Generalized vector bundles on curves

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Introduction

In their paper [14] G. Harder and M.S. Narasimhan (and independently D. Quillen) have constructed a canonical flag of subbundles on any vector bundle on a complete smooth algebraic curve over a field. This flag measures how far away from semistability the vector bundle is.

Influenced by this D. Grayson and the third author of this paper have studied the corresponding situation over number fields [9, 10], [20, 25]. Here one is looking at lattices in some euclidean or hermitean vector space. It turns out that one has again a canonical filtration by a flag of sublattices.

If one compares both situations more carefully one sees that the second is more general.

This leads one to consider what we call generalized vector bundles in this paper. The idea is to replace locally or better over the completion of the local rings of the curve the lattices given by the stalks of the locally free sheaf associated to the vector bundle by the real valued norms given by the lattices on the associated vector spaces.

This relation between norms and lattices is not new and has been considered for example in [8] or for a particularly nice and more recent exposition [7].

In the more general context of reductive groups over local field it corresponds to the relation of the Bruhat-Tits building as a simplicial complex to its topological realisation. It turns out that the H-N-filtration exists also in this more general context. This is done in section two of this paper and we could follow for this more or less completely the exposition in [11] with the minor difference that we have to be more careful with questions of existence here, because the degree of a generalized vector bundle can be an arbitrary real number. We have included in this section a study of the H-N-filtration where the vector bundle is deformed in a family. This is related to the well known paper of S. Shatz [19] where he studies algebraic deformations of vector bundles. The two results however do not compare directly because our deformations are definitely not algebraic.

Looking at normed vector spaces (usually finite dimensional here) over complete discretely valued fields in section one as a preparation for the following we could not resist to study the relevant additive category in greater detail. We construct an embedding of this category into an abelian category which includes so to say torsion objects. This is quite natural and it would be possible to do a similar thing for the category of generalized vector bundles on an algebraic curve.

The third section contains some applications and further results: If one studies generalized vector bundles over an affine curve one obtains a direct sum decomposition into line bundles as in the classical case. Using some input from reduction theory we have a Grothendieck-type decomposition theorem for vector bundles over the projective line.

The main motivation for us to consider this kind of generalisation comes from reduction theory of the general linear group. It allows to consider invariants of reduction theory given usually by the H-N-filtration which are a priori only defined for the vertices of the relevant building on the whole topological realisation of the building. It might be possible that this point of view is sometimes more flexible.

1. Normed vector spaces

A) In this section F denotes a discretely valued complete field with a valuation

$$| \, | : F \to \mathbb{R}_{\geq 0}$$

satisfying the usual rules

i) $|a| = 0$ iff $a = 0$

ii) $|a + b| \leq \text{Max}\,\{|a|, |b|\}$

iii) $|ab| = |a||b|$

for arbitrary elements $a, b \in F$.

The valuation ring is $\mathcal{O} := \{x \in F : |x| \leq 1\}$ with maximal ideal $m := \{x \in F : |x| < 1\}$ and residue field $k := \mathcal{O}/m$.
$\pi \in \mathcal{O}$ denotes a uniformising element.

We consider F-vector spaces V equipped with a *seminorm*

$$\| \, \| : V \to \mathbb{R}$$

satisfying

i) $\|x\| \geq 0$

ii) $\|x + y\| \leq \text{Max}\,\{\|x\|, \|y\|\}$

iii) $\|\alpha x\| = |\alpha| \|x\|$

for arbitrary elements $x, y \in V$, $\alpha \in F$. $\| \, \|$ is a *norm* iff $\|x\| > 0$ for all $x \in V \setminus \{0\}$.

1.1. Examples

1) The standard vector space $V = F^n$ can be equipped with the maximum norm that is for $x = (x_1, \ldots, x_n) \in F^n$ one has

$$\|x\| := \text{Max}\,\{|x_i| : 1 \leq i \leq n.\}$$

2) V denotes an arbitrary F-vector space, $L \subset V$ an $\mathcal{O}$-submodule satisfying $F \cdot L = V$.
We can associate with L the seminorm

$$\|x\|_L := \text{Inf}\,\{|a| : a \in F, \, x \in aL\}$$

In particular, $\| \, \|$ will be a norm on the vector space V iff for any $x \in V$ the intersection $Fx \cap L$ is a finitely generated $\mathcal{O}$-module.
Conversely if $L = V$ then the seminorm $\| \, \|_L = 0$.

1.2. Remarks: We have the following well known properties of seminormed vector spaces. See for example [2], 2.4.4, Prop. 2.

1) Any linear subspace $W \subset V$ of a seminormed vector space $(V, \| \, \|)$ obtains an induced seminorm $\| \, \|_W$.

2) For W, $(V, \| \, \|)$ as in 1), the quotient vector space V/W obtains the seminorm

$$\|\bar{x}\| := \operatorname{Inf}\{\|x\| : x \in \bar{x}\},$$

where $\bar{x} \in V/W$.

3) If $(V_i, \| \, \|_i)$ for $i = 1, \dots, n$ are seminormed vector spaces then the finite direct sum $V = \bigoplus_{i=1}^{n} V_i$ obtains the seminorm

$$\|x\| := \operatorname{Max}\{\|x_i\|_i : 1 \leq i \leq n\}$$

where $x = \sum_{i=1}^{n} x_i \in V$, the $x_i \in V_i$.

If an arbitrary seminormed vector space $(V, \| \, \|)$ can be decomposed as above by subspaces $(V_i, \| \, \|_i)$ where

$$\| \, \|_i = \| \, \|_V \, |_{V_i} \quad (i = 1, \dots, n)$$

then we say that $(V, \| \, \|)$ is the orthogonal sum of the subspaces $(V_i, \| \, \|_i)$.

4) For a seminormed vector space $(V, \| \, \|)$ the subspace

$$\operatorname{rad}(V) := \{x \in V : \|x\| = 0\}$$

is an F-subspace of V.
One has the orthogonal decomposition

$$V = \operatorname{rad}(V) \oplus V/\operatorname{rad}(V).$$

5) The unit ball $L := \{x \in V : \|x\| \leq 1\}$ of the seminormed vector space $(V, \| \, \|)$ is an $\mathcal{O}$-module.
For finite dimensional seminormed vector spaces $(V, \| \, \|)$ one has particularly nice properties, namely

6) Any one dimensional seminormed space $(V, \| \, \|)$ is isomorphic (in the obvious sense) to a seminormed space $(F, \| \, \|_\lambda)$ where the seminorm $\| \, \|_\lambda$ is given by $\|1\|_\lambda = \lambda$, $\lambda \in \mathbb{R}$ and $\lambda \geq 0$, for the norm of the unit element $1 \in F$.

7) Any finite dimensional seminormed space $(V, \| \, \|)$ is the orthogonal sum of one dimensional spaces.

8) Suppose, $(V, \| \, \|)$ is now again an arbitrary seminormed vector space. Then a finite dimensional subspace (quotient space) in the sense of 1) (resp. 2)) is always a direct summand such that one has an orthogonal decomposition

$$(V, \| \, \|) \cong (W, \| \, \|) \oplus (V/W, \| \, \|)$$

resp.

$$(V, \| \|) \cong (\ker(V \to W), \| \|) \oplus (W, \| \|).$$

For a proof of 7) and 8) which are less trivial then the other statements see [2]. 2.4.1, Prop. 5.

9) Given two seminormed vector spaces $(V_1, \| \|_1)$ and $(V_2 \| \|_2)$ which we assume to be finite dimensional (this suffices for our purposes) we can associate other seminormed vector spaces, in particular

i) $\mathrm{Hom}_F(V_1, V_2)$ equipped with the norm

$$\|\varphi\| := \mathrm{Inf}\{c \in \mathbb{R} : \|\varphi(v)\|_2 \leq c\|v\|_1 \quad \text{for all } v \in V_1\}.$$

In particular, if $(V_2, \| \|_2) = (F, \| \|_{\mathcal{O}})$, then this makes the dual space V_1^* into a seminormed vector space.

ii) The tensor product $V_1 \otimes_F V_2$ can be made into a seminormed vector space by

$$\begin{aligned} \|v\| \;&:=\; \mathrm{Inf}\Big\{\mathrm{Max}\,\|v_i^{(1)}\|_1 \cdot \|v_i^{(2)}\|_2 : 1 \leq i \leq n \\ &= \sum_{i=1}^{n} v_i^{(1)} \otimes v_i^{(2)} \text{ any representation of } v \text{ in } V_1 \otimes_F V_2\Big\} \end{aligned}$$

10) Suppose, $F_1 \supset F$ is a finite extension of the discretely valued complete field F with valuation $| \, |$. There is a unique extension of the valuation $| \, |$ to an extension $| \, |_1$ on F_1. This makes $(F_1, | \, |_1 =: \| \|)$ into a normed F-vector space. If $(V, \| \|)$ is an arbitrary finite dimensional seminormed F-vectorspace then $F_1 \otimes_F V$ is, using ii), in a natural way a seminormed F_1-vector space.

The class of seminormed vector spaces will be made into a category in the following way:

1.3. Definition: A morphism

$$\varphi : (V, \| \|_V) \to (W, \| \|_W)$$

between seminormed spaces is an F-linear map satisfying

$$\|\varphi(x)\|_W \leq \|x\|_V.$$

1.4. Remarks:

i) The seminormed vector spaces together with contracting F-linear maps as above form an additive category where even further the Hom-groups $\mathrm{Hom}((V, \| \|_V), (W, \| \|_W))$ are $\mathcal{O}$-modules.

ii) One has a functor from the category of seminormed F-vector spaces to the category of $\mathcal{O}$-modules given by

$$(V, \| \|) \mapsto \{x \in V : \|x\| \leq 1\} =: V(1)$$

which associates with a seminormed vector space its unit ball.

Conversely given a torsion free $\mathcal{O}$-module M one can associate a normed vector space by posing

$$V := F \otimes_{\mathcal{O}} M$$

with the natural embedding

$$M \mapsto V, \quad m \mapsto m \otimes 1$$

and for $x \in V$ one has

$$\|x\| := \mathrm{Inf}\{|a| : a \in F,\ x \in aM \subset F \otimes_{\mathcal{O}} M\}.$$

Obviously the composed functor

$$M \mapsto (F \otimes_{\mathcal{O}} M, \| \, \|) \mapsto (F \otimes_{\mathcal{O}} M, \| \, \|)(1)$$

is the identity. The other composition

$$(V, \| \, \|) \mapsto V(1) \mapsto F \otimes_{\mathcal{O}} V(1)$$

is not the identity.

(B) Relation to the Bruhat-Tits building of the group $GL_n(F)$:

The Bruhat-Tits building $\underline{BT}(\mathrm{GL}_n/F)$ of the general linear group $\mathrm{GL}_n(F)$ is a simplicial complex, which has the following concrete description:
Its vertices are similarity classes $\langle L \rangle$ of $\mathcal{O}$-lattices $L \subset F^n$. Here we mean by an $\mathcal{O}$-lattice $L \subset F^n$ an $\mathcal{O}$-submodule of F^n, free of rank n over $\mathcal{O}$. Two such lattices L_1, L_2 are similar iff there is $a \in F^{\times}$ such that $aL_1 = L_2$.

The r-simplices of this simplicial complex $\underline{BT}(\mathrm{GL}_n/F)$ are given as $\langle L_0, L_1, \ldots, L_r \rangle$ where the L_j $(j = 0, \ldots, r)$ are lattices such that

$$L_0 \subsetneqq L_1 \subsetneqq \cdots \subsetneqq L_r \subsetneqq \pi^{-1} L_0$$

holds where π is some uniformizing element of $\mathcal{O}$.

This leads to a contractible simplicial complex whose topological realisation is denoted as $|\underline{BT}(\mathrm{GL}_n/F)|$.
We call two norms $\| \, \|_1$ and $\| \, \|_2$ on F^n *similar* iff they differ by a scaling factor $a \in \mathbb{R}$, $a > 0$ such that

$$a\| \, \|_1 = \| \, \|_2$$

holds. The set of such similarity class of norms can be made into a metric space X by defining a distance

$$d(\overline{\| \, \|}_1; \overline{\| \, \|}_2) := \log_c \sup_{v \in F^d \setminus \{0\}} \frac{\|v\|_1}{\|v\|_2} \cdot \sup_{v \in F^d \setminus \{0\}} \frac{\|v\|_2}{\|v\|_1}$$

between two classes of norms $\overline{\| \, \|}_1$ and $\overline{\| \, \|}_2$.
Here c equals $|\pi|^{-1}$ for some uniformising element $\pi \in m$.

$\mathrm{GL}_n(F)$ is acting on X by

$$\|v\|_{g(x)} := \|g^{-1}(v)\|_x$$

where $\| \,\|_x \in X$ is a norm (up to similarity), $g \in \mathrm{GL}_n(F)$ and $v \in F^n$. The two maps φ, ψ to be constructed below give a $\mathrm{GL}_n(F)$-equivariant isometry between X and the realisation $|\underline{BT}(\mathrm{GL}_n/F)|$ of the Bruhat-Tits building. This fact was first proved in [8].
Consider a point

$$y \in |\langle L_0, \ldots, L_r\rangle| \subset |\underline{BT}(\mathrm{GL}_n(F))|$$

where the r-simplex $\langle L_0, \ldots, L_r\rangle$ is given by the lattices

$$L_0 \subset L_1 \subset \cdots \subset L_r \subset \pi^{-1} L_0$$

in F^n. The point y is given inside the simplex $|\langle L_0, \ldots, L_r\rangle|$ by barycentric coordinates $(\lambda_0(y), \ldots, \lambda_r(y))$ satisfying

$$\lambda_i(y) \geq 0 \quad (i = 0, \ldots, r)$$

$$\sum_{i=0}^{r} \lambda_i(y) = 1$$

Then the norm $\varphi(y) := x \in X$ is given by interpolating

$$\| \,\|_x := \sum_{i=0}^{r} \lambda_i(y) \| \,\|_{L_i}.$$

This obviously gives a $\mathrm{GL}_n(F)$-equivariant map

$$\varphi : |\underline{BT}(\mathrm{GL}_n/F)| \to X.$$

The inverse map

$$\psi : X \to |\underline{BT}(\mathrm{GL}_n/F)|$$

is obtained as follows:

If $x \in X$ is given by the norm $\| \,\|_x$ (up to similarity) we consider the set of balls

$$L(\lambda) = \{v \in F^n : \|v\|_x \leq \lambda\}$$

for $\lambda \in \mathbb{R}_{>0}$.

These are all $\mathcal{O}$-lattices satisfying

i) $$L(\lambda) \subseteq L(\mu) \text{ if } \lambda \leq \mu$$

ii) $$L(|\pi|\lambda) = \pi L(\lambda)$$

It follows easily (looking for example at an orthogonal basis of $(F^n; \|\,\|_x)$ that the $\{L(\lambda) : \lambda \in \mathbb{R}_{>0}\}$ define an r-simplex

$$\langle L_0, \ldots, L_r \rangle$$

given by the $\mathcal{O}$-lattices

$$L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r \subset \pi^{-1} L_0$$

The barycentric coordinates of the point

$$\psi(x) = y \in |\langle L_0, \ldots, L_r \rangle| \subset |\underline{BT}(\mathrm{GL}_n/F)|$$

are computed from the equation (up to similarity $\sim$)

$$\|\,\|_x \sim \sum_{i=0}^{r} \lambda_i(x) \|\,\|_{L_i}$$

by evaluating on an orthogonal basis $\{e_1, \ldots, e_n\}$ of $(F^n, \|\,\|_x)$ which is also an orthogonal basis of all the $\|\,\|_{L_j}$ $(j = 0, \ldots, r)$ which follows by computing the L_j explicitly making use of the basis $\{e_1, \ldots, e_n\}$ of $(F^n, \|\,\|_x)$.

C) Embedding into an abelian category

We consider in this section the category of finite dimensional seminormed F-vector spaces and contracting morphisms and will embed it in a natural way into an abelian category. The idea of course is to immitate the description of a finitely generated $\mathcal{O}$-module as a quotient of finitely generated free modules. In fact we will be slightly more general and will obtain an embedding of the category $\mathcal{O} - \mathrm{mod}^*$ of subquotients of the $\mathcal{O}$-modules isomorphic to F^d, $d \in \mathbb{N}$.

Therefore $\mathcal{V}$ denotes the category of finite dimensional seminormed F-vector spaces with contracting F-linear maps as morphisms. For $V \in \mathrm{obj}(\mathcal{V})$ the associated seminorm usually is denoted as $\|\,\|_V$ or if no confusion can arise just $\|\,\|$.

We consider complexes of length one in $\mathcal{V}$,

$$0 \to V_1 \xrightarrow{\partial} V_0 \to 0$$

where we assume furthermore that ∂ is *injective*.

We denote such a complex often as $[V_1 \to V_0]$ or even shorter as $V.$, ∂ sometimes will also be denoted by ∂_V.

Morphisms $f. : V. \to W.$ are given by a commutative diagram in $\mathcal{V}$

$$\begin{tikzcd} 0 \arrow[r] & V_1 \arrow[r, "\partial_{V.}"] \arrow[d, "f_1"] & V_0 \arrow[r] \arrow[d, "f_0"] & 0 \\ 0 \arrow[r] & W_1 \arrow[r, "\partial_{W.}"] & W_0 \arrow[r] & 0 \end{tikzcd}$$

Two morphisms $f_., g_. : V_. \to W_.$ are *homotopic* if one has a morphism $s_1 : V_0 \to W_1$ satisfying the relations

$$\begin{aligned} f_1 - g_1 &= s_1 \circ \partial_{V.} \\ f_0 - g_0 &= \partial_{W.} \circ s_1 \end{aligned}$$

1.5. Remark: In fact, the first relation above is a consequence of the second one.

Objects of our new category $\mathcal{V}^*$ are complexes of $\mathcal{V}$ in the sense above, morphisms are homotopy classes of morphisms of complexes in the sense above. We have the following theorem concerning the properties of the category $\mathcal{V}^*$:

1.6. Theorem:

(1) *$\mathcal{V}^*$ is an additive category whose Hom-groups are even $\mathcal{O}$-modules.*

(2) *One has a natural embedding functor*

$$\mathcal{V} \to \mathcal{V}^*$$

which associates with a seminormed finite dimensional F-vector space V the resolution of length one

$$0 \to V_1 := 0 \to V_0 := V \to 0.$$

(3) *One has a natural functor of the above mentioned category $\mathcal{O} - \mathrm{mod}^*$,*

$$\mathcal{O} - \mathrm{mod}^* \to \mathcal{V}^*$$

given as follows:
If $L_1 \subseteq L_0 \subseteq F^d$ are $\mathcal{O}$-submodules, then the functor maps L_0/L_1 to the object

$$[(F \cdot L_1, \| \, \|_{L_1}) \to (F \cdot L_0, \| \, \|_{L_0})]$$

of $\mathcal{V}^*$.

(4) *$\mathcal{V}^*$ is an abelian category.*

Proof. Only (4) is not trivialy, we have to show here

i) $\mathcal{V}^*$ has kernels

ii) $\mathcal{V}^*$ has cokernels

iii) the canonical morphism α deduced from an arbitrary morphism

$$f_. : V_. \to W_.$$

as

$$\alpha : \mathrm{cok}\,(\ker(f_.) \to V_.) \to \ker(W_. \to \mathrm{cok}\,(f))$$

is an isomorphism.

Ad i) We equip the vector space $W_1 \times_{W_0} V_0$ with the seminorm induced from the seminormed vector space $W_1 \times V_0 \cong W_1 \oplus V_0$.

We will see below that

$$\ker(f_\cdot : V_\cdot \to W_\cdot) \cong [V_1 \xrightarrow{(f_1, \partial_{V_\cdot})} W_1 \times_{W_0} V_0].$$

To see this we have to check

1) the morphism in $\mathcal{V}^*$ obtained by composition

$$[V_1 \xrightarrow{(f_1, \partial_{V_\cdot})} W_1 \times_{W_0} V_0] \to [W_1 \to W_0]$$

is the zero morphism in $\mathcal{V}^*$. This can be seen immediately by using the projection onto the first factor

$$s_1 := \mathrm{pr}_1 : W_1 \times_{W_0} V_0 \to W_1$$

as homotopy.

2) We have to show the factorisation property of the kernel. Suppose therefore given a morphism

$$g_\cdot : U_\cdot \to V_\cdot$$

in $\mathcal{V}^*$ such that the composition

$$f_\cdot \circ g_\cdot = 0.$$

One has to find a factorisation in $\mathcal{V}^*$

$$U_0 \to \ker(f_\cdot) = [V_1 \to W_1 \times_{W_0} V_0]$$

By definition the equation $f_\cdot \circ g_\cdot = 0$ implies the existence of a homotopy

$$s_1 : U_0 \to W_1$$

satisfying the relations

$$s_1 \circ \partial_{U_\cdot} = f_1 \circ g_1$$

$$\partial_{W_\cdot} \circ s_1 = f_0 \circ g_0$$

From the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & U_1 \arrow[r] \arrow[d, "g_1"] & U_0 \arrow[r] \arrow[d, "g_0"] & 0 \\
0 \arrow[r] & V_1 \arrow[r, "s_1"] \arrow[d, "f_1"] & V_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & W_1 \arrow[r] & W_0 \arrow[r] & 0
\end{tikzcd}$$

it follows that one has a canonical morphism

$$U_0 \to W_1 \times_{W_0} V_0.$$

But this is exactly the factorisation

$$U. \to [V_1 \to W_1 \times_{W_0} V_0] = \ker(f.).$$

The necessary properties are checked easily. $\square$

Ad ii) $\mathcal{V}^*$ has cokernels:
This is not more difficult than i). For the convenience of the reader we give an outline of the proof.

Given again a morphism

$$f. : V. \to W.$$

in $\mathcal{V}^*$ represented (up to homotopy) by a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{\partial_{V.}} & V_0 & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \\ 0 & \longrightarrow & W_1 & \xrightarrow{\partial_{W.}} & W_0 & \longrightarrow & 0 \end{array}$$

We have a canonical map in $\mathcal{V}$ of the orthogonal direct sum

$$\partial_W + f_0 : W_1 \oplus V_0 \to W_0$$

Then $\ker(\partial_W + f_0)$ as a subspace and $\mathrm{coim}(\partial_W + f_0)$ as a quotient space of $W_1 \oplus V_0$ are seminormed vector spaces. One should observe that $\mathrm{coim}(\partial_W + f_0)$ is in general not a subspace of the seminormed space W_0 but there is a canonical contracting monomorphism

$$\mathrm{coim}(\partial_W + f_0) \to W..$$

From the composition

$$g_1 : W_1 \to W_1 \oplus V_0 \to \mathrm{coim}(\partial_W + f_0)$$

we obtain altogether the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{\partial_{V.}} & V_0 & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \\ 0 & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \\ 0 & \longrightarrow & \mathrm{coim}(\partial_W + f_0) & \longrightarrow & W_0 & \longrightarrow & 0 \end{array}$$

from which we will show now that it represents $\mathrm{coker}(f.)$ in $\mathcal{V}^*$.

Again we have to check the following points 1) and 2):

1) The composition $g. \circ f. : V. \to$ "$\mathrm{coker}(f.)$" is the zero morphism:
We have the composition of morphisms

$$\begin{array}{ccccc} V_0 & \to & W_1 \oplus V_0 & \to & \mathrm{coim}(\partial_W + f_0) \\ v & \mapsto & (0, v) & \mapsto & \text{class of } (0, v) \text{ in the quotient space} \end{array}$$

It is immediate to check that this defines a homotopy of $g. \circ f.$ with the zero morphism.

2) Next we assume a morphism

$$\tilde{g} : W. \to U.$$

in $\mathcal{V}^*$ such that the composition

$$\tilde{g}. \circ f. : V. \to U.$$

is zero. We have to show that we can factorize over

$$\text{“Coker}(f)'' = [\mathrm{coim}(\partial_W + f_0) \to W_0]$$

Because $\tilde{g} \circ f = 0$ in $\mathcal{V}^*$, by definition we have a morphism in $\mathcal{V}$

$$s_1 : V_0 \to U_1$$

satisfying the relevant relations for a homotopy.

We have the diagram

$$\begin{tikzcd} & W_1 \oplus V_0 \arrow[r, "(\partial_W + f_0)"] \arrow[d] & W_0 \arrow[d, equal] & \\ 0 \arrow[r] & \mathrm{coim}(\partial_W + f_0) \arrow[r] \arrow[d, "h_1(\text{to be constructed})"] & W_0 \arrow[r] \arrow[d, "\tilde{g}_0 =: h_0"] & 0 \\ 0 \arrow[r] & U_1 \arrow[r] & U_0 \arrow[r] & 0 \end{tikzcd}$$

We consider

$$(\tilde{g}_1 + s_1) : W_1 \oplus V_0 \to U_1.$$

One checks that $\tilde{g}_1 + s_1$ vanishes on $\ker(\partial_W + f_0) \subset W_1 \oplus V_0$, therefore one obtains a factorisation of $\tilde{g}_1 + s_1$ as

$$W_1 \oplus V_0 \to \mathrm{coim}(\partial_W + f_0) \xrightarrow{=: h_1} U_1$$

This proves 2)

It remains to check the last defining property of an abelian category [17], 4.2. namely given a morphism

$$f. : V. \to W.$$

in $\mathcal{V}^*$ then one has a canonical isomorphism

$$\mathrm{coker}(\ker(f.) \to V.) \xrightarrow{\sim} \ker(W. \to \mathrm{coker}(f.))$$

One sees easily that

i) $\mathrm{coker}(\ker(f.) \to V.) \cong [W_1 \times_{W_0} V_0 \xrightarrow{\mathrm{pr}_2} V_0]$

ii) $\ker(W.) \to \mathrm{coker}(f.))V.) \cong [W_1 \to \mathrm{coim}(\partial_W + f_0)]$.

Choosing a decomposition

$$W_1 \oplus V_0 \underset{\varphi}{\overset{\sim}{\to}} (W_1 \times_{W_0} V_0) \oplus \mathrm{coim}(\partial_W + f_0)$$

one has the following commutative diagram from which one will obtain all the morphisms needed below:

$$\begin{tikzcd}
0 \arrow[r] & W_1 \times_{W_0} V_0 \arrow[rr] \arrow[dd] & & V_0 \arrow[r] \arrow[dd] & 0 \\
& & W_1 \oplus V_0 & & \\
0 \arrow[r] & W_1 \arrow[rr] & & \mathrm{coim}(\partial_W + f_0) \arrow[r] & 0
\end{tikzcd}$$

All the morphisms coming up above are obtained from projections and embeddings into direct sums using $f.$ and φ in an obvious sense.

The canonical morphism

$$\mathrm{coker}(...) \to \ker(...)$$

to be considered in iii) is then given in the diagram

$$\begin{tikzcd}
0 \arrow[r] & W_1 \times_{W_0} V_0 \arrow[r] \arrow[d, "\bar{f}_1", "g_1"'] & V_0 \arrow[r] \arrow[d, "\bar{f}_0", "g_0"'] & 0 \\
0 \arrow[r] & W_1 \arrow[r] & \mathrm{coim}(\partial_W + f_0) \arrow[r] & 0
\end{tikzcd}$$

by the morphism $\bar{f}., g. = (g_1, g_0)$ is obtained from the diagram above by obvious composition.

It remains to show in the category $\mathcal{V}^*$, that is up to homotopy

$$\begin{aligned} g. \circ \bar{f}. &= \mathrm{id} \\ \bar{f}. \circ g. &= \mathrm{id} \end{aligned}$$

For this we have to give the relevant homotopies

$$\begin{aligned} s_1 &: V_0 \to W_1 \times_{W_0} V_0 \\ t_1 &: \mathrm{coim}(\partial_W + f_0) \to W_1 \end{aligned}$$

which are obtained by forming appropriate compositions from diagram. The necessary equations for a homotopy are checked immediately. This finishes iii) and therefore the proof of (4) of theorem 1.6. □

5) The category $\mathcal{V}^*$ has enough projective and injective objects, its homological dimension is one.

Proof: It is immediate to see that the injective objects are precisely the seminormed vector spaces $(V, \| \| \equiv 0)$ with trivial seminorm.

The projective objects are the normed vector spaces $(V, \| \|)$. Any object has a resolution of length one by projective objects.

2. Generalized vector bundles and Harder-Narasimhan filtration

X is a regular, irreducible algebraic curve over the field of constants k. Changing our notation from section 1, $F = k(X)$ denotes from now on the field of rational functions on X.

For any closed point $x \in X$ we have the discrete valuation ring $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}_{X,x}$ and residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. The degree of the field extension $(k(x) : k)$ is the residue degree of x.

For $a \in F$, $a \neq 0$, $\deg_x(a)$ denotes the order of vanishing of a at $x \in |X|$, where $|X|$ is the set of closed points of X.

Choosing a real number $c > 1$ we obtain a valuation $|\,|_x$ on F by

$$|a|_x := c^{-\deg_x(a)(k(x):k)}$$

F_x is a completion of F with respect to $|\,|_x$, $\mathcal{O}_x$ the valuation ring with maximal ideal $\mathfrak{m}_x$ and residue field $k(x)$. One has the product formula

$$\prod_{x \in |X|} |a|_x = 1$$

for all $a \in F^\times$ if X is additionally complete.

2.1. Definition: A generalized vector bundle E on the algebraic curve X is given by

i) a finite dimensional F-vector space E_η

ii) for any closed point $x \in |X|$ a norm $\|\,\|_x$ on the F_x-vector space $E_x := E_\eta \otimes_F F_x$ such that the following *consistency condition* is fullfilled:

iii) Given any basis $\{e_1, \ldots, e_n\}$ of the F-vector space E_η, all except finitely many of the norms $\|\,\|_x$ on E_x are equal to the standard norms given by the $\mathcal{O}_x$-lattices

$$\mathcal{O}_x e_1 + \cdots + \mathcal{O}_x e_n \subset E_x$$

as explained in section 1.

2.2. Definition: $n := \dim_F(E_\eta)$ denotes the rank $\mathrm{rk}\,(E)$ of the gen. vector bundle E.

2.3. Definition: A morphism $\varphi : E_1 \to E_2$ between gen. vector bundles E_1 and E_2 is an F-linear map

$$\varphi_\eta : E_{1,\eta} \to E_{2,\eta}$$

such that the induced F_x-linear mappings

$$\varphi_x : E_{1,x} \to E_{2,x}$$

are contracting for all $x \in |X|$ (that is are morphisms of seminormed vector spaces in the sense of section 1).

2.4. Definition: A gen. line bundle L on X is a gen. vector bundle of rank one.

2.5. Remarks:

i) The gen. vector bundles and morphisms in the sense above form an additive category where the Hom-groups are even k-vector spaces.

ii) A subbundle E_1 of a gen. vector bundle E is given by

1. an F-linear subspace $E_{1,\eta}$ of E_η.
2. for all closed points $x \in |X|$, $E_{1,x} = E_{1,\eta} \otimes_F F_x$ has a norm obtained by restricting $\| \|_x$ on E_x to $E_{1,x}$.

It is easy to see that in particular $(E_{1,\eta}; E_{1,x}$ for $x \in |X|)$ is itself a gen. vector bundle

iii) in a similar way one obtains the quotient vector bundle E/E_1 as the gen. vector bundle

$$(E_\eta/E_{1,\eta}; E_x/E_{1,x} \quad \text{for} \quad x \in |X|)$$

where the $E_x/E_{1,x}$ are quotients in the sense of seminormed vector spaces.

iv) A sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

of gen. vector bundles on X is exact iff the sequence

$$0 \to E_{1,\eta} \to E_\eta \to E_{2,\eta} \to 0$$

of F-vector spaces is exact and additionally for all closed points $x \in |X|$ $E_{1,x}$ is identified with a subspace of the normed vector space E_x, $E_{2,x}$ with the quotient space $E_x/E_{1,x}$ as a normed vector space.

v) For gen. vector bundles one has the usual operations like forming the dual bundle, direct sums, tensor products and Hom-bundles. In particular one has the determinant line bundle which for a bundle $E = (E = (E_\eta; E_x, \| \|_x$ for $x \in |X|)$ is given by

$$\det(E) = (\det(E_\eta); \det(E_x))$$

From now on, if not said otherwise, the curve X will be complete.

2.6. Definition:

i) The degree $\deg(L)$ of a gen. line bundle L on X is given by

$$\deg(L) := -\sum_{x \in |X|} \log_c \|a\|_x$$

where $a \in L_\eta$, $a \neq 0$, is a rational section of the line bundle L.

ii) The degree of a gen. vector bundle $E = (E_\eta; E_x$ for $x \in |X|)$ is given by

$$\deg(E) := \deg(\det(E))$$

iii) The invariant $\mu(E)$ for a nontrivial bundle E is the quotient

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}\,(E)}$$

One has the following properties

(1) If

$$0 \to E_1 \to E \to E_2 \to 0$$

is an exact sequence of gen. vector bundles then

$$\deg(E) = \deg(E_1) + \deg(E_2).$$

(2) If $E = E_1 \otimes E_2$, then

$$\begin{aligned} \deg(E) &= \deg(E_1)\,\mathrm{rk}\,(E_2) + \deg(E_2)\,\mathrm{rk}\,(E_1) \\ \mu(E) &= \mu(E_1) + \mu(E_2). \end{aligned}$$

(3) For the dual bundle E^* of a gen. vector bundle E one has

$$\begin{aligned} \deg E^* &= -\deg(E) \\ \mu(E^*) &= -\mu(E) \end{aligned}$$

These are proved in the same way as for vector bundles in the usual sense.

It is plausible that the argument leading to the Harder-Narasimhan-filtration go through in this generalized situation. For the convenience of the reader we repeat the main arguments in our context (see in particular [11], and [14]).

2.7. Definition: A gen. vector bundle E is called stable (resp. semistable) iff for all nonzero subbundles E_1 of E one has $\mu(E_1) < \mu(E)$ (resp. $\mu(E_1) \leq \mu(E)$).

2.8. Remarks:

i) Stability can equivalently be decribed by quotient bundles namely: E is stable (resp. semistable) iff for all quotient bundles E/E_1 one has

$$\mu(E) > \mu(E/E_1)$$

resp.

$$\mu(E) \geq \mu(E/E_1)).$$

ii) The degree of subbundles is bounded above. Again we follow [11] to see this:
We fix a flag of gen. vector bundles

$$0 \subset E_1 \subset \cdots \subset E_n = E$$

where E_i is a subbundle of E of rank i.
If $E' \subset E$ is an arbitrary subbundle one considers the intersection

$$0 \subset E_1 \cap E' \subset \cdots \subset E_n \cap E' = E'.$$

The successive quotients

$$E_j \cap E'/E_{j-1} \cap E'$$

are either 0 or line bundles.
One has canonical morphisms

$$E_j \cap E'/E_{j-1} \cap E' \longrightarrow E_j/E_{j-1},$$

which are obviously contracting for all $x \in |X|$.
Therefore

$$\deg(E_j \cap E'/E_{j-1} \cap E') \leq \deg(E_j/E_{j-1})$$

2.9. Lemma: If E_1 and E_2 are gen. semistable vector bundles on X and $\mathrm{Hom}\,(E_1, E_2) \neq 0$ then $\mu(E_1) \leq \mu(E_2)$.

Proof: Suppose, $\varphi : E_1 \to E_2$ is a nonzero homomorphism. We then have the exact sequence of gen. vector bundles

$$0 \to \ker(\varphi) \to E_1 \to \mathrm{coim}(\varphi) \to 0.$$

Because E_1 is semistable we have

$$\mu(E_1) \leq \mu(\mathrm{coim}(\varphi))$$

Denoting $\bar{E}_1 \subset E_2$ the subbundle of E_2 given by the F-subspace $\varphi(E_{1,\eta}) \subset E_{2,\eta}$ we have the morphism

$$\mathrm{coim}(\varphi) \to \bar{E}_1$$

which gives $\mu(\mathrm{coim}(\varphi)) \leq \mu(\bar{E}_1)$. Because E_2 is semistable we obtain $\mu(\bar{E}_1) \leq \mu(E_2)$. Therefore $\mu(E_1) \leq \mu(E_2)$. □

2.10. Lemma:

i) Let E be a gen. vector bundle on X. Then the set

$$\{\mu(E') : E' \subset E \quad \text{a subbundle}\}$$

has a maximum which is called $\mu_{\max}(E)$.

ii) The set

$$\{\mu(E/E') : E' \subset E \quad \text{a subbundle}\}$$

of μ-values of the quotient bundles of E has a minimum which is called $\mu_{\min}(E)$

Proof: By 2.8. ii) the set of real numbers

$$\{\mu(E') : 0 \neq E' \subset E\}$$

is bounded above.
We have to show that the supremum is obtained.
We argue by induction. The case $\mathrm{rk}\,(E) = 1$ is trivial.
For the induction step we can assume that E is not semistable, otherwise $\mu(E)$

would be a maximum. Therefore we find a subbundle $E_1 \subset E$ such that $\mu(E_1) > \mu(E)$. We can assume that there are subbundles $E_2 \subset E$ satisfying

$$\mu(E_2) > \mu(E_1) > \mu(E),$$

otherwise $\mu(E_1)$ is maximal and we are ready.
The gen. vector bundle E/E_1 is not semistable for if it would be, we would obtain the following contradiction:
Because

$$\mu(E_2) > \mu(E_1) > \mu(E) > \mu(E/E_1),$$

by lemma 2.9. the composed morphism $E_2 \to E/E_1$ must be trivial. But then we obtain $E_2 \subset E_1$ and can conclude that E_1 is not semistable contradicting our assumption on E_1.
Therefore E/E_1 is not semistable. We can conclude from this that there exists a nontrivial quotient bundle $\bar{E}$,

$$E/E_1 \to \bar{E},$$

which is semistable itself and satisfies

$$\mu(E/E_1) > \mu(\bar{E})$$

and therefore also

$$\mu(E_2) > \mu(\bar{E}).$$

Again applying lemma 2.9. above, we conclude that the composed morphism

$$E_2 \to \bar{E}$$

must be zero. Therefore we obtain

$$E_2 \subset \mathrm{pr}^{-1}(\ker(E/E_1 \to \bar{E})) =: E'$$

where pr denotes the projection morphism

$$\mathrm{pr} : E \to E/E_1.$$

Because $\mathrm{rk}\,(E') < \mathrm{rk}\,(E)$, it follows by the induction hypothesis that the set of real numbers

$$\{\mu(E_2) : 0 \neq E_2 \subset E\,,\ \mu(E_2) > \mu(E_1)\}$$

$=$

$$\{\mu(E_2) : 0 \neq E_2 \subset E',\ \mu(E_2) > \mu(E_1)\}$$

has a maximum. $\square$

2.11. Proposition: ([11], [14]) A gen. vector bundle E on X has a unique flag of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

satisfying the following properties (1) and (2):

(1) E_i/E_{i-1} is semistable for each possible i.
(2) $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for each possible i.
Moreover this flag also satisfies:
(3) E_i/E_{i-1} is the largest subbundle of E/E_{i-1} such that $\mu(E_i/E_{i-1}) = \mu_{\max}(E/E_{i-1})$.
(4) E_i/E_{i-1} is the largest quotient bundle of E_i such that $\mu(E_i/E_{i-1}) = \mu_{\min}(E_i)$.

Proof: This can be taken from [11].
For the convenience of the reader we show (3) and (4) as in [11]:

Let E' be a subbundle of E/E_{i-1} with $\mu(E') = \mu_{\max}(E/E_{i-1})$; it is enough to show $E' \subset E_i/E_{i-1}$ because E_i/E_{i-1} is semistable and therefore

$$\mu(E') \leq \mu(E_i/E_{i-1}) \leq \mu_{\max}(E/E_{i-1}) = \mu(E').$$

Therefore we have equality everywhere above and 3) would follow.
Clearly E' has to be semistable and $\mu(E') > \mu(E_j/E_{j-1})$ for all $j > i$. Therefore by descending induction and Lemma 2.9. above it follows that $E' \subset E_{j-1}/E_{i-1}$ for all $j > i$. This proves (3).

4) This is just dual to 3) and can be obtained by applying (3) to the dual bundle E^*. □

We also have the following useful corollary from [11].

2.12. Corollary: *Suppose E' is a subbundle of the gen. vector bundle E, assume that $E'_1 \subset \cdots \subset E'_r$ is the H-N-filtration of E' and $E_1/E' \subset \cdots \subset E_{s-1}/E'$ is the H-N-filtration of the bundle E/E'. We assume furthermore*

$$\mu_{\max}(E/E') < \mu_{\min}(E')$$

Then the H-N-filtration of E is the flag of subbundles

$$0 \subset E'_1 \subset \cdots \subset E'_{r-1} \subset E' \subset E_1 \subset \cdots \subset E_{s-1} \subset E$$

Proof: Immediate from the Proposition using 1) and 2). □

The result above can be applied to study the behavior of the H-N-filtration under continuous deformation of the norm as explained below.

We consider the following situation:
T is a topological space with a base point $o \in T$.
$\infty \in |X|$ is a closed point of the algebraic curve X. $\{E^{(t)} : t \in T\}$ is a family of gen. vector bundles parametrized by T such that

i) $E^{(t)}_\eta = E^{(0)}_\eta$ for the underlying F-vector spaces and therefore also

$$E^{(t)}_x = E^{(0)}_x \quad (x \in |X|)$$

for the associated F_x-vector spaces.

ii) If $\| \ \|_x^{(t)}$ is the norm on the vector space $E_x^{(t)} = E_x^{(0)}$, then

$$\| \ \|_x^{(t)} = \| \ \|_x^{(0)} \quad \text{for} \quad x \neq \infty,$$

$\| \ \|_\infty^{(t)}$ depends continuously on $t \in T$ in $o \in T$.

This means explicitly:

For any given real number $\varepsilon > 0$ there is a neighbourhood $U = U(\varepsilon; o)$ of the base point o in T such that for any vector $v \in E_\infty^{(t)} = E_\infty^{(0)}$ the inequalities

$$(1+\varepsilon)^{-1}\|v\|_\infty^{(0)} \leq \|v\|_\infty^{(t)} \leq (1+\varepsilon)\|v\|_\infty^{(0)}$$

hold.
For an F_∞-linear subspace $\tilde{E}_\infty \subset E_\infty^{(t)} = E_\infty^{(0)}$ one obtains for the volume elements the inequalities

$$\begin{aligned}(1+\varepsilon)^{-\dim_{F_\infty}(\tilde{E}_\infty)}\mathrm{vol}\,((\tilde{E}_\infty : \| \ \|_\infty^{(0)} \mid_{\tilde{E}_\infty}) \ &\leq \ \mathrm{vol}\,(\tilde{E}_\infty; \| \ \|_\infty^{(t)} \mid_{\tilde{E}_\infty}) \\ &\leq \ (1+\varepsilon)^{-\dim_{F_\infty}(\tilde{E}_\infty)}\mathrm{vol}\,(\tilde{E}_\infty; \| \ \|_\infty^{(0)} \mid_{\tilde{E}_\infty})\end{aligned}$$

for all $t \in U(\varepsilon; 0)$. Therefore for any F-subspace

$$\tilde{E}_\eta \subset E_\eta^{(t)} = E_\eta^{(0)}$$

one obtains for the associated subbundles $\tilde{E}^{(t)}$ of $E^{(t)}$ resp. $\tilde{E}^{(0)}$ of $E^{(0)}$:

$$|\deg(\tilde{E}^{(t)}) - \deg(\tilde{E}^{(0)})| < \eta$$

for all $t \in U(\varepsilon; 0)$ and where

$$\eta = \dim_F((L_\eta) \log_c(1+\varepsilon)$$

is positive and arbitrary close to zero. These preparations suffice to prove the following

2.13. Proposition: *If a subspace of $E^{(0)}$ given by an F-vector subspace $\tilde{E}_\eta \subset E_\eta^{(0)}$ occurs in the H-N-filtration of $E^{(0)}$ then it occurs in the H-N-filtration of all deformed gen. vector bundles $E^{(t)}$ in the sense above for a sufficiently small neighborhood of o in T.*

Proof: Because $\tilde{E}^{(0)}$ occurs in the H-N-filtration of $E^{(0)}$ it follows that

$$\mu_{\max}(E^{(0)}/\tilde{E}^{(0)}) < \mu_{\min}(\tilde{E}^{(0)}).$$

We call the difference $\varepsilon > 0$.
By the considerations made above we can find a neighbourhood $U(\varepsilon; 0)$ of o in T such that

$$|\mu_{\max}(E^{(t)}/\tilde{E}^{(t)}) - \mu_{\max}(E^{(0)}/\tilde{E}^{(0)})| < \frac{\varepsilon}{2}$$

$$|\mu_{\min}(\tilde{E}^{(t)}) - \mu_{\min}(\tilde{E}^{(0)})| < \frac{\varepsilon}{2}$$

Using corollary 2.12. it follows that $\tilde{E}^{(t)}$ occurs in the H-N-filtration of $E^{(t)}$ for all $t \in U(\varepsilon; o)$. $\square$

3. Complements

A) We first consider gen.line bundles and divisors.

$L = (L_\eta; \|\,\|_x)$ denotes a gen.line bundle on X with underlying F-vector space L_η and norms $\|\,\|_x$ on $L_x = L_\eta \otimes_F F_x$.

$a \in L_\eta$, $a \neq 0$, is a rational section of L.

3.1. Definition:

i) The $\mathbb{R}$-vectorspace of gen.divisors on X is the direct sum

$$\bigoplus_{x \in |X|} \mathbb{R} \cdot (x) =: \mathrm{Div}^*(X)$$

of one dimensional spaces $\mathbb{R} \cdot (x)$.

ii) The gen. divisor associated to (L, a) is

$$\mathrm{div}(a) := -\sum_{x \in |X|} (\log_c \|a\|_x) \cdot (x)$$

as an element of $\mathrm{Div}^*(X)$.

Conversely, given an element

$$D = \sum_{x \in |X|} \alpha_x (x) \in \mathrm{Div}^*(X)$$

we can associate a gen. line bundle given as $L = \mathcal{O}_x(D) = (F; \|\,\|_x)$ where

$$\|1\|_x := c^{-\alpha_x}, \quad \text{where} \quad x \in |X|.$$

Denoting $\mathrm{Pic}^*(X)$ the gen. Picard group consisting of gen. line bundles with group multiplication the tensor product, one easily obtains the following diagram

$$\begin{tikzcd}
1 \arrow[r] & F^\times \arrow[r] & \mathrm{Div}^*(X) \arrow[r] & \mathrm{Pic}^*(X) \arrow[r] & 0 \\
1 \arrow[r] & F^\times \arrow[r] \arrow[u, equal] & \mathrm{Div}(X) \arrow[r] \arrow[u] & \mathrm{Pic}(X) \arrow[r] \arrow[u] & 0\,.
\end{tikzcd}$$

Here $\mathrm{Div}(X), \mathrm{Pic}(X)$ are the groups of ordinary divisors and line bundles, the vertical maps are the obvious ones. From this we obtain the exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}^*(X) \to \bigoplus_{x \in |X|} \mathbb{R}/\mathbb{Z} \to 0$$

B) The next point we want to consider concerns gen. vector bundles on affine curves. Of course these should have similar properties as finitely generated projective modules over Dedekind rings which are direct sums of rank one projective modules.

3.2. Proposition

i) *X denotes an affine smooth curve over the field of constants k,*

$$0 \to E^{(1)} \xrightarrow{\ i\ } E \xrightarrow{\ \mathrm{pr}\ } E^{(2)} \to 0$$

is an exact sequence of gen. vector bundles on X. Then the sequence above splits.

ii) *Any gen. vector bundle E on X is a direct sum of line bundles.*

Proof. We choose an F-linear section

$$s : E^{(2)}_\eta \to E_\eta$$

such that $\mathrm{pr} \circ s = \mathrm{id}$.

It is clear that for all except finitely many $x \in |X|$ the induced morphism

$$s \otimes \mathrm{id} =: s_x : E^{(2)}_x \to E_x$$

is contracting. We call this exceptional set S. The set of all sections as above is given by the affine space over F,

$$s + \mathrm{Hom}_F(E^{(2)}_\eta, E^{(1)}_\eta).$$

Because for any $x \in |X|$ the exact sequence of normed vector spaces

$$0 \to E^{(1)}_x \to E_x \to E^{(2)}_x \to 0$$

can be split, we can find in particular elements

$$\varphi_x \in \mathrm{Hom}_{F_x}(E^{(2)}_x, E^{(1)}_x) \quad (x \in S)$$

such that the homomorphisms

$$s + \varphi_x : E^{(2)}_x \to E_x$$

are contracting and split the exact sequence

$$0 \to E^{(1)}_x \to E_x \to E^{(2)}_x \to 0$$

for all $x \in S$.

Using strong approximation for the F-vector space $\mathrm{Hom}_F(E^{(2)}_\eta, E^{(1)}\eta)$ we can find an element

$$\varphi \in \mathrm{Hom}_F(E^{(2)}_\eta, E^{(1)}_\eta)$$

satisfying

i) $\|\varphi\| \leq 1$ as an element of the normed vector space $\mathrm{Hom}_{F_x}(E^{(2)}_x, E^{(1)}_x)$ for $x \in |X| \setminus S$

ii) $\|\varphi - \varphi_x\| < 1$ as an element of the normed vector space $\mathrm{Hom}_{F_x}(E^{(2)}_x, E^{(1)}_x)$ for $x \in S$.

It follows that

$$s + \varphi : E_\eta^{(2)} \to E_\eta$$

is a section which additionally satisfies $\|s+\varphi\| \leq 1$ as an element of the normed F_x-vector space $\mathrm{Hom}_{F_x}(E_x^{(2)}, E_x)$ for all $x \in |X|$. This means exactly that $(s+\varphi)$ induces a morphism of gen. vector bundles

$$s + \varphi : E^{(2)} \to E$$

which splits the exact sequence

$$0 \to E^{(1)} \to E \to E^{(2)} \to 0.$$

This proves i).
ii) is an immediate consequence of i). □

The line bundle $\mathcal{O}$ is given by the zero divisor. Another description is as

$$\mathcal{O} = (F, \|\ \|_x),$$

where

$$\|1\|_x = 1 \quad \text{for all} \quad x \in |X|.$$

3.3. Lemma: Let X denote again an affine smooth curve over k, L_1 and L_2 two gen. line bundles on X. Then one has an isomorphism of gen. vector bundles

$$L_1 \oplus L_2 \cong \mathcal{O} \oplus (L_1 \otimes L_2).$$

Proof. We can easily reduce to the case that $L_1, L_2 \subset \mathcal{O}$ and L_1, L_2 are *coprime*, that is the canonical morphism of gen. vector bundles

$$L_1 \oplus L_2 \to \mathcal{O},\ (u_1, u_2) \mapsto u_1 + u_2 \quad \text{for } u_1 \in L_1,\ u_2 \in L_2,$$

is a surjective morphism.
By proposition 3.2. we can split and obtain a direct decomposition of vector bundles

$$L_1 \oplus L_2 \cong L' \oplus \mathcal{O}.$$

Comparing the determinant bundles of both sides it follows that

$$L_1 \otimes L_2 \cong L'.$$

□

3.4. Proposition. Any gen. vector bundle E on an affine smooth curve X over k of rank n is isomorphic to $\mathcal{O}^{n-1} \oplus \det(E)$.

Proof. Immediate from Proposition 3.2. and lemma 3.3. □

C) As a further application we prove Grothendieck's theorem for gen. vector bundles over $\mathbb{P}^1$.

3.5. Proposition: *A gen. vector bundle E over the projective line $\mathbb{P}^1$ over k is a direct sum of line bundles*

$$E = \bigoplus_{i=1}^{n} L_i.$$

This decomposition is unique up to isomorphism.

Proof. Tensorizing E by the line bundle $\det(E)^{-1/n}$ we can assume $\det(E) \cong \mathcal{O}$ because the statement of the proposition is invariant against tensorisation by a line bundle. Looking at the restriction of the gen. vector bundle E to the affine line

$$\mathbf{A}^1/k = \mathbb{P}^1/k \setminus \{\infty\}$$

we obtain by proposition that $E|_{\mathbf{A}^1} \cong \mathcal{O}^n$ is the trivial bundle of rank n.
E therefore is uniquely given by a norm $\| \, \|_\infty$ on $E_\infty = F_\infty^n$.
Up to similarity this gives by section 1 a point $x \in |\underline{BT}(\mathrm{GL}_n/F_\infty)|$ which is in the interior of a unique simplex $\Delta(x)$ of the simplicial complex $\underline{BT}(\mathrm{GL}_n/F_\infty)$. Let p denote a vertex of $\Delta(x)$ which is given by an $\mathcal{O}_\infty$-lattice L in F_∞^n up to similarity. Conversely a point x as above determines a vector bundle on X up to tensorisation by a gen. line bundle given by a gen. divisor on X supported at $\infty \in X$ such that $E(x) = (\mathcal{O}^n, \| \, \|_x)$.

By Grothendieck's (and others) theorem on vector bundles (in the usual sense) on $\mathbb{P}^1$ one has for the vector bundles $E(p)$, associated to p

$$E(p) = \bigoplus_{i=1}^{n} \mathcal{O}(d_i),$$

a direct sum of line bundles $\mathcal{O}(d_i)$ of degree d_i where $d_1 \geq \cdots \geq d_n$. Denoting $\mathcal{O}(d_i)_\eta = F \cdot e_i \subset F^n$ $(i = 1 \cdots n)$ for the underlying F-vector spaces the decomposition

$$F^n = \bigoplus_{i=1}^{n} F \cdot e_i$$

defines an appartment A in the sense of [4], [7] in the Bruhat-Tits building $\underline{BT}(\mathrm{GL}_n/F_\infty)$ which contains the vertex p..
The link of p in $\underline{BT}(\mathrm{GL}_n/F_\infty)$ can be identified with the Tits building [7] of the group $\mathrm{GL}_n(L/\pi_\infty L)$. The simplex $\Delta(x)$ introduced above is uniquely given by a simplex $\bar{\Delta}$ in the Tits building of $\mathrm{GL}_n(L/\pi_\infty L)$. The appartment A above defines an appartment $\bar{A}$ of the Tits building of $GL_n(L/\pi_\infty L)$ by the direct sum decomposition

$$L/\pi_\infty L = \bigoplus_{i=1}^{n} L \cap F_\infty e_i / \pi_\infty (L \cap F_\infty e_i).$$

Now the automorphism group $\mathrm{Aut}\, E(p)$ of $E(p)$ which can be easily determined explicitely acts on the building of $\mathrm{GL}_n(L/\pi_\infty L)$. It is immediate to see that any simplex of the Tits building is equivalent to a simplex of the appartment $\bar{A}$ under the action of the automorphism group Aut $E(p)$. It follows that the simplex $\Delta(x)$ is equivalent under Aut $E(p)$ to a simplex in the appartment A. But this means that the point x itself is equivalent to a point x' in A. By section 1 this means that

$$F_\infty^n = \bigoplus_{i=1}^{n} F_\infty e_i$$

is also an orthogonal decomposition of F_∞^n with respect to the norm $\| \ \|_{x'}$ on F_∞^n. This implies that $E(x')$ is isomorphic to the direct sum of line bundles given by the F-subspaces $F \cdot e_i \subset E(x')_\eta = F^n$ for $i = 1, \ldots, n$.

Because $E(x)$ and $E(x')$ are isomorphic by construction of $E(x')$ it follows that $E(x)$ is direct sum of gen. line bundles. $\square$

D) In part 3) we have used already the relation to reduction theory. The concept is a follows:

X denotes again a smooth complete curve over k, $X' = \mathrm{Spec}(A) \subset X$ an open affine subscheme, itself a smooth curve over k such that $X' = X \setminus S$ where S is a finite set of closed points. We consider the product

$$|\underline{BT}^{(S)}| := \prod_{x \in S} |\underline{BT}(\mathrm{GL}_n / F_x)|$$

For any point $p = (p_x)_{x \in S} \in |\underline{BT}^{(S)}|$ we can asssociate a gen. vector bundle

$$E(p) := (F^n ; \| \ \|_x \text{ for } x \in |X|)$$

where

i) $\| \ \|_x$ for $x \notin S$ is the norm on F_x^n given by the lattice $\sum_{i=1}^{n} \mathcal{O}_x e_i$ in the sense of section 1.

ii) $\| \ \|_x = \| \ \|_{p_x}$ for $x \in S$.

Of course $E(p)$ is determined by this only up to similarity that is more precisely up to tensorisation by a gen. line bundle L supported by a gen. divisor contained in S.

The group $\prod_{x \in S} \mathrm{GL}_n(F_x)$ is acting on X as explained in section 1 and the groups $\mathrm{GL}_n(A)$ or $\mathrm{GL}_n(F)$ are acting on $|\underline{BT}^{(S)}|$ via the diagonal embedding into $\prod_{x \in S} \mathrm{GL}_n(F_x)$. Two points $p^{(1)}, p^{(2)} \in |\underline{BT}^{(S)}|$ are equivalent under the action of the group $\mathrm{GL}_n(A)$ iff the gen. vector bundles $E(p^{(1)}), E(p^{(2)})$ are isomorphic up to tensorisation by a gen. line bundle $L = \bigotimes_{x \in S} L(x)$ where $L(x)$ is a gen. line bundle on X given by a gen. divisor with support in $x \in S$.

3.6. Remarks:

i) With any point $p \subset |\underline{BT}^{(S)}|$ one can therefore associate a gen. vector bundle $E(p)$. Any F-linear subspace $V \subset F^n$ defines a gen. subbundle $E' \subset E(p)$ such that one can define functions

$$\mu(p; V) := \mu(E')$$

depending on $p \in |\underline{BT}^{(S)}|$ and $V \subset F^n$.

ii) As the H-N-filtration is invariant against tensorisation, it can be defined via the gen. vector bundles $E(p)$ and gives a canonical flag of F-subspaces in F^n which are convenient to describe fundamental domains.

iii) If p is in the interior of a polysimplex $|\Delta| \subset |\underline{BT}^{(S)}|$ the value of $\mu(p : V)$ can be interpolated from the values of $\mu(p_j : V)$ where the $\{p_j\}$ are the vertices of the polysimplex Δ

E) Recently K. Behrend has given in [1] a somewhat different treatment of the H-N-filtration in the context of smooth reductive group schemes G over a curve X as above. There he associates with G a canonical parabolic subgroup of G. Because for a gen. vector bundle E over X the associated group scheme $\mathrm{GL}\,(E)/X$ of the general linear group is not smooth in general, his results do not apply directly.

However it is easy to write down explicitly in our context what is done in [1].

So, $E = (E_\eta; \| \, \|_x)$ is a gen. vector bundle on the complete, smooth curve X over k.

With any flag $\mathcal{F}$ of subbundles

$$0 \subset E_1 \subsetneqq \cdots \subsetneqq E_r = E$$

one associates a degree given as

$$\deg(\mathcal{F}) := \sum_{\infty \leq \rangle < | \leq \nabla} \deg \underline{\mathrm{Hom}}\,(\mathcal{E}_| / \mathcal{E}_{|-\infty}\,,\ \mathcal{E}_\rangle / \mathcal{E}_{\rangle-\infty})$$

3.7. Remark: This is the degree of the unipotent radical (or better its Lie algebra scheme) of the parabolic group scheme $P = P(\mathcal{F})$ associated to the flag $\mathcal{F}$.

We consider those flags $\mathcal{F}$ satisfying

i) $\deg(\mathcal{F})$ is maximal

(ii) There is no subflag $\mathcal{F}'$ of $\mathcal{F}$ such that $\deg(\mathcal{F}') = \deg(\mathcal{F})$ holds.

Of course in our situation it is not a priori clear that such a flag $\mathcal{F}$ exists.
In any case one can show easily:

3.8. Lemma: *If $\mathcal{F}$ is a flag satisfying i) and ii) above, it follows that*

(1) *all succesive quotients E_i/E_{i-1} $(i = 1, \ldots, r)$ are semistable gen. vector bundles.*

(2) *For all* $i \in \{1, \ldots, r-1\}$ *one has*

$$\mu(E_{i+1}/E_i) \leq \mu(E_i/E_{i-1})$$

3.9. Remark: It follows therefore that $\mathcal{F}$, if it exists, is exactly the H-N-flag. Conversely it is not difficult to show that the H-N-flag satisfies the conditions i) and ii) above.

In [1] there is also a nice geometric description of the canonical flag which gives a good uniqueness proof of the H-N-filtration.

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