*Journal de Théorie des Nombres*
*de Bordeaux* 00 (XXXX), 000–000

# More cubic surfaces violating the Hasse principle

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Résumé. Nous généralisons la construction due à L. J. Mordell des surfaces cubiques pour lesquelles la principe de Hasse est fausse.

Abstract. We generalize L. J. Mordell's construction of cubic surfaces for which the Hasse principle fails.

## 1. Introduction and main result

Sir Peter Swinnerton-Dyer [4] was the first to construct a cubic surface over $\mathbb{Q}$ for which the Hasse principle provably fails. Swinnerton-Dyer's construction had soon been generalized by L. J. Mordell [3] who found two series of such examples. The starting points of Mordell's construction are the cubic number fields contained in $\mathbb{Q}(\zeta_p)$ for $p = 7$ and $p = 13$, respectively.

In this note, we will show that Mordell's construction may be generalized to an arbitrary prime $p \equiv 1 \pmod 3$.

**Notation.** i) We denote by $K/\mathbb{Q}$ the unique cubic field extension contained in the cyclotomic extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$.
ii) We fix the explicit generator $\theta \in K$ given by $\theta := \mathrm{tr}_{\mathbb{Q}(\zeta_p)/K}(\zeta_p - 1)$. More concretely, $\theta = -n + \sum\limits_{i \in (\mathbb{F}_p^*)^3} \zeta_p^i$ for $n := \frac{p-1}{3}$.

**Theorem 1.1.** *Consider the cubic surface $X \subset \mathbf{P}^3_{\mathbb{Q}}$, given by*

$$T_3(a_1T_0 + d_1T_3)(a_2T_0 + d_2T_3) = \prod_{i=1}^{3} \left(T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2\right).$$

*Here, $a_1, a_2, d_1, d_2$ are integers and $\theta^{(i)}$ are the images of $\theta$ under* $\mathrm{Gal}(K/\mathbb{Q})$.
i) *Then, the reduction $X_p$ of $X$ at $p$ is given by*

$$T_3(a_1T_0 + d_1T_3)(a_2T_0 + d_2T_3) = T_0^3\,.$$

*Over the algebraic closure, $X_p$ is the union of three planes. These are given by*

$$T_3/T_0 = s_1, \quad T_3/T_0 = s_2, \quad T_3/T_0 = s_3$$

*for $s_i$ the zeroes of $T(a_1+d_1T)(a_2+d_2T)-1$, considered as a polynomial over $\mathbb{F}_p$.*

ii) *Suppose $p\nmid d_1d_2$ and $\gcd(d_1,d_2)=1$. Then, for every*

$$(t_0:t_1:t_2:t_3)\in X(\mathbb{Q})$$

*the term $s:=(t_3/t_0 \bmod p)$ admits the property that*

$$\frac{a_1+d_1s}{s}$$

*is a cube in $\mathbb{F}_p^*$.*

*In particular, if $(a_1+d_1s_i)/s_i\in\mathbb{F}_p^*$ is a non-cube for every $i$ such that $s_i\in\mathbb{F}_p$ then $X(\mathbb{Q})=\emptyset$.*

iii) *Assume that $p\nmid d_1d_2$ and that $\gcd(a_1,d_1)$ and $\gcd(a_2,d_2)$ contain only prime factors that completely split in $K$. Suppose further that $T(a_1+d_1T)(a_2+d_2T)-1\in\mathbb{F}_p[T]$ has at least one simple zero in $\mathbb{F}_p$. Then, $X(\mathbb{A}_\mathbb{Q})\neq\emptyset$.*

**Remarks.** i) $K/\mathbb{Q}$ is an abelian cubic field extension. It is totally ramified at $p$ and unramified at all other primes. A prime $q\neq p$ is split in $K$ if and only if $q$ is a cube modulo $p$.

ii) We will write $\mathfrak{p}$ for the prime ideal in $K$ lying above $(p)$. Note that $\mathfrak{p}=(\theta)$ by virtue of our definition of $\theta$.

iii) We have $\prod_{i=1}^3\big(T_0+\theta^{(i)}T_1+(\theta^{(i)})^2T_2\big)=N_{K/\mathbb{Q}}(T_0+\theta T_1+\theta^2T_2)\,.$

**Remark.** For $p=7$ and $13$, we recover exactly the result of L. J. Mordell. The original example of Sir Peter Swinnerton-Dyer reappears for $p=7$, $a_1=d_1=a_2=1$, and $d_2=2$.

## 2. The proofs

**Observations 2.1.** i) *For $s$ any solution of $T(a_1+d_1T)(a_2+d_2T)-1=0$, the expression $(a_1+d_1s)/s$ is well defined and non-zero.*

ii) *No $\mathbb{Q}_p$-valued point on $X$ reduces to the triple line "$T_0=T_3=0$".*

iii) *For every $(t_0:t_1:t_2:t_3)\in X(\mathbb{Q}_p)$, the fraction $(a_1t_0+d_1t_3)/t_3$ is a $p$-adic unit.*

**Proof.** i) By assumption, we have $s\neq0$ and $a_1+d_1s\neq0$.

ii) Suppose, $(t_0:t_1:t_2:t_3)\in X(\mathbb{Q}_p)$ is a point reducing to the triple line. We may assume $t_0,t_1,t_2,t_3\in\mathbb{Z}_p$ are coprime. Then $\nu_p(t_0)\geq1$ and $\nu_p(t_3)\geq1$ together imply that $\nu_p\big(t_3(a_1t_0+d_1t_3)(a_2t_0+d_2t_3)\big)\geq3$. On the other hand,

$$\nu_p\bigg(\prod_{i=1}^3(t_0+\theta^{(i)}t_1+(\theta^{(i)})^2t_2)\bigg)$$

equals 1 or 2 since $t_1$ or $t_2$ is a unit and $\nu_p(\theta^{(i)}) = \frac{1}{3}$.

iii) Again, assume $t_0, t_1, t_2, t_3 \in \mathbb{Z}_p$ to be coprime. Assertion ii) implies that $t_3$ is a $p$-adic unit. Hence, $(a_1t_0 + d_1t_3)/t_3 \in \mathbb{Z}_p$. Further,

$$\Big(\frac{a_1t_0 + d_1t_3}{t_3} \bmod p\Big) = \frac{a_1 + d_1s}{s}$$

for $s := (t_3/t_0 \bmod p)$ a solution of $T(a_1 + d_1T)(a_2 + d_2T) - 1 = 0$. □

**Lemma 2.1.** *Let $\nu$ be any valuation of $\mathbb{Q}$ different from $\nu_p$ and $w$ an extension of $\nu$ to $K$. Further, let $X$ be as in Theorem 1.1.ii) and $(t_0 : t_1 : t_2 : t_3) \in X(\mathbb{Q}_\nu)$.*

*Then, $(a_1t_0+d_1t_3)/t_3 \in \mathbb{Q}_\nu^*$ is in the image of the norm map $N\colon K_w \to \mathbb{Q}_\nu$.*

**Proof.** *First step:* Elementary cases.

If $q$ is a prime split in $K$ then every element of $\mathbb{Q}_q^*$ is a norm. The same applies to the infinite prime.

*Second step:* Preparations.

It remains to consider the case that $q$ remains prime in $K$. Then, an element $x \in \mathbb{Q}_q^*$ is a norm if and only if $3|\nu(x)$ for $\nu := \nu_q$.

It might happen that $\theta$ is not a unit in $K_w$. However, as $K_w/\mathbb{Q}_q$ is unramified, there exists some $t \in \mathbb{Q}_q^*$ such that $\underline{\theta} := t\theta \in K_w$ is a unit. The surface $\widetilde{X}$ given by

$$T_3(a_1T_0 + d_1T_3)(a_2T_0 + d_2T_3) = \prod_{i=1}^{3} \big(T_0 + \underline{\theta}^{(i)}T_1 + (\underline{\theta}^{(i)})^2T_2\big)$$

is isomorphic to $X \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_q$. Even more, the map

$$\iota\colon X \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_q \to \widetilde{X}, \quad (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : \tfrac{t_1}{t} : \tfrac{t_2}{t^2} : t_3)$$

is an isomorphism that leaves the rational function $(a_1T_0 + d_1T_3)/T_3$ unchanged. Hence, we may assume without restriction that $\theta \in K_w$ is a unit.

*Third step:* The case that $\theta$ is a unit.

Assume that $t_0, t_1, t_2, t_3 \in \mathbb{Z}_q$ are coprime. If

$$\nu\big(t_3(a_1t_0 + d_1t_3)(a_2t_0 + d_2t_3)\big) = 0$$

then $(a_1t_0 + d_1t_3)/t_3$ is a $q$-adic unit, hence clearly a norm. Otherwise, we have

$$\nu\bigg(\prod_{i=1}^{3}(t_0 + \theta^{(i)}t_1 + (\theta^{(i)})^2t_2)\bigg) > 0\,.$$

This means that one of the factors $t_0 + \theta^{(i)}t_1 + (\theta^{(i)})^2t_2$ vanishes after reduction to the residue field $\mathbb{F}_{q^3}$. As $\theta^{(i)}$ is reduced to a generator of the extension $\mathbb{F}_{q^3}/\mathbb{F}_q$, this implies that $\nu(t_0)$, $\nu(t_1)$, $\nu(t_2) > 0$. Consequently, $t_3$ must be a unit.

From the equation of $X$, we deduce $\nu(d_1d_2) > 0$. If $\nu(d_2) > 0$ then, according to the assumption, $d_1$ is a unit. This shows $\nu(a_1t_0 + d_1t_3) = 0$, from which the assertion follows.

Thus, assume $\nu(d_1) > 0$. Then, $d_2$ is a unit and, therefore, $\nu(a_2t_0 + d_2t_3) = 0$. Further, we note that

$$3 \,|\, \nu\Big(\prod_{i=1}^{3}(t_0 + \theta^{(i)}t_1 + (\theta^{(i)})^2t_2)\Big)$$

since the product is a norm. By consequence,

$$3 \,|\, \nu\big(t_3(a_1t_0 + d_1t_3)(a_2t_0 + d_2t_3)\big).$$

Altogether, we see that $3 \,|\, \nu(a_1t_0+d_1t_3)$ and $3 \,|\, \nu\big((a_1t_0+d_1t_3)/t_3\big)$. The assertion follows. $\square$

*Proof* **of Theorem 1.1.ii).** According to Lemma 2.1, $(a_1t_0+d_1t_3)/t_3 \in \mathbb{Q}^*$ is a local norm at every prime except $p$. Global class field theory [5, Theorem 5.1 together with 6.3] shows that it must necessarily be a norm at that prime, too.

By Observation 2.1.iii), $(a_1t_0 + d_1t_3)/t_3$ is automatically a $p$-adic unit. $(p) = \mathfrak{p}^3$ is a totally ramified prime. A $p$-adic unit $u$ is a norm if and only if $\overline{u} := (u \bmod p)$ is a cube in $\mathbb{F}_p^*$. As $(a_1+d_1s)/s = \big((a_1t_0+d_1t_3)/t_3 \bmod p\big)$, this is exactly the assertion. $\square$

*Proof* **of Theorem 1.1.iii).** We have to show that $X(\mathbb{Q}_\nu) \neq \emptyset$ for every valuation of $\mathbb{Q}$. $X(\mathbb{R}) \neq \emptyset$ is obvious. For a prime number $q$, in order to prove $X(\mathbb{Q}_q) \neq \emptyset$, we use Hensel's lemma. It is sufficient to verify that the reduction $X_q$ has a smooth $\mathbb{F}_q$-valued point. Thereby, we may replace $X$ by a $\mathbb{Q}_q$-scheme $\widetilde{X}$ isomorphic to $X \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_q$.

*Case 1: $q = p$.*

Then, the reduction $X_p$ is the union of three planes meeting in the line given by $T_0 = T_3 = 0$. By assumption, one of the planes appears with multiplicity one and is defined over $\mathbb{F}_p$. It contains $p^2$ smooth points.

*Case 2: $q \neq p$.*

Assume without restriction that $\theta$ is a $w$-adic unit. There are two subcases.

a) $q \nmid d_1d_2$. It suffices to show that there is a smooth $\mathbb{F}_q$-valued point on the intersection $X'_q$ of $X_q$ with the hyperplane "$T_0 = 0$". This curve is given by

$$\overline{d}_1\overline{d}_2T_3^3 = \overline{\theta}^{(1)}\overline{\theta}^{(2)}\overline{\theta}^{(3)}\prod_{i=1}^{3}(T_1 + \overline{\theta}^{(i)}T_2)\,.$$

If $q \neq 3$ then this equation defines a smooth genus one curve. It has an $\mathbb{F}_q$-valued point by Hasse's bound.

If $q = 3$ then the projection $X'_q \to \mathbf{P}^1$ given by $(T_1 : T_2 : T_3) \mapsto (T_1 : T_2)$ is one-to-one on $\mathbb{F}_q$-valued points. At least one of them is smooth since $\prod_{i=1}^3 (T + \overline{\theta}^{(i)})$ is a separable polynomial.

b) $q | d_1 d_2$. Then, $X'_q := X_q \cap$ "$T_0 = 0$" is given by

$$0 = \overline{\theta}^{(1)} \overline{\theta}^{(2)} \overline{\theta}^{(3)} \prod_{i=1}^{3} (T_1 + \overline{\theta}^{(i)} T_2) \,.$$

In particular, $x = (0 : 0 : 0 : 1) \in X_q(\mathbb{F}_q)$. We may assume that $x$ is singular.

Then, $X_q$ is given as $Q(T_0, T_1, T_2)T_3 + K(T_0, T_1, T_2) = 0$ for $Q$ a quadratic form and $K$ a cubic form. If $Q \not\equiv 0$ then there is an $\mathbb{F}_q$-rational line $\ell$ through $x$ such that $Q|_\ell \neq 0$. Hence, $\ell$ meets $X_q$ twice in $x$ and once in another $\mathbb{F}_q$-valued point that is smooth.

Otherwise, ($F$ mod $q$) does not depend on $T_3$. I.e., the left hand side of the equation of $X$ vanishes modulo $q$. This means that one of the factors must vanish. We have, say, $a_1 \equiv d_1 \equiv 0 \pmod q$. Then, by assumption, $q$ splits completely in $K$. At such a prime, $X'_q$ is the union of three lines that are defined over $\mathbb{F}_q$, different from each other, and meet in one point. There are plenty of smooth points on $X'_q$. These points are smooth on $X_q$, as well. $\square$

## 3. Examples

**Example.** For $p = 19$, a counterexample to the Hasse principle is given by

$$\begin{aligned} T_3(19T_0 + 5T_3)(19T_0 + 4T_3) &= \prod_{i=1}^{3} \big(T_0 + \theta^{(i)} T_1 + (\theta^{(i)})^2 T_2\big) \\ &= T_0^3 - 19T_0^2T_1 + 133T_0^2T_2 + 114T_0T_1^2 \\ &\quad - 1\,539\, T_0T_1T_2 + 5\,054\, T_0T_2^2 - 209T_1^3 \\ &\quad + 3\,971\, T_1^2T_2 - 23\,826\, T_1T_2^2 + 43\,681\, T_2^3 \,. \end{aligned}$$

Indeed, in $\mathbb{F}_{19}$, the cubic equation $T^3 - 1 = 0$ has the three solutions 1, 7, and 11. However, in any case $(a_1 + d_1 s)/s = 5$, which is a non-cube.

**Example.** Put $p = 19$. Consider the cubic surface $X$ given by

$$T_3(T_0 + T_3)(12T_0 + T_3) = \prod_{i=1}^{3} \big(T_0 + \theta^{(i)} T_1 + (\theta^{(i)})^2 T_2\big) \,.$$

Then, $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$. $X$ violates the Hasse principle.

Indeed, in $\mathbb{F}_{19}$, the cubic equation $T(1 + T)(12 + T) - 1 = 0$ has the three solutions 12, 15, and 17. However, in $\mathbb{F}_{19}$, $13/12 = 9$, $16/15 = 15$, and $18/17 = 10$, which are three non-cubes.

**Example.** For $p = 19$, consider the cubic surface $X$ given by

$$T_3(T_0 + T_3)(2T_0 + T_3) = \prod_{i=1}^{3} \left(T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2\right).$$

Then, for $X$, the Hasse principle fails.

Indeed, in $\mathbb{F}_{19}$, the cubic equation $T(1 + T)(2 + T) - 1 = 0$ has $T = 5$ as its only solution. The two other solutions are conjugate to each other in $\mathbb{F}_{19^2}$. However, in $\mathbb{F}_{19}$, $6/5 = 5$ is a non-cube.

**Example.** Put $p = 19$ and consider the cubic surface $X$ given by

$$T_3(T_0 + T_3)(6T_0 + T_3) = \prod_{i=1}^{3} \left(T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2\right).$$

There are $\mathbb{Q}$-rational points on $X$ but weak approximation fails.

Indeed, in $\mathbb{F}_{19}$, the cubic equation $T(1+T)(6+T)-1 = 0$ has the three solutions 8, 9, and 14. However, in $\mathbb{F}_{19}$, $10/9 = 18$ is a cube while $9/8 = 13$ and $15/14 = 16$ are non-cubes. The smallest $\mathbb{Q}$-rational point on $X$ is $(14 : 15 : 2 : (-7))$. Observe that, in fact, $T_3/T_0 = -7/14 \equiv 9 \pmod{19}$.

**Remark.** From each of the examples given, by adding multiples of $p$ to the coefficients $a_1$, $d_1$, $a_2$, and $d_2$, a family of surfaces arises, which are of similar nature.

**Remark** (Lattice basis reduction)**.** The norm form in the $p = 19$ examples produces coefficients that are rather large. An equivalent form with smaller coefficients may be obtained using lattice basis reduction. In its simplest form, this means the following.

For the rank 2 lattice in $\mathbb{R}^3$, generated by $v_1 := (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})$ and $v_2 := \left((\theta^{(1)})^2, (\theta^{(2)})^2, (\theta^{(3)})^2\right)$, in fact $\{v_1, v_2 + 7v_1\}$ is a reduced basis. Therefore, the substitution $T_1' := T_1 - 7T_2$ simplifies the norm form. We find

$$\begin{aligned}\prod_{i=1}^{3}\left(T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2\right) = T_0^3 &- 19T_0^2T_1' + 114T_0T_1'^2 + 57T_0T_1'T_2 \\ &- 133T_0T_2^2 - 209T_1'^3 - 418T_1'^2T_2 \\ &+ 1045T_1'T_2^2 - 209T_2^3\,.\end{aligned}$$

**Remark** (Brauer-Manin obstruction)**.** It was observed by Yu. I. Manin [2] that a class $\alpha \in \mathrm{Br}(X)$ in the Grothendieck-Brauer group may be responsible for the failure of the Hasse principle or of weak approximation. In fact, all the examples given above may be explained more conceptually in this way.

In short, this may be seen is as follows. We consider the rational function $f \in \mathbb{Q}(X)$ given by $(a_1T_0+d_1T_3)/T_3$. The principal divisor $\mathrm{div}(f)$ is the norm of a divisor $E \in \mathrm{Div}(X_K)$. Indeed, it is the sum over the three conjugate lines given by $a_1T_0 + d_1T_3 = T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2 = 0$ minus the sum of the three conjugate lines given by $T_3 = T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2 = 0$.

By Manin's formula [2, Proposition 31.3], such a principal divisor induces a class in $H^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{Pic}(X_K)) \subseteq H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(X_{\overline{\mathbb{Q}}}))$. Furthermore, for cubic surfaces, the Hochschild-Serre spectral sequence shows that the latter Galois cohomology group is isomorphic to $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$.

Hence, associated to $f$, we have a Brauer class $\alpha \in \mathrm{Br}(X)$, which is unique up to addition of some element from $\mathrm{Br}(\mathbb{Q})$. To evaluate $\alpha \in \mathrm{Br}(X)$ at an adelic point $x \in X(\mathbb{A}_\mathbb{Q})$ essentially means to evaluate $f$ and to apply the norm-residue-homomorphisms $\mathbb{Q}_\nu^*/NK_w^* \to \mathbb{Q}/\mathbb{Z}$ [2, 45.2]. This is exactly what we did in the proof of Theorem 1.1.ii).

More details on this approach are given in the author's Habilitation thesis [1, Sec. III.5].

## References


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