On the smallest point on a diagonal cubic surface

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Abstract

For diagonal cubic surfaces S, we study the behaviour of the height $\mathrm{m}(S)$ of the smallest rational point versus the Tamagawa type number $\tau(S)$ introduced by E. Peyre. We determined both quantities for a sample of 849 781 diagonal cubic surfaces. Our methods to do this are explained in some detail. The results suggest an inequality of the type $\mathrm{m}(S) < C(\varepsilon)/\tau(S)^{1+\varepsilon}$. We conclude the article by the construction of a sequence of diagonal cubic surfaces showing that the inequality $\mathrm{m}(S) < C/\tau(S)$ is wrong, in general.

1 Introduction

1.1. —— Let $S \subseteq \mathbf{P}^n_\mathbb{Q}$ be a Fano variety defined over $\mathbb{Q}$. If $S(\mathbb{Q}_\nu) \neq \emptyset$ for every $\nu \in \mathrm{Val}(\mathbb{Q})$ then it is natural to ask whether $S(\mathbb{Q}) \neq \emptyset$. When this is the case, S is said to fulfill the Hasse principle. Further, it would be desirable to have an *a-priori* upper bound for the height of the smallest $\mathbb{Q}$-rational point on S as this would allow to effectively decide whether $S(\mathbb{Q}) \neq \emptyset$ or not.

When S is a conic, Legendre's theorem on zeroes of ternary quadratic forms proves the Hasse principle and, moreover, yields an effective bound for the smallest point. For quadrics of arbitrary dimension, the same is true by an observation due to J. W. S. Cassels [Ca]. Further, there is a theorem of C. L. Siegel [Si, Satz 1] which provides a generalization to hypersurfaces defined by norm equations.

For more general Fano varieties, no theoretical upper bound is known for the smallest height of a $\mathbb{Q}$-rational point. This already applies to diagonal cubic surfaces. Furthermore, some of these fail the Hasse principle [CTKS].

In this article, we present some theoretical and experimental results concerning the smallest height of a $\mathbb{Q}$-rational point on diagonal cubic surfaces in $\mathbf{P}^3_\mathbb{Q}$.

Key words and phrases. Diagonal cubic surface, Diophantine equation, smallest solution, naive height, E. Peyre's Tamagawa-type number

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1.2. —— A conjecture, due to Yu. I. Manin, asserts that the number of $\mathbb{Q}$-rational points of anticanonical height $< B$ on a Fano variety S is asymptotically equal to $\tau B \log^{\operatorname{rk}\operatorname{Pic}(S)-1} B$, for $B \to \infty$.

In the particular case of a cubic surface, the anticanonical height is the same as the naive height. Further, the coefficient $\tau \in \mathbb{R}_{\geq 0}$ equals the Tamagawa-type number $\tau(S)$ introduced by E. Peyre in [Pe]. Hence, at least $\sim \tau(S)B$ points of height $< B$ are supposed to exist. Assume that their heights are equally distributed within $[0, B)$. Then, the height of the smallest point is $< \frac{1}{\tau(S)}$.

Therefore, one might generally expect that $\mathrm{m}(S)$, the height of the smallest $\mathbb{Q}$-rational point on S, is bounded by $\frac{C}{\tau(S)}$ for a certain absolute constant C.

1.3. —— To test this expectation, we computed the Tamagawa number and ascertained the smallest $\mathbb{Q}$-rational point for each of the cubic surfaces given by

$$ax^3 + by^3 + 2z^3 + w^3 = 0$$

for $a = 1, \ldots, 3000$ and $b = 1, \ldots, 300$.

We restricted our considerations to the case that

i) a and b are odd,

ii) there exists an odd prime p dividing a but not b such that $3 \nmid \nu_p(a)$,

or

iii) there exists an odd prime p dividing b but not a such that $3 \nmid \nu_p(b)$.

This guarantees that we are in the "First Case" according to the classification of J.-L. Colliot-Thélène and his coworkers [CTKS].

In addition, we assume that $a > b+3$. The inequality $a \geq b$ is necessary in order to avoid duplications. Further, surfaces such that $|a - b| \leq 3$ trivially have rational points of uncharacteristically small height.

The results are summarized by the diagram below.

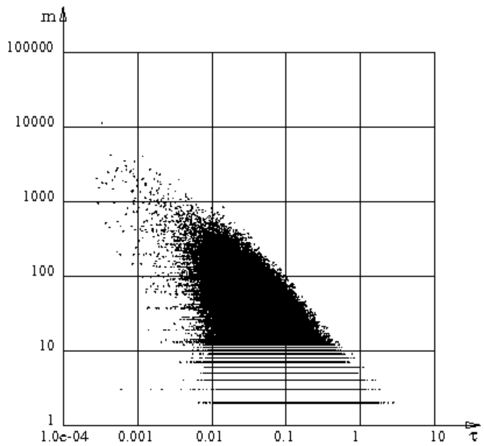

Figure 1: Height of smallest point versus Tamagawa number

Actually, the sample described consists of 849 781 surfaces. Among them, 802 891 turn out to have rational points. Each such surface is marked at its proper place $(\tau(S), \mathrm{m}(S))$ in the diagram.

It is apparent that the experiment agrees with the expectation. The slope of a line tangent to the top right of the scatter plot is indeed near (-1). However, as we will show in Theorem 8.2, the inequality $\mathrm{m}(S) < \frac{C}{\tau(S)}$ does not hold, in general. The following remains a logical possibility.

1.4. Question. —— For every $\varepsilon > 0$, does there exist a constant $C(\varepsilon)$ such that, for each cubic surface,

$$\mathrm{m}(S) < \frac{C(\varepsilon)}{\tau(S)^{1+\varepsilon}}\,?$$

1.5. Plan of the article. —— Sections 2 through 7 will be devoted to the computations which led to the diagram shown as Figure 1.

In section 2, we will recall Peyre's constant $\tau(S)$. The next four sections will discuss the factors this constant is composed of. First, in section 3, we will use the Lefschetz trace formula in order to estimate the product over all local factors $\tau_p(S)$ at the good primes p, uniformly over all cubic surfaces.

Then, sections 4 and 5 will be concerned with the L-factors. In section 4, for diagonal cubic surfaces, we will give an explicit decomposition of the Galois representation $\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{C}$ into irreducible components. As an application of this, we efficiently computed the values of the corresponding Artin L-functions at 1. Our method will be presented in detail in section 5. In section 6, we will explain our approach to deal with the factor $\tau_\infty(S)$. This requires numerical integration. Further, we will describe our computations of the Euler products over all non-Archimedean primes. Finally, in section 7, we will describe our method to find the smallest point on every surface in the sample.

Section 8 will be more theoretical in nature. We will construct a sequence $\{S^{(q)}\}_{q\in\mathbb{N}}$ of diagonal cubic surfaces such that $\mathrm{m}(S^{(q)})\tau(S^{(q)})$ is unbounded.

1.6. Notation. —— Let $\mathfrak{a} = (a_0, \ldots, a_3) \in (\mathbb{Z}\setminus\{0\})^4$ be a vector. Then, we denote by $S^{\mathfrak{a}}$ the cubic surface in $\mathbf{P}^3_{\mathbb{Q}}$ given by $a_0x_0^3 + \ldots + a_3x_3^3 = 0$.

2 Peyre's constant

2.1. —— Recall that E. Peyre's Tamagawa-type number is defined [PT, Definition 2.4] as

$$\tau(S) := \alpha(S)\cdot\beta(S)\cdot\lim_{s\to 1}(s-1)^t L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})\cdot\tau_H\big(S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}\big)$$

for $t = \mathrm{rk}\,\mathrm{Pic}(S)$. Here, $\mathrm{Pic}(S)$ denotes the Picard group of the $\mathbb{Q}$-scheme S.

The factor $\beta(S)$ is simply defined as

$$\beta(S) := \#H^1\big(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})\big).$$

Further, $\alpha(S)$ is given as follows [Pe, Définition 2.4]. Let $\Lambda_{\text{eff}}(S) \subset \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ be the cone generated by the effective divisors. Identify $\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}^t$ via a mapping induced by an isomorphism $\mathrm{Pic}(S) \xrightarrow{\cong} \mathbb{Z}^t$. Consider the dual cone $\Lambda^\vee_{\text{eff}}(S) \subset (\mathbb{R}^t)^\vee$. Then,

$$\alpha(S) := t \cdot \mathrm{vol}\, \{\, x \in \Lambda^\vee_{\text{eff}}(S) \mid \langle x, -K \rangle \leq 1 \,\}\,.$$

$L(\;\cdot\;, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})$ denotes the Artin L-function of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ which contains the trivial representation t times as a direct summand. Therefore, $L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) = \zeta(s)^t \cdot L(s, \chi_P)$ and

$$\lim_{s \to 1} (s-1)^t L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) = L(1, \chi_P)$$

where ζ denotes the Riemann zeta function and P is a representation which does not contain trivial components. [Mu, Corollary 11.5 and Corollary 11.4] show that $L(s, \chi_P)$ has neither a pole nor a zero at $s = 1$.

Finally, τ_H is the *Tamagawa measure* on the set $S(\mathbb{A}_\mathbb{Q})$ of adelic points on S and $S(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \subseteq S(\mathbb{A}_\mathbb{Q})$ denotes the subset of the adelic points which are Brauer-Manin unobstructed.

2.2. —— As S is projective, we have

$$S(\mathbb{A}_\mathbb{Q}) = \prod_{\nu \in \mathrm{Val}(\mathbb{Q})} S(\mathbb{Q}_\nu).$$

τ_H is defined to be a product measure $\tau_H := \prod_{\nu \in \mathrm{Val}(\mathbb{Q})} \tau_\nu$.

For a prime number p, the local measure τ_p is given as follows. Let $a \in S(\mathbb{Z}/p^k\mathbb{Z})$ and put $\mathfrak{U}_a^{(k)} := \{\, x \in S(\mathbb{Q}_p) \mid x \equiv a \pmod{p^k} \,\}$. Then,

$$\tau_p(\mathfrak{U}_a^{(k)}) := \det(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}) \cdot \lim_{n \to \infty} \frac{\#\{\, y \in S(\mathbb{Z}/p^n\mathbb{Z}) \mid y \equiv a \pmod{p^k} \,\}}{p^{n \dim S}}\,.$$

Here, $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}$ denotes the fixed module under the inertia group.

The measure τ_∞ is described in [Pe, Lemme 5.4.7]. In the case of a hypersurface of degree d, defined by the equation $f = 0$, this yields

$$\tau_\infty(U) = \frac{n+1-d}{2} \int\limits_{\substack{CU \\ |x_0|, \dots, |x_n| \leq 1}} \omega_{\text{Leray}}$$

for every Borel set $U \subset S(\mathbb{R})$. Here, ω_{Leray} is the *Leray measure* on the cone $CS(\mathbb{R}) \subset \mathbb{R}^{n+1}$ associated to the equation $f = 0$. It is given by the differential form $\frac{1}{|\partial f/\partial x_0|}\, dx_1 \wedge \ldots \wedge dx_n$.

2.3. Remark. —— The Leray measure differs from the "surface area" which is typically introduced for hypersurfaces in $\mathbb{R}^{n+1}$ in multivariable calculus. It is related to that measure by the formula $\omega_{\text{Leray}} = \frac{1}{\| \operatorname{grad} f \|} \, \omega_{\text{hyp}}$.

2.4. Remark. —— At least for diagonal cubic surfaces, the reciprocal $\frac{1}{\tau(S)}$ admits a fundamental finiteness property. More precisely, for each $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\frac{1}{\tau(S^{\mathfrak{a}})} \geq C(\varepsilon) \cdot \mathrm{H}_{\text{naive}}\big(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3}\big)^{\frac{1}{3} - \varepsilon}$$

for every $\mathfrak{a} \in (\mathbb{Z} \backslash \{0\})^4$. In particular, there is an estimate for $\mathrm{m}(S)$ in terms of $\tau(S)$. Details on this are given in [EJ4].

3 A technical lemma

In this section, we will give estimates for the factors $\tau_p(S(\mathbb{Q}_p))$ of Peyre's constant at the primes p of good reduction. These are, in fact, regularized versions of the numbers $\#S(\mathbb{F}_p)$. Our main tool will be the Lefschetz trace formula.

3.1. Sublemma. —— a) (Good reduction)
If $p \nmid 3a_0 \cdot \ldots \cdot a_3$ then the sequence $\big(\#S^{(a_0,\ldots,a_3)}(\mathbb{Z}/p^n\mathbb{Z})/p^{2n}\big)_{n \in \mathbb{N}}$ is constant.
b) (Bad reduction)
i) *If p divides $a_0 \cdot \ldots \cdot a_3$ but not 3 then the sequence $\big(\#S^{(a_0,\ldots,a_3)}(\mathbb{Z}/p^n\mathbb{Z})/p^{2n}\big)_{n \in \mathbb{N}}$ becomes stationary as soon as p^n does not divide any of the coefficients $a_0, \ldots, a_3$.*
ii) *If $p = 3$ then the sequence $\big(\#S^{(a_0,\ldots,a_3)}(\mathbb{Z}/p^n\mathbb{Z})/p^{2n}\big)_{n \in \mathbb{N}}$ becomes stationary as soon as 3^n does not divide any of the numbers $3a_0, \ldots, 3a_3$.*

3.2. Lemma. —— a) *For every $a_0, \ldots, a_3 \in \mathbb{Z} \backslash \{0\}$, the infinite product*

$$\prod_{p \text{ prime}} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big)$$

is absolutely convergent.
b) *There are two positive constants C_1 and C_2 such that, for all $a_0, \ldots, a_3 \in \mathbb{Z} \backslash \{0\}$,*

$$C_1 < \prod_{\substack{p \text{ prime} \\ p \nmid 3a_0 \cdot \ldots \cdot a_3}} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \ < C_2 \, .$$

Proof. For a prime p of good reduction, Sublemma 3.1 shows

$$\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) = \det\big(1 - p^{-1} \operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})\big) \cdot \frac{\#S^{(a_0,\ldots,a_3)}(\mathbb{F}_p)}{p^2} .$$

Further, for the number of points on a non-singular cubic surface over a finite field, the Lefschetz trace formula can be made completely explicit [Ma, Theorem 27.1]. It shows $\# S^{(a_0,\ldots,a_3)}(\mathbb{F}_p) = p^2 + p \cdot \mathrm{tr}\big(\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})\big) + 1$.

Denoting the eigenvalues of the Frobenius on $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})$ by $\lambda_1, \ldots, \lambda_7$, we find

$$\begin{aligned}\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) &= (1-\lambda_1 p^{-1})(1-\lambda_2 p^{-1}) \cdot \ldots \cdot (1-\lambda_7 p^{-1}) \\ &\qquad\qquad \cdot [1 + (\lambda_1 + \cdots + \lambda_7)p^{-1} + p^{-2}] \\ &= (1 - \sigma_1 p^{-1} + \sigma_2 p^{-2} \mp \ldots - \sigma_7 p^{-7})(1 + \sigma_1 p^{-1} + p^{-2}) \\ &= 1 + (1 - \sigma_1^2 + \sigma_2)p^{-2} - (\sigma_1 - \sigma_1\sigma_2 + \sigma_3)p^{-3} \pm \\ &\qquad \pm \ldots - (\sigma_5 - \sigma_1\sigma_6 + \sigma_7)p^{-7} + (\sigma_6 - \sigma_1\sigma_7)p^{-8} - \sigma_7 p^{-9}\end{aligned}$$

where σ_i denote the elementary symmetric functions in $\lambda_1, \ldots, \lambda_7$.

We know $|\lambda_i| = 1$ for all i. Estimating very roughly, we have $|\sigma_j| \leq \binom{7}{j} \leq 7^j$ and see

$$\begin{aligned}1 - 99p^{-2} - 7{\cdot}99p^{-3} - \ldots - 7^7{\cdot}99p^{-9} \leq \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \leq& \\ \leq 1 + 99p^{-2} + 7{\cdot}99p^{-3} + \ldots + 7^7 \cdot 99p^{-9}&.\end{aligned}$$

I.e.,

$$1 - 99p^{-2}\frac{1}{1-7/p} < \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) < 1 + 99p^{-2}\frac{1}{1-7/p}.$$

The infinite product over all $1 - 99p^{-2}\frac{1}{1-7/p}$ (respectively $1 + 99p^{-2}\frac{1}{1-7/p}$) is convergent.

The left hand side is positive for $p > 13$. For the small primes remaining, we need a better lower bound. For this, note that a cubic surface over a finite field $\mathbb{F}_p$ always has at least one $\mathbb{F}_p$-rational point. This yields $\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \geq (1-1/p)^7/p^2 > 0$. $\square$

3.3. Remark. —— The convergence generating factors

$$\det\big(1 - p^{-1}\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}\big)$$

are all positive. Indeed, for a pair of complex conjugate eigenvalues, we have $(1 - \lambda p^{-1})(1 - \overline{\lambda} p^{-1}) = |1 - \lambda p^{-1}|^2 > 0$ and an eigenvalue of 1 or (-1) contributes a factor $1 \pm p^{-1} > 0$. Consequently, we always have

$$\Big(1 - \frac{1}{p}\Big)^7 < \det\big(1 - p^{-1}\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}\big) < \Big(1 + \frac{1}{p}\Big)^7.$$

4 Splitting the Picard group

4.1. Motivation. —— In the case of the diagonal cubic surface $S^{(a_0,\dots,a_3)} \subset \mathbf{P}^3_{\mathbb{Q}}$, given by $a_0x_0^3 + \ldots + a_3x_3^3 = 0$ for $a_0, \dots, a_3 \in \mathbb{Z}\backslash\{0\}$, the 27 lines on $S^{(a_0,\dots,a_3)}$ may easily be written down explicitly. Indeed, for each pair $(i, j) \in (\mathbb{Z}/3\mathbb{Z})^2$, the system

$$\begin{aligned} \sqrt[3]{a_0}\, x_0 + \zeta_3^i \sqrt[3]{a_1}\, x_1 &= 0 \\ \sqrt[3]{a_2}\, x_2 + \zeta_3^j \sqrt[3]{a_3}\, x_3 &= 0 \end{aligned}$$

of equations defines a line on $S^{(a_0,\dots,a_3)}$. Decomposing the index set $\{0, \dots, 3\}$ differently into two subsets of two elements each yields all the lines. In particular, we see that the 27 lines may be defined over $L = \mathbb{Q}\big(\zeta_3, \sqrt[3]{a_1/a_0}, \sqrt[3]{a_2/a_0}, \sqrt[3]{a_3/a_0}\,\big)$.

It is classically known that the classes of the 27 lines on a smooth cubic surface generate its Picard group. Consequently, $\mathrm{Pic}(S^{(a_0,\dots,a_3)})$ is acted upon by the Galois group $\mathrm{Gal}(L/\mathbb{Q})$. The goal of this section is to study the Galois module structure on $\mathrm{Pic}(S^{(a_0,\dots,a_3)})$ more closely.

4.2. Fact. —— *Let p be a prime number and $a_0, \dots, a_3$ be integers not divisible by p. Then,*

$$\#S^{(a_0,\dots,a_3)}(\mathbb{F}_p) = \begin{cases} p^2 + \big(1 + \chi_3(a_0a_1a_2^2a_3^2) + \chi_3(a_0^2a_1^2a_2a_3) \\ \qquad + \chi_3(a_0a_1^2a_2a_3^2) + \chi_3(a_0^2a_1a_2^2a_3) \\ \qquad + \chi_3(a_0a_1^2a_2^2a_3) + \chi_3(a_0^2a_1a_2a_3^2)\big)\, p + 1 & \text{if } p \equiv 1 \pmod 3, \\ p^2 + p + 1 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Here, in the case $p \equiv 1 \pmod 3$, $\chi_3 \colon \mathbb{F}_p^* \to \mathbb{C}$ denotes a cubic residue character.

Proof. If $p \equiv 2 \pmod 3$ then every residue class modulo p has a unique cubic root. This immediately shows $\#S^{(a_0,\dots,a_3)}(\mathbb{F}_p) = p^2 + p + 1$.

For $p \equiv 1 \pmod 3$, the number of $\mathbb{F}_p$-rational points on S may be determined using Jacobi sums. The formula given follows directly from [IR, Chapter 10, Theorem 2] together with the well-known relation $g(\chi_3)g(\chi_3^2) = p$ for cubic Gauß sums. $\square$

4.3. Lemma. —— *Let $a_0, \dots, a_3 \in \mathbb{Z}\setminus\{0\}$. Then, for each prime p such that $p \nmid 3a_0 \cdot \ldots \cdot a_3$,*

$$\begin{aligned} \chi_{\mathrm{Pic}(S_{\mathbb{Q}}^{(a_0,\dots,a_3)})}(\mathrm{Frob}_p) &= \mathrm{tr}\,\big(\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}}^{(a_0,\dots,a_3)}) \otimes_{\mathbb{Z}} \mathbb{C}\big) \\ &= \begin{cases} \chi_3(a_0a_1a_2^2a_3^2) + \chi_3(a_0^2a_1^2a_2a_3) \\ + \chi_3(a_0a_1^2a_2a_3^2) + \chi_3(a_0^2a_1a_2^2a_3) \\ + \chi_3(a_0a_1^2a_2^2a_3) + \chi_3(a_0^2a_1a_2a_3^2) + 1 & \text{if } p \equiv 1 \pmod 3, \\ 1 & \text{if } p \equiv 2 \pmod 3. \end{cases} \end{aligned}$$

Proof. As we have good reduction, the trace of Frob_p on $\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ is the same as that of Frob on $\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{F}}_p}) \otimes_{\mathbb{Z}} \mathbb{C}$. Further, the Lefschetz trace formula [Ma, Theorem 27.1] shows

$$\#S^{(a_0,\ldots,a_3)}(\mathbb{F}_p) = p^2 + p \cdot \mathrm{tr}\left(\mathrm{Frob} \mid \mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{F}}_p}) \otimes_{\mathbb{Z}} \mathbb{C}\right) + 1.$$

The explicit formulas for the numbers of points given in Fact 4.2 therefore yield the assertion. $\square$

4.4. Notation. —— Let A be an integer, $K := \mathbb{Q}(\zeta_3, \sqrt[3]{A})$, $G := \mathrm{Gal}(K/\mathbb{Q})$, $H := \mathrm{Gal}(K/\mathbb{Q}(\zeta_3))$, and $\chi\colon H \to \mathbb{C}^*$ a primitive character. Then, we write $\nu^K := \mathrm{ind}_H^G(\chi)$ for the induced character and $\mathbf{V}^K$ for the corresponding G-representation.

If K is of degree three over $\mathbb{Q}(\zeta_3)$ then $\mathbf{V}^K$ is an irreducible rank two representation of $G \cong \mathfrak{S}_3$. Otherwise, $K = \mathbb{Q}(\zeta_3)$. Then, $\mathbf{V}^K \cong \mathbb{C} \oplus M$ splits into the direct sum of a trivial and a non-trivial one-dimensional representation of $H \cong \mathbb{Z}/2\mathbb{Z}$.

We will freely consider $\mathbf{V}^K$ as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation.

4.5. Lemma. —— *Let A be any integer. Then, for a prime p not dividing A, we have*

$$\nu^{\mathbb{Q}(\zeta_3,\sqrt[3]{A})}(\mathrm{Frob}_p) = \begin{cases} \chi_3(A) + \overline{\chi}_3(A) & \text{if } p \equiv 1 \pmod 3, \\ 0 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. The primitive character is unique up to conjugation by an element of G. Therefore, the induced character λ is well-defined.

The Kummer pairing allows to make a definite choice for χ as follows. Fix an embedding $\sigma\colon \mathbb{Q}(\zeta_3) \to \mathbb{C}$. Then, put $\chi(g) := \sigma\big(g(\sqrt[3]{A})/\sqrt[3]{A}\big)$.

If $p \equiv 2 \pmod 3$ then p remains prime in $\mathbb{Q}(\zeta_3)$. This means, Frob_p acts non-trivially on $\mathbb{Q}(\zeta_3)$. I.e., $\mathrm{Frob}_p \in G \backslash H$. Since H is a normal subgroup in G, the induced character vanishes on such an element.

For $p \equiv 1 \pmod 3$, we have that (p) splits in $\mathbb{Q}(\zeta_3)$. Let us write $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. The choice of $\mathfrak{p}$ is equivalent to the choice of a homomorphism $\iota\colon \langle\zeta_3\rangle \to \mathbb{F}_p^*$. The Frobenius Frob_p is determined only up to conjugation, we may choose $\mathrm{Frob}_p = \mathrm{Frob}_{\mathfrak{p}} \in H$. Then, directly by the definition of an induced character, $\nu^{\mathbb{Q}(\zeta_3,\sqrt[3]{A})}(\mathrm{Frob}_p) = \chi(\mathrm{Frob}_{\mathfrak{p}}) + \overline{\chi}(\mathrm{Frob}_{\mathfrak{p}})$. We need to show that $\chi(\mathrm{Frob}_{\mathfrak{p}}) = \chi_3(A)$ or $\chi(\mathrm{Frob}_{\mathfrak{p}}) = \overline{\chi}_3(A)$.

For this, by the choice made above, we have $\chi(\mathrm{Frob}_{\mathfrak{p}}) := \sigma\big(\mathrm{Frob}_{\mathfrak{p}}(\sqrt[3]{A})/\sqrt[3]{A}\big)$. After reduction modulo $\mathfrak{p}$, we may write $\mathrm{Frob}(\sqrt[3]{A})/\sqrt[3]{A} = (\sqrt[3]{A})^p/\sqrt[3]{A} = A^{\frac{p-1}{3}}$. Therefore, $\mathrm{Frob}_{\mathfrak{p}}(\sqrt[3]{A})/\sqrt[3]{A} = \iota^{-1}(A^{\frac{p-1}{3}})$ which shows $\chi(\mathrm{Frob}_p) = \sigma(\iota^{-1}(A^{\frac{p-1}{3}}))$. That final formula is a definition for a cubic residue character at A. $\square$

4.6. Theorem. —— *Let $a_0, \ldots, a_3 \in \mathbb{Z}\backslash\{0\}$. Then, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ splits into the direct sum*

$$\mathrm{Pic}\big(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}}\big) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C} \oplus \mathbf{V}^{K_1} \oplus \mathbf{V}^{K_2} \oplus \mathbf{V}^{K_3}$$

for $K_1 := \mathbb{Q}(\zeta_3, \sqrt[3]{a_0 a_1 a_2^2 a_3^2})$, $K_2 := \mathbb{Q}(\zeta_3, \sqrt[3]{a_0 a_1^2 a_2 a_3^2})$, *and* $K_3 := \mathbb{Q}(\zeta_3, \sqrt[3]{a_0 a_1^2 a_2^2 a_3})$.

Proof. We will show that the representations on both sides have the same character. For that, by virtue of the Cebotarev density theorem, it suffices to consider the values at the Frobenii Frob_p for $p \nmid 3a_0 \cdot \ldots \cdot a_3$.

For the representation on the left hand side, $\chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}}^{(a_0,\ldots,a_3)})}(\mathrm{Frob}_p)$ has been computed in Lemma 4.3. For the representation on the right hand side, Lemma 4.5 shows that exactly the same formula is true. $\square$

4.7. Corollary. — *Let* $a_0, \ldots, a_3 \in \mathbb{Z}\backslash\{0\}$ *be integers, consider*

$$V^{(a_0,\ldots,a_3)} := \mathrm{Pic}\big(S_{\overline{\mathbb{Q}}}^{(a_0,\ldots,a_3)}\big) \otimes_{\mathbb{Z}} \mathbb{C},$$

as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representation, and let* $\chi^{(a_0,\ldots,a_3)}$ *be the associated character. Put* $K_1 := \mathbb{Q}(\zeta_3, \sqrt[3]{a_0 a_1 a_2^2 a_3^2})$, $K_2 := \mathbb{Q}(\zeta_3, \sqrt[3]{a_0 a_1^2 a_2 a_3^2})$, *and* $K_3 := \mathbb{Q}(\zeta_3, \sqrt[3]{a_0 a_1^2 a_2^2 a_3})$. *Then, for the Artin conductor* $N_{\chi^{(a_0,\ldots,a_3)}}$ *of* $\chi^{(a_0,\ldots,a_3)}$, *we have*

$$N_{\chi^{(a_0,\ldots,a_3)}}^2 = \mathbf{D}(K_1)\,\mathbf{D}(K_2)\,\mathbf{D}(K_3)/(-27),$$

where

$$\mathbf{D}(K) := \begin{cases} \mathrm{Disc}(K/\mathbb{Q}) & \text{if } [K\colon \mathbb{Q}(\zeta_3)] = 3, \\ -27 & \text{if } K = \mathbb{Q}(\zeta_3). \end{cases}$$

Proof. We have to show $N_{\nu^K}^2 = \mathbf{D}(K)/(-3)$. Assume first that $[K : \mathbb{Q}(\zeta_3)] = 3$. Then, the conductor-discriminant formula [Ne, Chapter VII, Section (11.9)] shows $\mathrm{Disc}(K/\mathbb{Q}) = N_{\mathbb{C}} N_M N_{\nu^K}^2$ and $-3 = \mathrm{Disc}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) = N_{\mathbb{C}} N_M$ which together yield the assertion. In the opposite case, we have $\mathbf{V}^K = \mathbb{C} \oplus M$ and $N_{\nu^K} = N_{\mathbb{C}} N_M = -3$. $\square$

4.8. Lemma. — *Let* a *and* b *be integers different from zero. Then,*

$$\big|\,\mathrm{Disc}\,\big(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q}\big)\big| \leq 3^9 a^4 b^4.$$

Proof. We have, at first,

$$\begin{aligned} \big|\,\mathrm{Disc}\,\big(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q}\big)\big| &\leq \big|\,\mathrm{Disc}\,\big(\mathbb{Q}(\zeta_3)/\mathbb{Q}\big)\big|^3 \cdot \mathrm{Disc}\,\big(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q}\big)^2 \\ &= 27 \cdot \mathrm{Disc}\,\big(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q}\big)^2. \end{aligned}$$

Further, by [Mc, Chapter 2, Exercise 41], we know $\big|\,\mathrm{Disc}\,\big(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q}\big)\big| \leq 3^3 a^2 b^2$. This shows $\big|\,\mathrm{Disc}\,\big(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q}\big)\big| \leq 3^9 a^4 b^4$. $\square$

4.9. Corollary. — *Let* $a_0, \ldots, a_3 \in \mathbb{Z}\backslash\{0\}$ *be integers and* $\chi^{(a_0,\ldots,a_3)}$ *the character associated to the* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representation*

$$V^{(a_0,\ldots,a_3)} := \mathrm{Pic}\big(S_{\overline{\mathbb{Q}}}^{(a_0,\ldots,a_3)}\big) \otimes_{\mathbb{Z}} \mathbb{C}\,.$$

Then, for the Artin conductor $N_{\chi^{(a_0,\dots,a_3)}}$, we have the estimate

$$|N_{\chi^{(a_0,\dots,a_3)}}| \leq 3^{12}(a_0 \cdot \ldots \cdot a_3)^6.$$

Proof. Lemma 4.8 shows $|\,\mathbf{D}(K_i)| \leq 3^9(a_0 \cdot \ldots \cdot a_3)^4$ for $i = 1$, 2, and 3. The assertion follows immediately from this. $\square$

5 The computation of the L-function at 1

We now return to the particular diagonal cubic surfaces treated in the numerical experiment. Cf. Section 1.3 for a description of our sample.

5.1. Lemma. —— *For $a, b \in \mathbb{Z}\backslash\{0\}$, consider in $\mathbf{P}^3_\mathbb{Q}$ the diagonal cubic surface $S = S^{(a,b,2,1)}$. Assume that S fulfills condition 1.3.i), ii), or iii).*

i) *Then,* $\operatorname{rk}\operatorname{Pic}(S) = 1$.

ii) *Furthermore, there is the relation*

$$\lim_{s\to 1}(s-1)L(s, \chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})}) = L(1, \nu^{K_1})L(1, \nu^{K_2})L(1, \nu^{K_3})$$

for $K_1 = \mathbb{Q}(\zeta_3, \sqrt[3]{4ab})$, $K_2 = \mathbb{Q}(\zeta_3, \sqrt[3]{2ab^2})$, and $K_3 = \mathbb{Q}(\zeta_3, \sqrt[3]{4ab^2})$.

Proof. i) The assumptions imply that $4ab$, $2ab^2$, and $4ab^2$ are three non-cubes. In particular, the $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations $\mathbf{V}^{K_1}$, $\mathbf{V}^{K_2}$, and $\mathbf{V}^{K_3}$ are irreducible of rank two.

Further, a standard application of the Hochschild-Serre spectral sequence ensures that $\operatorname{Pic}(S) \subseteq \operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is always a subgroup of finite index. Therefore, it suffices to verify that $\operatorname{rk}\operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = 1$. For this, we note that, by Theorem 4.6, $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})^{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \otimes_\mathbb{Z} \mathbb{C}$ splits into a trivial and three irreducible $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations.

ii) Note again that $\chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})} = 1 + \nu^{K_1} + \nu^{K_2} + \nu^{K_3}$. The assertion follows directly from [Ne, Chapter VII, Theorem (10.4).ii)]. $\square$

5.2. Observations. —— i) The character ν^{K_i} is induced by a non-trivial character of the group $\operatorname{Gal}(K_i/\mathbb{Q}(\zeta_3))$ of order three. Therefore, by [Ne, Chapter VII, Theorem (10.4).iv)], we may understand $L(s, \nu^{K_i})$ as the Artin L-function over $\mathbb{Q}(\zeta_3)$ associated to that character.

ii) Further, $K_i/\mathbb{Q}(\zeta_3)$ is an abelian extension. Then, [Ne, Chapter VII, Theorem (10.6)] shows that $L(s, \nu^{K_i})$ coincides with the Hecke L-function given by the generalized Dirichlet character of order three modulo $4ab$, $2ab^2$, or $4ab^2$ over $\mathbb{Q}(\zeta_3)$. An elementary proof of this fact requires the cubic reciprocity low [IR].

5.3. Remarks. —— i) As $L(1, \nu^{K_i})$ is not given by an absolutely convergent series, we cannot evaluate it directly.

ii) One could apply the analytic class number formula to compute $L(1, \nu^{K_i})$. This approach is, however, not practical for half a million L-functions.

5.4. Notation. —— From now on, we will denote the generalized Dirichlet character of order three modulo A by ν_A and its conductor by $m \in \mathbb{Z}[\zeta_3]$. Further, we write $N\colon \mathbb{Q}(\zeta_3) \to \mathbb{Q}$ for the norm map.

5.5. —— We complete the L-function by putting

$$\Lambda(s, \nu_A) := (-3N(m))^{s/2} \frac{2}{(2\pi)^s} \Gamma(s) L(s, \nu_A)\,.$$

The completed L-function is connected with a theta function via a Mellin transform. One has

$$\Lambda(s, \nu_A) = \int\limits_0^\infty f(t)\, t^{s/2}\, \frac{dt}{t}$$

where f is the function defined by

$$f(t) := \frac{1}{6} \sum_{a \in \mathbb{Z}[\zeta_3]} \nu_A(a) e^{-\frac{2\pi}{|3m|} N(a) \sqrt{t}}$$

for $t > 0$. The connection to the Hecke theta-function associated to $\mathbb{Z}[\zeta_3]$ and ν_A is given by

$$f(t) := \frac{1}{6}\, \theta(i\sqrt{t}, \nu_A)\,.$$

Inspecting the convergence properties of the series, we see that it converges very rapidly for $t \gg 0$ while convergence is arbitrarily slow for t close to zero.

The functional equation

$$\theta(-1/z, \nu_A) = \frac{z}{i}\, \theta(z, \overline{\nu}_A)$$

interchanges the ranges of good and bad convergence. Hence, this equation should be used to compute $f(t)$ for t small.

To be more precise, we split the half line $[0, \infty)$ into two parts and write

$$\Lambda(s, \nu_A) = \int\limits_0^u f(t)\, t^{s/2}\, \frac{dt}{t} + \int\limits_u^\infty f(t)\, t^{s/2}\, \frac{dt}{t}\,.$$

Applying the functional equation of the Hecke theta function to the first summand yields

$$\Lambda(s,\nu_A) = \frac{1}{6}\sum_{a\in\mathbb{Z}[\zeta_3]} 2\nu_A(a)\Bigg(\left[\frac{|3m|}{2\pi N(a)}\right]^{1-s}\int\limits_{\frac{2\pi N(a)}{|3m|}\frac{1}{\sqrt{u}}}^{\infty} e^{-x}x^{-s}dx + \left[\frac{|3m|}{2\pi N(a)}\right]^{s}\int\limits_{\frac{2\pi N(a)}{|3m|}\sqrt{u}}^{\infty} e^{-x}x^{s-1}dx\Bigg) \qquad (1)$$

for each $u > 0$. This is an absolutely convergent infinite series.

5.6. Remark. — The idea to evaluate an L-function at an arbitrary point $s \in \mathbb{C}$ using a series analogous to (1) goes back, at least, to A. F. Lavrik [La]. Descriptions of similar methods may also be found in [St], [Co, Section 10.3], and [Do].

5.7. Remark. — The relation of $\Lambda(s,\nu_A)$ to a theta function is a particular case of the very general [Ne, Chapter VII, Theorem (8.3)]. In comparison with the general case, many simplifications do occur, mainly because $\mathbb{Q}(\zeta_3)$ is an imaginary quadratic number field of class number 1. Note that $\mathbb{Q}(\zeta_3)$ has discriminant (-3) and precisely six units.

5.8. Remark. — In more generality, the functional equation of a Hecke theta function is of the form

$$\theta(-1/z,\nu) = \frac{\tau(\nu)}{\sqrt{N(m)}}\frac{z}{i}\,\theta(z,\overline{\nu})\,.$$

Here, $\tau(\nu)$ is the Gauß sum associated to the character ν [Ne, Chapter VII, Definition (7.4)].

In our case, it is immediate from the definition that $\tau(\nu_A)$ is real. Further, [Ne, Chapter VII, Theorem (7.7)] shows that $|\tau(\nu_A)| = \sqrt{N(m)}$ such that the coefficient of the functional equation is ± 1.

Actually, the sign is always positive. Indeed, a direct calculation shows

$$\zeta_{\mathbb{Q}(\sqrt[3]{A})}(s) = L(s,\nu_A)\zeta(s)\,.$$

Further, in the functional equation of the Dedekind zeta function, the sign is always positive [Ne, Chapter VII, Corollary (5.10)].

5.9. Remarks. — i) The convergence of the series (1) is optimal when u is close to 1. Calculations using different values of u may be used for checks [Do].

ii) The number of summands required for a numerical approximation is about $C|m|$. The constant C depends on the precision required.

5.10. Remark. —— There are a number of obvious ideas to optimize the computations.

i) The summand for a depends only on the ideal (a). Hence, the summands arise in groups of six. We calculate only once for each group.

ii) Both integrals depend only on $N(a)$ and $|m|$. Thus, we evaluate them only once for each pair $(N(a), |m|)$.

iii) The computation of the generalized Dirichlet characters ν_A is sped up using their multiplicativity in A. For a concrete value $a \in \mathbb{Z}[\zeta_3]$, we first use Euler's criterion to compute $\nu_p(a)$ for all prime numbers p less than 3000. Having tabulated these values, the calculation of all the characters ν_A at a is done rapidly.

Since we are interested in the evaluation of many L-functions at $s = 1$, some more possibilities for optimization do arise.

iv) Actually, the first integral is the integral exponential function and the second one is just an exponential function. The numerical evaluation of the integral exponential function could be done by a combination of the power series expansion with a continued fraction expansion [PFTV].

However, there is another method which is better. The arguments of the integral exponential function we meet lie in a rather small range. This range was split up into even smaller intervals. On each interval, we used a polynomial approximation.

5.11. —— We organized the computations as follows. In a first step, we enumerated all the radicands A for which $L(1, \nu_A)$ had to be computed. We sorted the list and eliminated all repetitions. In addition, for each radicand, we stored its prime decomposition for later use. The resulting list consisted of 557 270 radicands. Only 214 285 different conductors occurred.

Then, we evaluated $L(1, \nu_A)$ for all the radicands A occurring. We used formula (1) for $u = 1$ and $u = 1.2$. To evaluate the series numerically, we worked with 64-bit hardware floats and used backward summation. The differences between the two results were always negligible. The whole computation of the values of L took around four days on a 2.2 GHz Opteron processor.

In Table 1 below, we present a few of the values computed. The first two lines represent the absolutely largest and the absolutely smallest value of L, we found. The three other lines all correspond to conductor 5 380 206 which is the largest conductor appearing in our list. For this maximal conductor, we worked in the summation with all $a \in \mathbb{Z}[\zeta_3]$ such that $N(a) \leq 38\,276\,797$. For smaller conductors, according to Remark 5.9.ii), less summands were used.

Radicand A	$L(1,\nu_A)$ using $u=1$	$L(1,\nu_A)$ using $u=1.2$	$\ldots$ using class number formula
166 249	4.419 173 379 082 995	4.419 173 379 082 997	4.419 173 379 082 996 519 114 130
102 044 100	0.596 117 703 616 924	0.596 117 703 616 918	0.596 117 703 616 923 884 079 232
3 586 804	0.888 154 374 767 605	0.888 154 374 767 607	0.888 154 374 767 604 963 111 775
536 227 198	0.946 251 759 020 570	0.946 251 759 020 576	0.946 251 759 020 569 971 686 643
1 072 454 396	1.437 503 627 427 445	1.437 503 627 427 447	1.437 503 627 427 445 188 453 952

Table 1: Some values of the L-functions at $s=1$

6 Computing the Tamagawa numbers

6.1. Lemma. —— *For $a,b\in\mathbb{Z}\backslash\{0\}$, consider in $\mathbf{P}^3_{\mathbb{Q}}$ the diagonal cubic surface $S=S^{(a,b,2,1)}$. Assume that S fulfills condition 1.3.i), ii), or iii).*

i) *Then, $\alpha(S)=1$ and $\beta(S)=3$.*

ii) *Furthermore, one has precisely*

$$\tau_H\big(S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}\big)=\frac{1}{3}\,\tau_H\big(S(\mathbb{A}_{\mathbb{Q}})\big)\,.$$

Proof. i) On a cubic surface, the self-intersection number of the canonical divisor K is equal to 3 which is square-free. Therefore, $\operatorname{rk}\operatorname{Pic}(S)=1$ immediately implies that $\operatorname{Pic}(S)=\langle K\rangle$. This is enough to ensure $\alpha(S)=1$.

$\beta(S)$ can be computed using the method described in Yu. I. Manin's book [Ma, Proposition 31.3]. Let $F\subset\operatorname{Div}(S)$ the free abelian group over the 27 lines, $F_0\subset F$ the subset of principal divisors, and $N\colon F\to F$ the norm map under the operation of the Galois group G on F. Then, Yu. I. Manin states that

$$H^1\big(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\operatorname{Pic}(S_{\overline{\mathbb{Q}}})\big)\cong\operatorname{Hom}\big((NF\cap F_0)/NF_0,\mathbb{Q}/\mathbb{Z}\big).$$

We have a group G of order 6, 18, or 54. If $\#G=54$ then G decomposes the 27 lines into three orbits of nine lines each. In this case, an easy calculation shows that

$$\operatorname{Hom}\big((NF\cap F_0)/NF_0,\mathbb{Q}/\mathbb{Z}\big)\cong\mathbb{Z}/3\mathbb{Z}\,.$$

The smaller groups might lead to the decomposition types $[3,6,9,9]$ or $[3,3,3,6,6,6]$. A calculation in `GAP` shows $\operatorname{Hom}\big((NF\cap F_0)/NF_0,\mathbb{Q}/\mathbb{Z}\big)\cong\mathbb{Z}/3\mathbb{Z}$ in these cases, too.

ii) This is known by the work of J.-L. Colliot-Thélène and his coworkers [CTKS, Proof of Proposition 2]. □

6.2. Corollary. —— *For $a,b\in\mathbb{Z}\backslash\{0\}$, consider the diagonal cubic surface $S=S^{(a,b,2,1)}$. Assume that S fulfills condition 1.3.i), ii), or iii).*

Then, for E. Peyre's Tamagawa-type number, one has

$$\tau(S)=\lim_{s\to1}\,(s-1)L(s,\chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})})\cdot\prod_{p\,\mathrm{prime}}\tau_p\big(S(\mathbb{Q}_p)\big)\cdot\tau_\infty\big(S(\mathbb{R})\big)\,.$$

6.3. The factor at the infinite place. —— Since S is a diagonal cubic surface, the projection from the cone $CS(\mathbb{R})$ to the (y, z, w)-space is one-to-one. Therefore,

$$\tau_\infty\big(S(\mathbb{R})\big) = \frac{1}{6\sqrt[3]{a}} \iiint\limits_{\substack{(y,z,w)\in[-1,1]^3 \\ |x(y,z,w)|\leq 1}} \frac{1}{(by^3+2z^3+w^3)^{2/3}}\, dy\, dz\, dw\,.$$

Further, we have

$$|x(y,z,w)| = \sqrt[3]{\frac{|by^3+2z^3+w^3|}{a}} \leq \sqrt[3]{\frac{b|y|^3+2|z|^3+|w|^3}{a}}\,.$$

Since $|y| \leq 1$, $|z| \leq 1$, $|w| \leq 1$, and $a > b+3$, it turns out that the condition $|x(y,z,w)| \leq 1$ is actually empty. The integral in the formula for $\tau_\infty(S(\mathbb{R}))$ depends only on b. We are left with just 300 different integrals.

A linear substitution leads to 300 integrals of the same function on an increasing sequence of integration domains. Hence, this sequence can be computed incrementally. Doing this, the first integrals (for $b = 1$, 2, and 3) are critical since the integrand is singular in the domain of integration. Thus, they should not be computed naively. We evaluated them using the approach described in [EJ3].

6.4. Computation of the Euler product. —— By Lemma 3.2, the Euler product is absolutely convergent and, for the relative error, we have the estimate

$$\Bigg|\prod_{\substack{p\geq N \\ p\equiv 1 \pmod 3}} \left(1 \pm 99p^{-2}\frac{1}{1-7/p}\right) \cdot \prod_{\substack{p\geq N \\ p\equiv 2 \pmod 3}} \left(1-\frac{1}{p^3}\right) - 1\Bigg| \leq \frac{99/2}{N\log N} + O\!\left(\frac{1}{N\log^2 N}\right)$$

if all bad primes are below N. In particular, the approximation by the finite product over all primes up to 10^6 leads to a relative error of less than $4\cdot 10^{-6}$.

The computation of the Euler products was done according to their definition. An optimization which is worth a mention is that we ran the outer loop over the prime numbers and the inner loops over a and b. The whole computation of the Euler products took a quarter of an hour.

7 Searching for the smallest solution

7.1. —— We will now explain how we generated the data for Figure 1. In addition to computing the Tamagawa type numbers, we had to find the points of smallest height. I.e., the smallest solutions of the equations

$$ax^3 + by^3 + 2z^3 + w^3 = 0$$

where $a = 1, \ldots, 3000$ and $b = 1, \ldots, 300$ fulfill the conditions formulated in 1.3.

We applied a modification of the strategy due to M. Vallino [CTKS, p. 79/80]. The algorithms used are slight modifications of [EJ1, Algorithm 27]. We dealt with the decoupling $ax^3 + 2z^3 = -by^3 - w^3$.

7.2. Description of the method. —— i) In a first stage, we worked with a search bound of 100 and ran the algorithm simultaneously on all the 900 000 equations for $a = 1, \ldots, 3000$ and $b = 1, \ldots, 300$. For exactly 69 074 of these equations, no solution was found. Among them, 67 787 fulfilled the congruence conditions formulated in 1.3. In this list, there were only a few duplications. 65 314 of the equations obeyed the limitation $a > b + 3$, too.

For these, we ran a test for p-adic solvability. It turned out that only 18 424 of the remaining 65 314 equations were solvable in $\mathbb{Q}_p$ for every prime p.

ii) We executed the second stage with the corresponding pairs. They were read from a file. The searching algorithm was run separately for each equation. We worked with search bounds of 200, 400, and 800 and stopped when a solution was found.

Only 113 equations remained unsolved by that stage.

iii) In most of these cases, there was a prime p such that 2 is a cubic non-residue modulo p dividing both a and b. This enforces that both z and w must be divisible by p. We used these strong divisibility conditions when working with search bounds of 4000 and 20 000.

7.3. Remark. —— Actually, in the last stage, there were only three equations remaining for which no solution had been found with a search bound of $B = 4000$. They are represented by the pairs $(a, b) = (2321, 211)$, $(2331, 222)$, and $(a, b) = (2641, 278)$. The corresponding smallest solutions are $(-125, -884, 4220, -211)$, $(-389, 64, 4033, 1813)$, and $(-1023, -458, 11\,259, -695)$, respectively.

7.4. Remark. —— Altogether, there are exactly 849 781 cubic surfaces fulfilling the congruence conditions and limitations given in 1.3. It turned out that 46 890 of them are p-adically unsolvable for some prime $p \equiv 1 \pmod 3$. Each of the remaining cubic surfaces admits a $\mathbb{Q}$-rational point.

Thus, there are no counterexamples to the Hasse principle in our sample. This confirms to a conjecture of J.-L. Colliot-Thélène, cf. [CTS, Conjecture C].

7.5. Remark. —— It should be noticed that [EJ1, Algorithm 27] itself would not work very well on this problem, at least not on the first stage. The point is that there are some numbers which appear as values of the expressions $ax^3 + 2z^3$ and $(-by^3 - w^3)$, many times. Whether we chose one side or the other, we had a hash function which was quite far from being uniform.

Our idea to overcome this difficulty was to replace hashing by sorting. We generate sorted lists of all values taken by the expressions on the two sides. We look for coincidences by a procedure similar to a step of `Mergesort`.

8 A negative result

In this section, we will show that the inequality $\mathrm{m}(S) < \frac{C}{\tau(S)}$ is wrong, in general. We will construct a sequence $\{S^{(q)}\}_{q \in \mathbb{N}}$ of diagonal cubic surfaces such that $\mathrm{m}(S^{(q)})\tau(S^{(q)})$ is unbounded.

8.1. —— For an integer $q \neq 0$, denote by $S^{(q)} \subset \mathbf{P}^3_{\mathbb{Q}}$ the cubic surface given by $qx^3 + 4y^3 + 2z^3 + w^3 = 0$ and let

$$\mathrm{m}(S^{(q)}) := \min \{ \mathrm{H}_{\mathrm{naive}}(x : y : z : w) \mid (x : y : z : w) \in S^{(q)}(\mathbb{Q}) \}$$

be the smallest height of a $\mathbb{Q}$-rational point on $S^{(q)}$. We want to compare $\mathrm{m}(S^{(q)})$ with the Tamagawa type number $\tau^{(q)} := \tau(S^{(q)})$.

8.2. Theorem. —— *Assume the Generalized Riemann Hypothesis. Then, there is no constant C such that*

$$\mathrm{m}(S^{(q)}) < \frac{C}{\tau^{(q)}}$$

for all $q \in \mathbb{Z} \backslash \{0\}$.

Proof. We will construct a sequence $\{q_i\}_{i \in \mathbb{N}}$ of primes such that $q_i \equiv 1 \pmod{72}$ and $\mathrm{m}(S^{(q_i)})\, \tau^{(q_i)} \to \infty$ for $i \to \infty$. The proof will consist of several steps.

First step. It is sufficient to verify that

$$\mathrm{m}(S^{(q_i)}) \cdot \lim_{s \to 1} (s-1) L(s, \chi_{\mathrm{Pic}(S^{(q_i)}_{\overline{\mathbb{Q}}})}) \cdot \prod_{p\,\mathrm{prime}} \tau_p\big(S^{(q_i)}(\mathbb{Q}_p)\big) \cdot \tau_\infty\big(S^{(q_i)}(\mathbb{R})\big) \to \infty.$$

Since $q_i \equiv 1 \pmod{72}$, the prime q_i is odd. Hence, the surface S fulfills condition 1.3.ii). The claim follows directly from Corollary 6.2.

Second step. For the height of the smallest point, we have $\mathrm{m}(S^{(q)}) \geq \sqrt[3]{\frac{q}{7}}$.

There are no rational solutions of the equation $4y^3+2z^3+w^3 = 0$ as this is impossible, 2-adically. $|x| \geq 1$ yields $|4y^3 + 2z^3 + w^3| \geq q$ and $\max\{|y|, |z|, |w|\} \geq \sqrt[3]{\frac{q}{7}}$.

Third step. For $|q| \geq 7$, one has $\tau_\infty\big(S^{(q)}(\mathbb{R})\big) = \frac{1}{\sqrt[3]{|q|}} I$ where I is independent of q. This was shown in section 6, above.

Fourth step. There is a positive constant C such that $\prod\limits_{p\,\text{prime}} \tau_p\big(S^{(q)}(\mathbb{Q}_p)\big) > C$ for every prime $q \equiv 1 \pmod{72}$.
By Lemma 3.2, we have $C_1 > 0$ such that

$$\prod_{\substack{p\,\text{prime}\\ p\neq 2,3,q}} \tau_p\big(S^{(q)}(\mathbb{Q}_p)\big) > C_1.$$

It, therefore, remains to give lower bounds for the factors $\tau_2\big(S^{(q)}(\mathbb{Q}_2)\big)$, $\tau_3\big(S^{(q)}(\mathbb{Q}_3)\big)$, and $\tau_q\big(S^{(q)}(\mathbb{Q}_q)\big)$.

As $2 \nmid q$, by virtue of Sublemma 3.1 we have, $\tau_2\big(S^{(q)}(\mathbb{Q}_2)\big) = \frac{1}{2^7} \cdot \frac{\#S^{(q)}(\mathbb{Z}/8\mathbb{Z})}{64}$. Further, $\#S^{(q)}(\mathbb{Z}/8\mathbb{Z}) \geq 1$ since $q \equiv 1 \pmod 8$ implies $(1:0:0:(-1)) \in S^{(q)}(\mathbb{Z}/8\mathbb{Z})$.

Similarly, $\tau_3\big(S^{(q)}(\mathbb{Q}_3)\big) = (\frac{2}{3})^7 \cdot \frac{\#S^{(q)}(\mathbb{Z}/9\mathbb{Z})}{81}$. Again, $q \equiv 1 \pmod 9$ makes sure that $(1:0:0:(-1)) \in S^{(q)}(\mathbb{Z}/9\mathbb{Z})$ and $\#S^{(q)}(\mathbb{Z}/9\mathbb{Z}) \geq 1$.

For the prime q, we argue a bit differently. First,

$$\det\big(1 - q^{-1}\operatorname{Frob}_p \mid \operatorname{Pic}(S^{(q)}_{\overline{\mathbb{Q}}})^{I_q}\big) \geq (1-1/q)^7 \geq (72/73)^7.$$

Furthermore, the reduction of $S^{(q)}$ modulo q is the cone over the elliptic curve given by $4y^3 + 2z^3 + w^3 = 0$. Therefore, on $S^{(q)}$ there are at least $(q - 2\sqrt{q} + 1)(q-1)$ smooth points defined over $\mathbb{F}_q$. As Hensel's lemma may be applied to them, we get

$$\lim_{n\to\infty} \frac{\#S^{(q)}(\mathbb{Z}/q^n\mathbb{Z})}{q^{2n}} \geq \frac{(q-2\sqrt{q}+1)(q-1)}{q^2} > \Big(1 - \frac{2}{\sqrt{q}}\Big)\Big(1 - \frac{1}{q}\Big) \geq \frac{72}{73}\Big(1 - \frac{2}{\sqrt{73}}\Big).$$

Fifth step. There is a sequence $\{q_i\}_{i\in\mathbb{N}}$ of primes such that $q_i \equiv 1 \pmod{72}$ and $[\lim_{s\to 1}(s-1)L(s, \chi_{\operatorname{Pic}(S^{(q_i)}_{\overline{\mathbb{Q}}})})] \to \infty$ for $i \to \infty$.
Since $\operatorname{rk}\operatorname{Pic}(S^{(q_i)}) = 1$, the representation $\operatorname{Pic}\big(S^{(q_i)}_{\overline{\mathbb{Q}}}\big) \otimes_{\mathbb{Z}} \mathbb{C}$ contains exactly one trivial summand. Hence,

$$L\big(s, \chi_{\operatorname{Pic}(S^{(q_i)}_{\overline{\mathbb{Q}}})}\big) = \zeta(s) \cdot L(s, \chi_0^{(q_i)})$$

for $\chi_0^{(q_i)}$ the character of a representation $V_0^{(q_i)}$ not containing trivial components. Our goal is, therefore, to show $L(1, \chi_0^{(q_i)}) \to \infty$ for $i \to \infty$.

For each $i \in \mathbb{N}$, denote by P_i the i-th prime number p such that $p \equiv 1 \pmod 3$. We define q_i to be the smallest prime such that

$$q_i \equiv 1 \pmod{72P_1 \cdot \ldots \cdot P_i}.$$

From this, we clearly have that $q_i > 72P_1 \cdot \ldots \cdot P_i \to \infty$ for $i \to \infty$.

Furthermore, by Chebyshev, we know that

$$72P_1 \cdot \ldots \cdot P_i \leq 72e^{\theta(P_i)} < 72e^{(2\log 2)P_i}.$$

Hence, Linnik's Theorem in the version of R. Heath-Brown [HB] shows

$$q_i \leq C_1 \cdot (72e^{(2\log 2)P_i})^{5.5} = C_2 e^{(11\log 2)P_i}$$

for certain constants C_1 and C_2.

Corollary 4.9 gives us an estimate for the Artin conductor of the character $\chi^{(q_i,4,2,1)}$ which is the same as that of $\chi_0^{(q_i)}$. We see

$$N_{\chi_0^{(q_i)}} \leq 3^{12}(a_0 \cdot \ldots \cdot a_3)^6 = 3^{12} 8^6 q_i^6 \leq C_3\, e^{(66\log 2)P_i}$$

for another constant C_3. Consequently,

$$\log N_{\chi_0^{(q_i)}} \leq (66\log 2)P_i + \log C_3.$$

We observe that $(\log N_{\chi_0^{(q_i)}})^{1/2} \leq P_i$ for i sufficiently large. We assume from now on that this inequality is fulfilled.

Recall from Theorem 4.6 that $V_0^{(q_i)}$ is actually the direct sum of representations which are induced from one-dimensional characters. By consequence, it is known that the Artin L-function $L(\,\cdot\,, \chi_0^{(q_i)})$ is entire. Since we also assume the Generalized Riemann Hypothesis, we may apply the estimate of W. Duke [Du, Proposition 5]. It shows

$$\log L(1, \chi_0^{(q_i)}) = \sum_{p < (\log N_{\chi_0^{(q_i)}})^{1/2}} \chi_0^{(q_i)}(\mathrm{Frob}_p)\, p^{-1} + O(1).$$

Here,

$$\chi_0^{(q_i)}(\mathrm{Frob}_p) = \chi_{\mathrm{Pic}(S^{(q_i)}_{\overline{\mathbb{Q}}})}(\mathrm{Frob}_p) - 1.$$

For $p \equiv 2 \pmod 3$, this yields $\chi_0^{(q_i)}(\mathrm{Frob}_p) = 0$. On the other hand, for $p \equiv 1 \pmod 3$, we have, by virtue of Lemma 4.3,

$$\begin{aligned}
\chi_0^{(q_i)}(\mathrm{Frob}_p) &= \chi_3(16q) + \chi_3(32q^2) + \chi_3(32q) + \chi_3(16q^2) + \chi_3(64q) + \chi_3(8q^2) \\
&= \chi_3(q) + \chi_3(2q) + \chi_3(4q) + \chi_3(q^2) + \chi_3(2q^2) + \chi_3(4q^2) \\
&= \big(1 + \chi_3(2) + \chi_3(4)\big)\big(\chi_3(q) + \chi_3(q^2)\big)\,.
\end{aligned}$$

This may be written down in an explicit form as

$$\chi_0^{(q_i)}(\mathrm{Frob}_p) = \begin{cases} 0 & \text{if } p \equiv 2 \pmod 3\,, \\ 0 & \text{if } p \equiv 1 \pmod 3 \text{ and } \left(\frac{2}{p}\right)_3 \neq 1\,, \\ 6 & \text{if } p \equiv 1 \pmod 3,\ \left(\frac{2}{p}\right)_3 = 1, \text{ and } \left(\frac{q_i}{p}\right)_3 = 1\,, \\ -3 & \text{if } p \equiv 1 \pmod 3,\ \left(\frac{2}{p}\right)_3 = 1, \text{ and } \left(\frac{q_i}{p}\right)_3 \neq 1\,. \end{cases}$$

Modulo all primes $p \equiv 1 \pmod 3$, $p < (\log N_{\chi_0^{(q_i)}})^{1/2} \leq P_i$, the number q_i was constructed to be a cubic residue. Further,

$$\chi_0^{(q_i)}(\mathrm{Frob}_3)\, 3^{-1}$$

is of absolute value at most 2. Thus,

$$\log L(1,\chi_0^{(q_i)}) = 6 \sum_{\substack{p\equiv 1 \pmod 3 \\ \left(\frac{2}{p}\right)_3 = 1 \\ p < (\log N_{\chi_0^{(q_i)}})^{1/2}}} \frac{1}{p} + O(1)\,.$$

By the Cebotarev density theorem, the set of all primes such that $p \equiv 1 \pmod 3$ and $\left(\frac{2}{p}\right)_3 = 1$ is of density $\frac{1}{6}$. We, therefore, have $\log L(1,\chi_0^{(q_i)}) \to \infty$ as soon as we may guarantee $N_{\chi_0^{(q_i)}} \to \infty$.

Since only a trivial character is missing, we have, by Corollary 4.7,

$$N_{\chi_0^{(q_i)}} = N_{\chi_{\mathrm{Pic}(S^{(q_i)}_{\overline{\mathbb{Q}}})}} = |\,\mathbf{D}(K_1)\,\mathbf{D}(K_2)\,\mathbf{D}(K_3)/27\,|^{1/2} \geq |\,\mathbf{D}(K_3)/27\,|^{1/2}$$

where, by choice of the coefficients, $K_3 = \mathbb{Q}(\zeta_3, \sqrt[3]{64q_i}) = \mathbb{Q}(\zeta_3, \sqrt[3]{q_i})$. There is the estimate

$$\begin{aligned} |\,\mathbf{D}(K_3)| &= \big|\operatorname{Disc}\big(\mathbb{Q}(\zeta_3, \sqrt[3]{q_i})/\mathbb{Q}\big)\big| \\ &= \operatorname{Disc}\big(\mathbb{Q}(\sqrt[3]{q_i})/\mathbb{Q}\big)^2 \cdot \big|N\big(\operatorname{Disc}\big(\mathbb{Q}(\zeta_3, \sqrt[3]{q_i})/\mathbb{Q}(\sqrt[3]{q_i})\big)\big)\big| \\ &\geq \operatorname{Disc}\big(\mathbb{Q}(\sqrt[3]{q_i})/\mathbb{Q}\big)^2. \end{aligned}$$

According to [Mc, Chapter 2, Exercise 41], we know $\big|\operatorname{Disc}\big(\mathbb{Q}(\sqrt[3]{q_i})/\mathbb{Q}\big)\big| \geq 3q_i^2$. □

8.3. Remark. —— Note that the estimate for $L(1,\chi_0^{(q_i)})$ is the only point where we used the Generalized Riemann Hypothesis.

Observe, in particular, that we work with a version of Linnik's Theorem which is true, unconditionally. Here, the Generalized Riemann Hypothesis would lead to the much better exponent $2+\varepsilon$. This improvement is, however, not necessary for our particular application.

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