathbb{Q}$-rational point for each of the quartic threefolds $V_4^{(-a,b)} \subset \mathbf{P}^4_{\mathbb{Q}}$ given by $ax^4 = by^4 + z^4 + v^4 + w^4$ for $a, b = 1, \ldots, 1000$.

On 516 820 of these varieties, there are no $\mathbb{Q}$-rational points as the equation is unsolvable in $\mathbb{Q}_p$ for $p = 2$, 5, or 29. (Note that, for each prime p different from 2, 5, or 29, there are p-adic points already on the Fermat quartic given by $z^4+v^4+w^4 = 0$.) On each of the remaining varieties, $\mathbb{Q}$-rational points were found. In other words, there are no counterexamples to the Hasse principle in this family.

The methods to systematically search for solutions of Diophantine equations and to compute Tamagawa numbers we applied here are described in our earlier papers [EJ1, EJ2].

The results are summarized by the diagrams below.



$a, b = 1 \ldots 100$ $\qquad$ $a, b = 1 \ldots 1000$

Figure 1: Height of smallest point versus Tamagawa number

It is apparent from the diagrams that the experiment agrees with the expectation above. The slope of a line tangent to the top right of each of the scatter plots is indeed near (-1). However, we show in Section 2 that, in general, the inequality $\mathrm{m}(V) < \frac{C}{\tau(V)}$ does not hold. The following remains a logical possibility.

1.4. Question. —— For every $\varepsilon > 0$, does there exist a constant $C(\varepsilon)$ such that, for each quartic threefold,

$$\mathrm{m}(V) < \frac{C(\varepsilon)}{\tau(V)^{1+\varepsilon}}\,?$$

1.5. *Peyre's constant.* —— In our situation, $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}}) = 1$. In particular, there is no Brauer-Manin obstruction on V. Recall that, in this case, E. Peyre's Tamagawa-type number is defined [PT, Definition 2.4] as an infinite product $\tau(V) := \prod_{\nu\in\mathrm{Val}(\mathbb{Q})}\tau_\nu(V)$. For a prime number p, the definition of the local factor may be simplified to

$$\tau_p(V) := \Big(1 - \frac{1}{p}\Big)\cdot \lim_{n\to\infty}\frac{\#V(\mathbb{Z}/p^n\mathbb{Z})}{p^{3n}}\,.$$

$\tau_\infty(V)$ is described in [Pe, Lemme 5.4.7]. In the case of the diagonal quartic threefold $V^{(a_0,\ldots,a_4)}$ given by $a_0x_0^4 + \ldots + a_4x_4^4 = 0$ ($a_0 < 0, a_1, \ldots, a_4 > 0$) in $\mathbf{P}^4_{\mathbb{Q}}$, this yields

$$(*)\qquad \tau_\infty(V^{(a_0,\ldots,a_4)}) = \frac{1}{4\sqrt[4]{|a_0|}}\iiiint\limits_R \frac{1}{(a_1y^4 + a_2z^4 + a_3v^4 + a_4w^4)^{3/4}}\,dy\,dz\,dv\,dw$$

where the domain of integration is

$$R := \{(y,z,v,w) \in [-1,1]^4 \mid |a_1y^4 + a_2z^4 + a_3v^4 + a_4w^4| \leq |a_0|\}\,.$$

1.6. —— In the case of diagonal quartic threefolds, there is an estimate for $\mathrm{m}(V)$ in terms of $\tau(V)$. Namely, $\frac{1}{\tau(V)}$ admits a fundamental finiteness property. More precisely, in Section 3, we will show the following results.

Theorem. *Let $\mathfrak{a} = (a_0, \ldots, a_4)$ be a vector such that $a_0, \ldots, a_4 \in \mathbb{Z}$, $a_0 < 0$, and $a_1, \ldots, a_4 > 0$. Denote by $V^{\mathfrak{a}}$ the quartic in $\mathbf{P}^4_{\mathbb{Q}}$ given by $a_0x_0^4 + \ldots + a_4x_4^4 = 0$. Then, for each $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\frac{1}{\tau(V^{\mathfrak{a}})} \geq C(\varepsilon)\cdot \mathrm{H}_{\mathrm{naive}}\big(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\big)^{\frac{1}{4}-\varepsilon}.$$

Corollary (Fundamental Finiteness)**.** *For each $B > 0$, there are only finitely many quartics $V^{\mathfrak{a}}$: $a_0x_0^4 + \ldots + a_4x_4^4 = 0$ in $\mathbf{P}^4_{\mathbb{Q}}$ such that $a_0 < 0$, $a_1, \ldots, a_4 > 0$, and $\tau(V^{\mathfrak{a}}) > B$.*

Corollary (An inefficient search bound)**.** *There exists a monotonically decreasing function $S\colon (0,\infty) \to [0,\infty)$, the* search bound, *satisfying the following condition.* Every quartic $V^{\mathfrak{a}}$: $a_0x_0^4 + \ldots + a_4x_4^4 = 0$ such that $a_0 < 0$ and $a_1, \ldots, a_4 > 0$ admits, if $V^{\mathfrak{a}}(\mathbb{Q}) \neq \emptyset$, already a $\mathbb{Q}$-rational point of height $\leq S(\tau(V^{\mathfrak{a}}))$.

Proof. *One may simply put* $S(t) := \max\limits_{\substack{\tau(V^{\mathfrak{a}})\geq t\\ V^{\mathfrak{a}}(\mathbb{Q})\neq\emptyset}} \min\limits_{P\in V^{\mathfrak{a}}(\mathbb{Q})} h(P)$. □

In other words, we have $m(V^{\mathfrak{a}}) \leq S(\tau(V^{\mathfrak{a}}))$ as soon as $V^{\mathfrak{a}}(\mathbb{Q}) \neq \emptyset$.

2 A negative result

For $a \in \mathbb{N}$, let $V^{(-a)} \subset \mathbf{P}^4_{\mathbb{Q}}$ be given by $ax^4 = y^4 + z^4 + v^4 + w^4$ and let

$$\mathrm{m}(V^{(-a)}) := \min \{ \mathrm{H}_{\mathrm{naive}}(x:y:z:v:w) \mid (x:y:z:v:w) \in V^{(-a)}(\mathbb{Q}) \}$$

be the smallest height of a $\mathbb{Q}$-rational point on $V^{(-a)}$. We compare $\mathrm{m}(V^{(-a)})$ with the Tamagawa type number $\tau^{(-a)} := \tau(V^{(-a)})$.

2.1. Notation. — For a prime number p and integers $y, z, \ldots$, not all of which are equal to zero, we write $\gcd_p(y, z, \ldots)$ for the largest power of p dividing all of the $y, z, \ldots$.

2.2. Theorem. — *There is no constant C such that*

$$\mathrm{m}(V^{(-a)}) < \frac{C}{\tau^{(-a)}}$$

for all $a \in \mathbb{N}$.

Proof. The proof consists of several steps.

First step. For $a \geq 4$, one has $\tau_\infty^{(-a)} = \frac{1}{\sqrt[4]{a}} I$ where I is an integral independent of a. This follows immediately from formula $(*)$ above.

Second step. For the height of the smallest point, we have $\mathrm{m}(V^{(-a)}) \geq \sqrt[4]{\frac{a}{4}}$.
$|x| \geq 1$ yields $y^4 + z^4 + v^4 + w^4 \geq a$ and $\max\{|y|, |z|, |v|, |w|\} \geq \sqrt[4]{\frac{a}{4}}$.

Third step. There are two positive constants C_1 and C_2 such that, for all $a \in \mathbb{N}$,

$$C_1 < \prod_{\substack{p \text{ prime} \\ p>13, p\nmid a}} \tau_p^{(-a)} < C_2 \,.$$

For a prime p of good reduction, Hensel's lemma shows $\tau_p(V^{(-a)}) = (1-\frac{1}{p}) \cdot \frac{\#V^{(-a)}(\mathbb{F}_p)}{p^3}$. Further, for the number of points on a non-singular variety over a finite field, there are excellent estimates provided by the Weil conjectures, proven by P. Deligne. In our situation, [De, Théorème (8.1)] may be directly applied. It shows $\#V^{(-a)}(\mathbb{F}_p) = p^3 + p^2 + p + 1 + E^{(-a)}$ with an error-term $|E^{(-a)}| \leq 60p^{3/2}$. Note that $\dim H^3(V, \mathbb{R}) = 60$ for every smooth quartic threefold V in $\mathbf{P}^4_{\mathbb{C}}$.

Consequently, $1 - \frac{60(1-1/p)}{p^{3/2}} - \frac{1}{p^4} \leq \tau_p^{(-a)} \leq 1 + \frac{60(1-1/p)}{p^{3/2}} - \frac{1}{p^4}$. Here, the left hand side is positive for $p > 13$. The infinite product over all $1 - \frac{60(1-1/p)}{p^{3/2}} - \frac{1}{p^4}$ (respectively $1 + \frac{60(1-1/p)}{p^{3/2}} - \frac{1}{p^4}$) is convergent.

Fourth step. There is a sequence $\{a_i\}_{i\in\mathbb{N}}$ of natural numbers such that

$$\left\{ \prod_{\substack{p \text{ prime} \\ p\leq 13 \text{ or } p|a_i}} \tau_p^{(-a_i)} \right\}_{i\in\mathbb{N}}$$

is unbounded.

Let $C \in \mathbb{R}$ be given. We will show that

$$\prod_{\substack{p \text{ prime} \\ p \leq 13 \text{ or } p|a}} \tau_p^{(-a)} > C$$

when $a := p_1 \cdot \ldots \cdot p_r$ is a product of sufficiently many different primes $p_i \equiv 3 \pmod 4$ fulfilling $a \equiv 1 \pmod M$ for $M := 16 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

Let p be a prime such that $p|a$. We made sure that $p^4 \nmid a$. Then, for any point $(x : y : z : v : w) \in V^{(-a)}(\mathbb{Z}/p^n\mathbb{Z})$, the assumption $p \,|\, y, z, v, w$ would imply $p^4 | ax^4$ and $p|x$. Therefore, $\gcd_p(y, z, v, w) = 1$. Further, $y^4 + z^4 + v^4 + w^4 \equiv 0 \pmod p$.

As $p \equiv 3 \pmod 4$, the number of solutions of that congruence is the same as that of $y^2 + z^2 + v^2 + w^2 \equiv 0 \pmod p$. Since p remains prime in $\mathbb{Z}[i]$, this quadratic form is a direct sum of two norm forms. Its number of zeroes in $\mathbb{F}_p^4$ is therefore equal to $1 + (p-1)(p+1)^2$.

$\gcd_p(y, z, v, w) = 1$ implies that Hensel's lemma is applicable. It shows

$$\#V^{(-a)}(\mathbb{Z}/p^n\mathbb{Z}) = \frac{p^n \cdot (p-1)(p+1)^2 p^{3(n-1)}}{(p-1)p^{n-1}} = p^{3n-2}(p+1)^2.$$

Hence, $\tau_p^{(-a)} = (1 - \frac{1}{p})\frac{(p+1)^2}{p^2} = 1 + \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3}$. We may consequently write

$$\prod_{\substack{p \text{ prime} \\ p \leq 13 \text{ or } p|a}} \tau_p^{(-a)} = \prod_{i=1}^{r} \Big(1 + \frac{1}{p_i} - \frac{1}{p_i^2} - \frac{1}{p_i^3}\Big) \cdot \prod_{\substack{p \text{ prime} \\ p \leq 13}} \tau_p^{(-a)}.$$

The second product is over $p = 2$ and a finite number of primes of good reduction. The value of the product depends only on the residue of a modulo $M = 16 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ by virtue of [EJ2, Lemma 13.a) and b.ii)]. In particular, our assumption $a \equiv 1 \pmod M$ implies

$$T := \prod_{\substack{p \text{ prime} \\ p \leq 13}} \tau_p^{(-a)}$$

is a constant. It is clear that $T > 0$ as the equation $x^4 = y^4 + z^4 + v^4 + w^4$ admits a non-zero solution in $\mathbb{Z}/m\mathbb{Z}$ for any $m > 1$.

It remains to show that there exists a set $\{p_1, \ldots, p_r\}$ of primes $p_i \equiv 3 \pmod 4$ such that $p_i > 13$,

$$(+) \qquad \prod_{i=1}^{r} \Big(1 + \frac{1}{p_i} - \frac{1}{p_i^2} - \frac{1}{p_i^3}\Big) \geq \frac{C}{T},$$

and $p_1 \cdot \ldots \cdot p_r \equiv 1 \pmod M$.

Condition (+) is easily satisfied as the series $\sum_{p\equiv 3 \pmod 4} \frac{1}{p}$ diverges. We find a set $\{p_1, \ldots, p_s\}$ of prime numbers $p_i > 13$ of the form $p_i \equiv 3 \pmod 4$ such that $\prod_{i=1}^{s} \left(1 + \frac{1}{p_i} - \frac{1}{p_i^2} - \frac{1}{p_i^3}\right) \geq \frac{C}{T}$. Enlarging $\{p_1, \ldots, p_s\}$ makes that product even bigger. We may therefore assume $p_1 \cdot \ldots \cdot p_s \equiv 3 \pmod 4$.

The numbers M and $p_1 \cdot \ldots \cdot p_s$ are relatively prime. By Dirichlet's prime number theorem, there exists a prime p_{s+1}, larger than each of the $p_1, \ldots, p_s$, such that $p_1 \cdot \ldots \cdot p_{s+1} \equiv 1 \pmod M$. This shows, in particular, $p_{s+1} \equiv 3 \pmod 4$. The assertion follows.

Conclusion. The four steps together show that $\mathrm{m}(V^{(-a)}) \cdot \tau^{(-a)}$ is unbounded. $\square$

3 The fundamental finiteness property

3.1 An estimate for the factors at the finite places

3.1.1. Notation. — i) For a prime number p and an integer $x \neq 0$, we put $x^{(p)} := p^{\nu_p(x)}$. Note $x^{(p)} = 1/\|x\|_p$ for the normalized p-adic valuation.

ii) By putting $\nu(x) := \min\limits_{\substack{\xi \in \mathbb{Z}_p \\ x = (\xi \bmod p^r)}} \nu(\xi)$, we carry the *p-adic valuation* from $\mathbb{Z}_p$ over to $\mathbb{Z}/p^r\mathbb{Z}$.

Note that any $0 \neq x \in \mathbb{Z}/p^r\mathbb{Z}$ has the form $x = \varepsilon \cdot p^{\nu(x)}$ where $\varepsilon \in (\mathbb{Z}/p^r\mathbb{Z})^*$ is a unit. Clearly, ε is unique only in the case $\nu(x) = 0$.

3.1.2. Definition. — For $(a_0, \ldots, a_4) \in \mathbb{Z}^5$, $r \in \mathbb{N}$, and $\nu_0, \ldots, \nu_4 \leq r$, put

$$S^{(r)}_{\nu_0,\ldots,\nu_4;a_0,\ldots,a_4} := \{(x_0, \ldots, x_4) \in (\mathbb{Z}/p^r\mathbb{Z})^5 \mid \\ \nu(x_0) = \nu_0, \ldots, \nu(x_4) = \nu_4;\ a_0x_0^4 + \ldots + a_4x_4^4 = 0 \in \mathbb{Z}/p^r\mathbb{Z}\}\,.$$

For the particular case $\nu_0 = \ldots = \nu_4 = 0$, we will write $Z^{(r)}_{a_0,\ldots,a_4} := S^{(r)}_{0,\ldots,0;a_0,\ldots,a_4}$. I.e.,

$$Z^{(r)}_{a_0,\ldots,a_4} = \{(x_0, \ldots, x_4) \in [(\mathbb{Z}/p^r\mathbb{Z})^*]^5 \mid a_0x_0^4 + \ldots + a_4x_4^4 = 0 \in \mathbb{Z}/p^r\mathbb{Z}\}\,.$$

We will use the notation $z^{(r)}_{a_0,\ldots,a_4} := \#Z^{(r)}_{a_0,\ldots,a_4}$.

3.1.3. Sublemma. — *If $p^k | a_0, \ldots, a_4$ and $r > k$ then we have*

$$z^{(r)}_{a_0,\ldots,a_4} = p^{5k} \cdot z^{(r-k)}_{a_0/p^k,\ldots,a_4/p^k}\,.$$

Proof. Since $a_0x_0^4 + \ldots + a_4x_4^4 = p^k(a_0/p^k \cdot x_0^4 + \ldots + a_4/p^k \cdot x_4^4)$, there is a surjection

$$\iota\colon Z^{(r)}_{a_0,\ldots,a_4} \longrightarrow Z^{(r-k)}_{a_0/p^k,\ldots,a_4/p^k}\,,$$

given by $(x_0, \ldots, x_4) \mapsto ((x_0 \bmod p^{r-k}), \ldots, (x_4 \bmod p^{r-k}))$. The kernel of the homomorphism of modules underlying ι is $(p^{r-k}\mathbb{Z}/p^r\mathbb{Z})^5$. $\square$

3.1.4. Lemma. —— *Assume* $\gcd_p(a_0, \ldots, a_4) = p^k$*. Then, there is an estimate*

$$z^{(r)}_{a_0,\ldots,a_4} \leq 8p^{4r+k}.$$

Proof. Suppose first that $k = 0$. This means, one of the coefficients is prime to p. Without restriction, assume $p \nmid a_0$.

For any $(x_1, \ldots, x_4) \in (\mathbb{Z}/p^r\mathbb{Z})^4$, there appears an equation of the form $a_0x_0^4 = c$. For p odd, it cannot have more than four solutions in $(\mathbb{Z}/p^r\mathbb{Z})^*$ as this group is cyclic. On the other hand, in the case $p = 2$, we have $(\mathbb{Z}/2^r\mathbb{Z})^* \cong \mathbb{Z}/2^{r-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and up to eight solutions are possible.

The general case follows directly from Sublemma 3.1.3. Indeed, if $k < r$ then

$$z^{(r)}_{a_0,\ldots,a_4} = p^{5k} \cdot z^{(r-k)}_{a_0/p^k,\ldots,a_4/p^k} \leq p^{5k} \cdot 8p^{4(r-k)} = 8p^{4r+k}.$$

On the other hand, if $k \geq r$ then the assertion is completely trivial since

$$z^{(r)}_{a_0,\ldots,a_4} = \#Z^{(r)}_{a_0,\ldots,a_4} < p^{5r} \leq p^{4r+k} < 8p^{4r+k}. \qquad \square$$

3.1.5. Remark. —— The proof shows that in the case $p \neq 2$ the same inequality is true with coefficient 4 instead of 8. If $p \equiv 3 \pmod 4$ then one could even reduce the coefficient to 2. Unfortunately, these observations do not lead to a substantial improvement of our final result.

3.1.6. Lemma. —— *Let* $r \in \mathbb{N}$ *and* $\nu_0, \ldots, \nu_4 \leq r$*. Then,*

$$\#S^{(r)}_{\nu_0,\ldots,\nu_4;a_0,\ldots,a_4} = \frac{z^{(r)}_{p^{4\nu_0}a_0,\ldots,p^{4\nu_4}a_4} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_4})}{\varphi(p^r)^5}.$$

Proof. As $p^{4\nu_0}a_0x_0^4 + \ldots + p^{4\nu_4}a_4x_4^4 = a_0(p^{\nu_0}x_0)^4 + \ldots + a_4(p^{\nu_4}x_4)^4$, we have a surjection

$$\pi\colon Z^{(r)}_{p^{4\nu_0}a_0,\ldots,p^{4\nu_4}a_4} \longrightarrow S^{(r)}_{\nu_0,\ldots,\nu_4;a_0,\ldots,a_4},$$

given by $(x_0, \ldots, x_4) \mapsto (p^{\nu_0}x_0, \ldots, p^{\nu_4}x_4)$.

For $i = 0, \ldots, 4$, consider the mapping $\iota\colon \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z}$, $x_0 \mapsto p^{\nu_i}x_0$. If $\nu_i = r$ then ι is the zero map. All $\varphi(p^r) = (p-1)p^{r-1}$ units are mapped to zero. Otherwise, observe that ι is $p^{\nu_i} : 1$ on its image. Further, $\nu(\iota(x)) = \nu_i$ if and only if x is a unit. By consequence, π is $(K^{(\nu_0)} \cdot \ldots \cdot K^{(\nu_4)}) : 1$ when we put $K^{(\nu)} := p^\nu$ for $\nu \neq r$ and $K^{(r)} := (p-1)p^{r-1}$. Summarizing, we could have written $K^{(\nu)} := \varphi(p^r)/\varphi(p^{r-\nu})$. The assertion follows. $\square$

3.1.7. Corollary. —— *Let* $(a_0, \ldots, a_4) \in (\mathbb{Z}\setminus\{0\})^5$*. Then, for the local factor* $\tau_p^{(a_0,\ldots,a_4)} := \tau_p(V^{(a_0,\ldots,a_4)})$*, one has*

$$\tau_p^{(a_0,\ldots,a_4)} = \lim_{r\to\infty} \sum_{\nu_0,\ldots,\nu_4=0}^{r} \frac{z^{(r)}_{p^{4\nu_0}a_0,\ldots,p^{4\nu_4}a_4} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_4})}{p^{4r} \cdot \varphi(p^r)^5}.$$

Proof. [PT, Corollary 3.5] implies that

$$\tau_p^{(a_0,\dots,a_4)} = \lim_{r\to\infty} \sum_{\nu_0,\dots,\nu_4=0}^{r} \frac{\# S^{(r)}_{\nu_0,\dots,\nu_4;a_0,\dots,a_4}}{p^{4r}} \,.$$

Lemma 3.1.6 yields the assertion. $\square$

3.1.8. Proposition. — *Let $(a_0, \dots, a_4) \in (\mathbb{Z}\setminus\{0\})^5$. Then, for each ε such that $0 < \varepsilon < \frac{1}{4}$, one has*

$$\tau_p^{(a_0,\dots,a_4)} \leq 8\Big(\frac{1}{1-\frac{1}{p^{1-4\varepsilon}}}\Big)\Big(\frac{1}{1-\frac{1}{p^{\varepsilon}}}\Big)^4 \cdot \big(a_0^{(p)}\cdot\ldots\cdot a_3^{(p)}\big)^{\frac{1-\varepsilon}{4}} \big(a_4^{(p)}\big)^{\varepsilon}.$$

Proof. By Lemma 3.1.4,

$$\begin{aligned} z^{(r)}_{p^{4\nu_0}a_0,\,\dots\,,p^{4\nu_4}a_4}/p^{4r} &\leq 8\gcd\nolimits_p(p^{4\nu_0}a_0,\,\dots\,,p^{4\nu_4}a_4) \\ &= 8\gcd\big(p^{4\nu_0}a_0^{(p)},\,\dots\,,p^{4\nu_4}a_4^{(p)}\big). \end{aligned}$$

Writing $p^{k_i} := a_i^{(p)}$, we see

$$\begin{aligned} z^{(r)}_{p^{4\nu_0}a_0,\,\dots\,,p^{4\nu_4}a_4}/p^{4r} &\leq 8\gcd(p^{4\nu_0+k_0},\,\dots\,,p^{4\nu_4+k_4}) \\ &= 8p^{\min\{4\nu_0+k_0,\,\dots\,,4\nu_4+k_4\}}. \end{aligned}$$

We estimate the minimum by a weighted arithmetic mean with weights $\frac{1-\varepsilon}{4}$, $\frac{1-\varepsilon}{4}$, $\frac{1-\varepsilon}{4}$, $\frac{1-\varepsilon}{4}$, and ε,

$$\begin{aligned} \min\{4\nu_0+k_0,\,\dots\,,4\nu_4+k_4\} &\leq \frac{1-\varepsilon}{4}\cdot(4\nu_0+k_0) + \,\dots\, + \frac{1-\varepsilon}{4}\cdot(4\nu_3+k_3) \\ &\qquad + \varepsilon(4\nu_4+k_4) \\ &= (1-\varepsilon)(\nu_0+\,\dots\,+\nu_3) + 4\varepsilon\nu_4 \\ &\qquad + \frac{1-\varepsilon}{4}(k_0+\,\dots\,+k_3) + \varepsilon k_4\,. \end{aligned}$$

This shows

$$\begin{aligned} z^{(r)}_{p^{4\nu_0}a_0,\,\dots\,,p^{4\nu_4}a_4}/p^{4r} &\leq 8p^{(1-\varepsilon)(\nu_0+\ldots+\nu_3)+4\varepsilon\nu_4+\frac{1-\varepsilon}{4}(k_0+\ldots+k_3)+\varepsilon k_4} \\ &= 8p^{(1-\varepsilon)(\nu_0+\ldots+\nu_3)+4\varepsilon\nu_4}\cdot\big(a_0^{(p)}\ldots a_3^{(p)}\big)^{\frac{1-\varepsilon}{4}}\big(a_4^{(p)}\big)^{\varepsilon}. \end{aligned}$$

We may therefore write

$$\begin{aligned} \tau_p^{(a_0,\dots,a_4)} &\leq 8\big(a_0^{(p)}\ldots a_3^{(p)}\big)^{\frac{1-\varepsilon}{4}}\big(a_4^{(p)}\big)^{\varepsilon} \\ &\qquad \cdot \lim_{r\to\infty}\sum_{\nu_0,\dots,\nu_4=0}^{r}\frac{p^{(1-\varepsilon)(\nu_0+\ldots+\nu_3)+4\varepsilon\nu_4}\cdot\varphi(p^{r-\nu_0})\cdot\ldots\cdot\varphi(p^{r-\nu_4})}{\varphi(p^r)^5}\,. \end{aligned}$$

Here, the term under the limit is precisely the product of four copies of the finite sum

$$\sum_{\nu=0}^{r} \frac{p^{(1-\varepsilon)\nu} \cdot \varphi(p^{r-\nu})}{\varphi(p^r)} = \sum_{\nu=0}^{r-1} \frac{1}{(p^\varepsilon)^\nu} + \frac{p}{p-1} \frac{1}{(p^\varepsilon)^r}$$

and one copy of the finite sum

$$\sum_{\nu=0}^{r} \frac{p^{4\varepsilon\nu} \cdot \varphi(p^{r-\nu})}{\varphi(p^r)} = \sum_{\nu=0}^{r-1} \frac{1}{(p^{1-4\varepsilon})^\nu} + \frac{p}{p-1} \frac{1}{(p^{1-4\varepsilon})^r} \, .$$

For $r \to \infty$, geometric series do appear while the additional summands tend to zero. $\square$

3.1.9. Remark. — Unfortunately, the constants

$$C_p^{(\varepsilon)} := 8\Big(\frac{1}{1-\frac{1}{p^{1-4\varepsilon}}}\Big)\Big(\frac{1}{1-\frac{1}{p^\varepsilon}}\Big)^4$$

have the property that the product $\prod_p C_p^{(\varepsilon)}$ diverges. On the other hand, we have at least that $C_p^{(\varepsilon)}$ is bounded for $p \to \infty$, say $C_p^{(\varepsilon)} \leq C^{(\varepsilon)}$.

3.1.10. Lemma. — *Let $C > 1$ be any constant. Then, for each $\varepsilon > 0$, one has*

$$\prod_{\substack{p \text{ prime} \\ p|x}} C \leq c \cdot x^\varepsilon$$

for a suitable constant c (depending on ε).

Proof. This follows directly from [Na, Theorem 7.2] together with [Na, Section 7.1, Exercise 7]. $\square$

3.1.11. Proposition. — *For each $\varepsilon > 0$, there exists a constant c such that*

$$\prod_{p \text{ prime}} \tau_p^{(a_0,\ldots,a_4)} \leq c \cdot |a_0 \cdot \ldots \cdot a_4|^{\frac{1}{4}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,4} \|a_i\|_p^{\frac{1}{4}-\varepsilon}$$

for all $(a_0, \ldots, a_4) \in (\mathbb{Z}\backslash\{0\})^5$.

Proof. As already noticed in the third step of the proof of Theorem 2.2, the product over all primes of good reduction is bounded by consequence of the Weil conjectures. It, therefore, remains to show that

$$\prod_{\substack{p \text{ prime} \\ p|2a_0\cdots a_4}} \tau_p^{(a_0,\ldots,a_4)} \leq c \cdot |a_0 \cdot \ldots \cdot a_4|^{\frac{1}{4}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,4} \|a_i\|_p^{\frac{1}{4}-\varepsilon}.$$

For this, we may assume that ε is small, say $\varepsilon < \frac{1}{4}$.

Then, by Proposition 3.1.8, we have at first

$$\begin{aligned}\tau_p^{(a_0,\ldots,a_4)} &\leq C_p^{(\varepsilon)} \cdot \big(a_0^{(p)} \cdot \ldots \cdot a_3^{(p)}\big)^{\frac{1}{4}-\frac{\varepsilon}{5}} \cdot (a_4^{(p)})^{\frac{4}{5}\varepsilon} \\ &= C_p^{(\varepsilon)} \cdot \big(a_0^{(p)} \cdot \ldots \cdot a_3^{(p)} a_4^{(p)}\big)^{\frac{1}{4}-\frac{\varepsilon}{5}} \cdot (a_4^{(p)})^{-\frac{1}{4}+\varepsilon}.\end{aligned}$$

Here, the indices $0, \ldots, 4$ are interchangeable. Hence, it is even allowed to write

$$\begin{aligned}\tau_p^{(a_0,\ldots,a_4)} &\leq C_p^{(\varepsilon)} \cdot \big(a_0^{(p)} \cdot \ldots \cdot a_4^{(p)}\big)^{\frac{1}{4}-\frac{\varepsilon}{5}} \cdot \big(\max_i a_i^{(p)}\big)^{-\frac{1}{4}+\varepsilon} \\ &= C_p^{(\varepsilon)} \cdot \big(a_0^{(p)} \cdot \ldots \cdot a_4^{(p)}\big)^{\frac{1}{4}-\frac{\varepsilon}{5}} \cdot \min_i \|a_i\|_p^{\frac{1}{4}-\varepsilon}.\end{aligned}$$

Now, we multiply over all prime divisors of $2a_0 \cdot \ldots \cdot a_4$. Thereby, on the right hand side, we may twice write the product over all primes since the two rightmost factors are equal to one for $p \nmid a_0 \cdot \ldots \cdot a_4$, anyway.

$$\begin{aligned}\prod_{\substack{p \text{ prime} \\ p | 2a_0 \cdots a_4}} \tau_p^{(a_0,\ldots,a_4)} &\leq \prod_{\substack{p \text{ prime} \\ p | 2a_0 \cdots a_4}} C_p^{(\varepsilon)} \cdot \prod_{p \text{ prime}} \big(a_0^{(p)} \cdot \ldots \cdot a_4^{(p)}\big)^{\frac{1}{4}-\frac{\varepsilon}{5}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,4} \|a_i\|_p^{\frac{1}{4}-\varepsilon} \\ &= \prod_{\substack{p \text{ prime} \\ p | 2a_0 \cdots a_4}} C_p^{(\varepsilon)} \cdot |a_0 \cdot \ldots \cdot a_4|^{\frac{1}{4}-\frac{\varepsilon}{5}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,4} \|a_i\|_p^{\frac{1}{4}-\varepsilon}\end{aligned}$$

when we observe that $\prod_p a^{(p)} = |a|$. Further, we have $C_p^{(\varepsilon)} \leq C^{(\varepsilon)}$ and, by Lemma 3.1.10,

$$\prod_{\substack{p \text{ prime} \\ p | 2a_0 \cdots a_4}} C^{(\varepsilon)} \leq c \cdot |2a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{5}}.$$

We finally estimate $2^{\frac{\varepsilon}{5}}$ by a constant. The assertion follows. □

3.2 A bound for the factor at the infinite place

We want to estimate the integral $\tau_\infty^{(a_0,\ldots,a_4)} := \tau_\infty(V^{(a_0,\ldots,a_4)})$ described in Paragraph 1.5.

3.2.1. Lemma. — *There exist two constants C_1 and C_2 such that*

$$\tau_\infty^{(a_0,\ldots,a_4)} \leq \begin{cases} \frac{1}{\sqrt[4]{|a_0|a_1 \cdot \ldots \cdot a_4}} \big(C_1 \min(|a_0|, 2a_4)^{1/4}\big), & \text{if } |a_0| \leq a_4, \\ \frac{1}{\sqrt[4]{|a_0|a_1 \cdot \ldots \cdot a_4}} \big(C_1 \min(|a_0|, 2a_4)^{1/4} + C_2\, a_4^{1/4} \log \frac{\min(|a_0|, 3a_1)}{a_4}\big), & \text{otherwise,} \end{cases}$$

for all $(a_0, \ldots, a_4) \in \mathbb{R}^5$ satisfying $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1$ and $a_0 \leq -1$.

Proof. A linear substitution shows

$$\tau_\infty^{(a_0,\ldots,a_4)} = \frac{1}{4} \cdot \frac{1}{\sqrt[4]{|a_0|a_1 \cdot \ldots \cdot a_4}} \iiiint\limits_{R^{(0)}} \frac{1}{(y^4 + z^4 + v^4 + w^4)^{3/4}}\, dy\, dz\, dv\, dw$$

where

$$R^{(0)} := \{(y,z,v,w) \in [-a_1^{1/4}, a_1^{1/4}] \times \cdots \times [-a_4^{1/4}, a_4^{1/4}] \mid |y^4+z^4+v^4+w^4| \leq |a_0|\}\,.$$

The integrand is non-negative. We cover $R^{(0)} \subseteq R_1 \cup R_2$ by two sets as follows,

$$R_1 := \{(y,z,v,w) \in \mathbb{R}^4 \mid y^4+z^4+v^4+w^4 \leq \min(|a_0|, 2a_4)\}\,,$$
$$R_2 := \{(y,z,v,w) \in \mathbb{R}^4 \mid a_4 \leq y^4+z^4+v^4 \leq \min(|a_0|, 3a_1) \text{ and } w \in [-a_4^{1/4}, a_4^{1/4}]\}\,,$$

and estimate. In the case $|a_0| \leq a_4$, the domain of integration is covered by R_1 alone and we may omit R_2, completely.

By homogeneity, we have

$$\begin{aligned}\iiiint\limits_{R_1} \frac{1}{(y^4+z^4+v^4+w^4)^{3/4}}\,dy\,dz\,dv\,dw &= \omega_1 \cdot \int\limits_0^{\min(|a_0|,2a_4)^{1/4}} \frac{1}{r^3} \cdot r^3\,dr \\ &= \omega_1 \cdot \min(|a_0|, 2a_4)^{1/4}\end{aligned}$$

where ω_1 is the three-dimensional hypersurface measure of the l_4-unit hypersphere

$$S_1 := \{(y,z,v,w) \in \mathbb{R}^4 \mid y^4+z^4+v^4+w^4 = 1\}\,.$$

Further,

$$\iiiint\limits_{R_2} \frac{1}{(y^4+z^4+v^4+w^4)^{3/4}}\,dy\,dz\,dv\,dw \leq 2a_4^{1/4} \iiint\limits_{R_3} \frac{1}{(y^4+z^4+v^4)^{3/4}}\,dy\,dz\,dv$$

where

$$R_3 := \{(y,z,v) \in \mathbb{R}^3 \mid a_4 \leq y^4+z^4+v^4 \leq \min(|a_0|, 3a_1)\}\,.$$

The latter integral may be treated in much the same way as the one above. We see

$$\iiint\limits_{R_3} \frac{1}{(y^4+z^4+v^4)^{3/4}}\,dy\,dz\,dv = \omega_2 \cdot \int\limits_{a_4^{1/4}}^{\min(|a_0|,3a_1)^{1/4}} \frac{1}{r^3} \cdot r^2\,dr$$

where ω_2 is the usual two-dimensional hypersurface measure of the l_4-unit sphere

$$S_2 := \{(y,z,v) \in \mathbb{R}^3 \mid y^4+z^4+v^4 = 1\}\,.$$

Finally,

$$\int\limits_{a_4^{1/4}}^{\min(|a_0|,3a_1)^{1/4}} \frac{1}{r^3} \cdot r^2\,dr = \log \frac{\min(|a_0|, 3a_1)^{1/4}}{a_4^{1/4}} = \frac{1}{4} \log \frac{\min(|a_0|, 3a_1)}{a_4}\,. \qquad \square$$

3.2.2. Proposition. — *For every $\varepsilon > 0$, there exists a constant C such that*

$$\tau_\infty^{(a_0,\ldots,a_4)} \leq C \cdot |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}+\varepsilon} \cdot \min_{i=0,\ldots,4} \|a_i\|_\infty^{\frac{1}{4}}$$

for each $(a_0, \ldots, a_4) \in \mathbb{Z}^5$ satisfying $a_0 < 0$ and $a_1, \ldots, a_4 > 0$.

Proof. We assume without restriction that $a_1 \geq \ldots \geq a_4$. There are two cases to be distinguished.

First case. $|a_0| \leq a_4$.
Then, by Lemma 3.2.1, we have

$$\begin{aligned}
\tau_\infty^{(a_0,\ldots,a_4)} &\leq |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot C_1 \min\{|a_0|, 2a_4\}^{\frac{1}{4}} \\
&= C_1 \cdot |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot |a_0|^{\frac{1}{4}} \\
&= C_1 \cdot |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot \min_{i=0,\ldots,4} \|a_i\|_\infty^{\frac{1}{4}} .
\end{aligned}$$

Second case. $|a_0| > a_4$.
Here, Lemma 3.2.1 shows

$$\begin{aligned}
\tau_\infty^{(a_0,\ldots,a_4)} &\leq |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \left(C_1 \min\{|a_0|, 2a_4\}^{\frac{1}{4}} + C_2\, a_4^{1/4} \log \frac{\min\{|a_0|, 3a_1\}}{a_4} \right) \\
&\leq |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \left(C_1 (2a_4)^{\frac{1}{4}} + C_2\, a_4^{1/4} \log \frac{|a_0|}{a_4} \right) \\
&= |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot |a_4|^{\frac{1}{4}} \left(C_1 2^{\frac{1}{4}} + C_2 \log \frac{|a_0|}{a_4} \right) \\
&= |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot \min_{i=0,\ldots,4} \|a_i\|_\infty^{\frac{1}{4}} \cdot \left(C_1 2^{\frac{1}{4}} + C_2 \log \frac{|a_0|}{a_4} \right) \\
&\leq |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot \min_{i=0,\ldots,4} \|a_i\|_\infty^{\frac{1}{4}} \cdot \big(C_1 2^{\frac{1}{4}} + C_2 \log |a_0 \cdot \ldots \cdot a_4| \big) \\
&\leq |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}} \cdot \min_{i=0,\ldots,4} \|a_i\|_\infty^{\frac{1}{4}} \cdot \big(C_3 \cdot |a_0 \cdot \ldots \cdot a_4| \big)^{\varepsilon} .
\end{aligned}$$

□

3.3 The Tamagawa number

3.3.1. Proposition. — *For every $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\frac{1}{\tau^{(a_0,\ldots,a_4)}} \geq C \cdot \frac{\mathrm{H}_{\mathrm{naive}} \left(\frac{1}{a_0} : \ldots : \frac{1}{a_4} \right)^{\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\varepsilon}}$$

for each $(a_0, \ldots, a_4) \in \mathbb{Z}^5$ satisfying $a_0 < 0$ and $a_1, \ldots, a_4 > 0$.

Proof. By Proposition 3.2.2, we have

$$\tau_\infty^{(a_0,\ldots,a_4)} \leq C_1 \cdot |a_0 \cdot \ldots \cdot a_4|^{-\frac{1}{4}+\frac{\varepsilon}{2}} \cdot \min_{i=0,\ldots,4} \|a_i\|_\infty^{\frac{1}{4}} .$$

On the other hand, by Theorem 3.1.11,

$$\prod_{p\,\text{prime}} \tau_p^{(a_0,\dots,a_4)} \leq C_2 \cdot |a_0 \cdot \ldots \cdot a_4|^{\frac{1}{4}} \cdot \prod_{p\,\text{prime}} \min_{i=0,\dots,4} \|a_i\|_p^{\frac{1}{4}-\frac{\varepsilon}{2}}.$$

It follows that

$$\tau^{(a_0,\dots,a_4)} \leq C_3 \cdot |a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{2}} \cdot \prod_{p\,\text{prime}} \min_{i=0,\dots,4} \|a_i\|_p^{\frac{1}{4}} \cdot \min_{i=0,\dots,4} \|a_i\|_\infty^{\frac{1}{4}} \cdot \prod_{p\,\text{prime}} \Big[\min_{i=0,\dots,4} \|a_i\|_p\Big]^{-\frac{\varepsilon}{2}}$$

and

$$\begin{aligned}
\frac{1}{\tau^{(a_0,\dots,a_4)}} &\geq \frac{1}{C_3} \cdot \frac{\prod\limits_{p\,\text{prime}} \Big[\min\limits_{i=0,\dots,4} \|a_i\|_p\Big]^{-\frac{1}{4}} \cdot \Big[\min\limits_{i=0,\dots,4} \|a_i\|_\infty\Big]^{-\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p\,\text{prime}} \Big[\min\limits_{i=0,\dots,4} \|a_i\|_p\Big]^{-\frac{\varepsilon}{2}}} \\
&= \frac{1}{C_3} \cdot \frac{\prod\limits_{p\,\text{prime}} \max\limits_{i=0,\dots,4} \Big\|\frac{1}{a_i}\Big\|_p^{\frac{1}{4}} \cdot \max\limits_{i=0,\dots,4} \Big\|\frac{1}{a_i}\Big\|_\infty^{\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p\,\text{prime}} \Big[\max\limits_{i=0,\dots,4} a_i^{(p)}\Big]^{\frac{\varepsilon}{2}}} \\
&= \frac{1}{C_3} \cdot \frac{\mathrm{H}_{\text{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_4}\big)^{\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p\,\text{prime}} \Big[\max\limits_{i=0,\dots,4} a_i^{(p)}\Big]^{\frac{\varepsilon}{2}}}\,.
\end{aligned}$$

It is obvious that $\max\limits_{i=0,\dots,4} a_i^{(p)} \leq |a_0^{(p)} \cdot \ldots \cdot a_4^{(p)}|$ and $\prod\limits_{p\,\text{prime}} |a_0^{(p)} \cdot \ldots \cdot a_4^{(p)}| = |a_0 \cdot \ldots \cdot a_4|$. This shows

$$\begin{aligned}
\frac{1}{\tau^{(a_0,\dots,a_4)}} &\geq \frac{1}{C_3} \cdot \frac{\mathrm{H}_{\text{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_4}\big)^{\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{2}} \cdot |a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{2}}} \\
&= \frac{1}{C_3} \cdot \frac{\mathrm{H}_{\text{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_4}\big)^{\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\varepsilon}}\,. \qquad \square
\end{aligned}$$

3.3.2. Lemma. — *Let* $(a_0 : \ldots : a_4) \in \mathbf{P}^4(\mathbb{Q})$ *be any point such that* $a_0 \neq 0, \ldots, a_4 \neq 0$. *Then,*

$$\mathrm{H}_{\text{naive}}(a_0 : \ldots : a_4) \leq \mathrm{H}_{\text{naive}}\big(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\big)^4\,.$$

Proof. First, observe that $(a_0 : \ldots : a_4) \mapsto \big(\frac{1}{a_0} : \ldots : \frac{1}{a_4}\big)$ is a well-defined map. Hence, we may assume without restriction that $a_0, \ldots, a_4 \in \mathbb{Z}$ and $\gcd(a_0, \ldots, a_4) = 1$. This yields $\mathrm{H}_{\text{naive}}(a_0 : \ldots : a_4) = \max\limits_{i=0,\dots,4} |a_i|$.

On the other hand, $(\frac{1}{a_0} : \ldots : \frac{1}{a_4}) = (a_1a_2a_3a_4 : \ldots : a_0a_1a_2a_3)$. Consequently,

$$\mathrm{H}_{\text{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\right) \leq [\max_{i=0,\ldots,4} |a_i|]^4 = \mathrm{H}_{\text{naive}}(a_0 : \ldots : a_4)^4 \, .$$

From this, the asserted inequality emerges when the roles of a_i and $\frac{1}{a_i}$ are interchanged. $\square$

3.3.3. Corollary. —— *Let* $a_0, \ldots, a_4 \in \mathbb{Z}$ *such that* $\gcd(a_0, \ldots, a_4) = 1$. *Then,*

$$|a_0 \cdot \ldots \cdot a_4| \leq \mathrm{H}_{\text{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\right)^{20}.$$

Proof. Observe that $|a_0 \cdot \ldots \cdot a_4| \leq \max_{i=0,\ldots,4} |a_i|^5 = \mathrm{H}_{\text{naive}}(a_0 : \ldots : a_4)^5$ and apply Lemma 3.3.2. $\square$

3.3.4. Theorem. —— *For each* $\varepsilon > 0$, *there exists a constant* $C(\varepsilon) > 0$ *such that, for all* $(a_0, \ldots, a_4) \in \mathbb{Z}^5$ *satisfying* $a_0 < 0$ *and* $a_1, \ldots, a_4 > 0$,

$$\frac{1}{\tau^{(a_0,\ldots,a_4)}} \geq C(\varepsilon) \cdot \mathrm{H}_{\text{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\right)^{\frac{1}{4}-\varepsilon}.$$

Proof. We may assume that $\gcd(a_0, \ldots, a_4) = 1$. Then, by Proposition 3.3.1,

$$\frac{1}{\tau^{(a_0,\ldots,a_4)}} \geq C \cdot \frac{\mathrm{H}_{\text{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\right)^{\frac{1}{4}}}{|a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{20}}} \, .$$

Corollary 3.3.3 yields $|a_0 \cdot \ldots \cdot a_4|^{\frac{\varepsilon}{20}} \leq \mathrm{H}_{\text{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_4}\right)^{\varepsilon}$. $\square$

3.3.5. Corollary (Fundamental finiteness)**.** —— *For each* $B > 0$, *there are only finitely many quartics* $V^{(a_0,\ldots,a_4)}\colon a_0x_0^4 + \ldots + a_4x_4^4 = 0$ *in* $\mathbf{P}^4_{\mathbb{Q}}$ *such that* $a_0 < 0$, $a_1, \ldots, a_4 > 0$, *and* $\tau^{(a_0,\ldots,a_4)} > B$.

Proof. This is an immediate consequence of the comparison to the naive height established in Theorem 3.3.4. $\square$

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