On Weil polynomials of $K3$ surfaces

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Abstract. For $K3$ surfaces, we derive some conditions the characteristic polynomial of the Frobenius on the étale cohomology must satisfy. These conditions may be used to speed up the computation of Picard numbers and the decision of the sign in the functional equation**. Our investigations are based on the Artin-Tate formula.

1 Introduction

An algebraic integer such that all its conjugates have absolute value $\sqrt{r}$ is called an r-Weil number. Correspondingly, a possibly reducible monic polynomial $\Phi \in \mathbb{Z}[T]$ such that all roots have absolute value $\sqrt{r}$ is called an r-Weil polynomial.

Let q be a prime power and $r = q^k$. Then, for every smooth projective variety V over $\mathbb{F}_q$, the eigenvalues of the Frobenius endomorphism Frob on the étale cohomology $H^k_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ are r-Weil numbers [3, Lemme 1.7]. Conversely, every q^k-Weil number is an eigenvalue of Frob on $H^k_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ for a suitable smooth projective variety V over $\mathbb{F}_q$. Actually, this fact is a direct consequence of the results of T. Honda [9].

In this note, we will study the Weil numbers of $K3$ surfaces. As the second Betti number of a $K3$ surface is $b_2(V) = 22$ and q is always a root of the characteristic polynomial, the possible Weil numbers are of degree at most 20.

We will show that *not* all q^2-Weil polynomials $\Phi \in \mathbb{Z}[T]$ satisfying $\deg \Phi = 22$ and $\Phi(q) = 0$ occur as characteristic polynomials of Frob on the étale cohomology of $K3$ surfaces. Concerning $K3$ surfaces of fixed degree, even more restrictions result. Our investigations are based on the Artin-Tate formula which we will recall in section 3.

* The first author was partially supported by the Deutsche Forschungsgemeinschaft (DFG) through a funded research project.

** The computer part of this work was executed on the Sun Fire V20z Servers of the Gauß Laboratory for Scientific Computing at the Göttingen Mathematisches Institut. Both authors are grateful to Prof. Y. Tschinkel for the permission to use these machines as well as to the system administrators for their support.

An application. The characteristic polynomial of Frob may be computed by counting points over extensions of the ground field. Indeed, for V a $K3$ surface over $\mathbb{F}_q$, the Lefschetz trace formula [13, Ch. VI, §12] yields $\mathrm{tr}(\mathrm{Frob}^e) = \#V(\mathbb{F}_{q^e}) - q^{2e} - 1$.

When we denote the eigenvalues of Frob by $r_1, \ldots, r_{22}$, we have $\mathrm{tr}(\mathrm{Frob}^e) = r_1^e + \cdots + r_{22}^e =: \sigma_e(r_1, \ldots, r_{22})$. Newton's identity [20]

$$s_k(r_1, \ldots, r_{22}) = \frac{1}{k} \sum_{j=0}^{k-1} (-1)^{k+j+1} \sigma_{k-j}(r_1, \ldots, r_{22}) s_j(r_1, \ldots, r_{22})$$

shows that the knowledge of $\sigma_e(r_1, \ldots, r_{22})$, for $e = 1, \ldots, k$, is sufficient in order to determine the coefficient $(-1)^k s_k$ of T^{22-k} of the characteristic polynomial Φ of Frob. Further, there is the functional equation

$$q^{\deg \Phi} \Phi(T) = \pm T^{\deg \Phi} \Phi(q^2/T) \tag{1}$$

which, as $\deg \Phi = 22$, relates the coefficient of T^k with that of T^{22-k}.

Nevertheless, this method is time-consuming. The size of the fields to be considered grows exponentially. One would like to avoid point counting over large fields and, nevertheless, determine Φ sufficiently well in order to decide things such as the sign in (1). Algorithms of this type were presented in [6]. For example, Algorithm 22 of [6] verifies that the geometric Picard rank is 2, having counted points over $\mathbb{F}_p, \ldots, \mathbb{F}_{p^9}$ for p a prime number.

The main result of the present article leads to a more substantial approach to this problem. In fact, we will show that certain hypothetical characteristic polynomials are impossible, in general. This leads to an improvement of [6, Algorithm 22]. Sections 7 and 8 will be devoted to examples showing how this improvement works in practice.

Remark 1. A continuation of this application, which we have in mind, is the computation of the geometric Picard rank for $K3$ surfaces over $\mathbb{Q}$. Here, the general strategy is to use reduction modulo p. One applies the inequality

$$\mathrm{rk}\,\mathrm{Pic}(V_{\overline{\mathbb{Q}}}) \leq \mathrm{rk}\,\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})$$

which is true for every smooth variety V over $\mathbb{Q}$ and every prime p of good reduction. Then, the number of eigenvalues of Frob which are roots of unity is an upper bound for the Picard number. More details are given in [6] and [7].

2 The Galois group of a Weil polynomial

For a randomly chosen irreducible polynomial over $\mathbb{Q}$, one expects the Galois group to be the full symmetric group. In this sense, the irreducible factors of a Weil polynomial are not very random.

When we consider the operation of Frob on a cohomology group of even degree, cyclotomic factors do arise. They correspond to the algebraic part of the cohomology, i.e., to the image of the Picard group and its analogues in higher codimension. The corresponding Galois group is always abelian.

Concerning the remaining factors, still, there are restrictions on the Galois group. Note that, for each root of an irreducible r-Weil polynomial not of degree 1, the complex conjugate is a root, too. This means, the roots come in pairs. The product of each pair is equal to r. The Galois group therefore acts on the pairs. For a suitable integer n, it is a subgroup of the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \subset S_{2n}$. Here, each factor $(\mathbb{Z}/2\mathbb{Z})$ acts on one pair by complex conjugation. The complex conjugation itself belongs to the center of the group.

An experimental result. One could ask for further restrictions on the Galois group. For that, we computed the characteristic polynomial of Frob for a few thousand randomly chosen $K3$ surfaces. In each case, the factorization of that polynomial had precisely one irreducible factor which was not cyclotomic. This coincides with Zarhin's results [18] for ordinary $K3$ surfaces.

Furthermore, in the vast majority of the examples, the Galois group of the last factor was actually equal to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \subset S_{2n}$. For example, this was true for 875 out of 1 000 $K3$ surfaces of degree 2 over $\mathbb{F}_3$ and 923 out of 1 000 $K3$ surfaces of degree 2 over $\mathbb{F}_7$.

The resolvent algebra. Let $\Phi \in \mathbb{Q}[T]$ be a polynomial such that its set of roots is of the particular form $\{r_1, r'_1, \ldots, r_n, r'_n\}$ such that $r_1 r'_1 = \ldots = r_n r'_n =: r \in \mathbb{Q}$. Then, the sums $r_1 + r'_1, \ldots, r_n + r'_n$ are the roots of a polynomial $R \in \mathbb{Q}[T]$ of half the degree. We will call R the *resolvent polynomial* and $A := \mathbb{Q}[T]/R$ the *resolvent algebra* of Φ.

Remarks 2. a) When Φ is an r-Weil polynomial of even degree, the assumption is satisfied if and only if $\sqrt{r}$ is a root of even multiplicity (or no root) of Φ. In this case, $(-\sqrt{r})$ has even multiplicity, too.

In fact, this means exactly that Φ fulfills the functional equation (1) with the plus sign.

b) On the other hand, when one wants to verify that a given polynomial satisfying the functional equation is, in fact, a Weil polynomial, the resolvent is helpful. Observe that the roots of the initial polynomial are all of absolute value $\sqrt{r}$ if and only if the roots of the resolvent are all real and in the interval $[-2\sqrt{r}, 2\sqrt{r}]$. That property may easily be checked using Sturm's chain theorem.

This is a fast and exact replacement of [6, Algorithm 23].

3 The Artin-Tate formula

Let us recall the Artin-Tate conjecture in the special case of a $K3$ surface.

Conjecture 3 (Artin-Tate). *Let V be a $K3$ surface over a finite field $\mathbb{F}_q$. Denote by ρ the rank and by Δ the discriminant of the Picard group of V, defined over $\mathbb{F}_q$. Then,*

$$|\Delta| = \frac{\lim\limits_{T \to q} \frac{\Phi(T)}{(T-q)^\rho}}{q^{21-\rho} \#\mathrm{Br}(V)} .$$

Here, Φ denotes the characteristic polynomial of Frob *on* $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. *Finally,* $\mathrm{Br}(V)$ *is the Brauer group of* V.

Remarks 4. i) The characteristic polynomial Φ is independent of the choice of the auxiliary prime l as long as $l \neq p$ for $q = p^e$ [3, Théorème 1.6].

ii) For a general non-singular, projective surface, the exponent of q in the numerator is $b_2(V) - h_{02}(V) - \rho$. Here, $h_{02}(V)$ denotes the Hodge number.

iii) The Artin-Tate conjecture is proven for most $K3$ surfaces. Most notably, the Tate conjecture implies the Artin-Tate conjecture [11, Theorem 6.1].

iv) The Tate conjecture claims that all zeroes of Φ of the form $q\zeta$ for ζ a root of unity belong to the algebraic part of $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. I.e., it asserts that the transcendental part never generates a zero of this form.

The evidence for this is overwhelming as far as $K3$ surfaces are concerned. The Tate conjecture is proven for elliptic $K3$ surfaces [1] and ordinary $K3$ surfaces [15]. In characteristic different from 2 and 3, even more particular cases were successfully treated [16].

v) It is expected that $\mathrm{Br}(V)$ is always a finite group. This is actually equivalent to the Tate conjecture. In this case, $\#\mathrm{Br}(V)$ is automatically a perfect square. We may therefore compute the square class of Δ making use of the Artin-Tate conjecture.

An unconditional version of the Artin-Tate formula.

Notation 5. i) For n a positive integer, we will denote by μ_n the sheaf of n-th roots of unity with respect to the fppf topology. When l is a prime number, we put $H^d_{\text{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu) := \varprojlim_e H^d_{\text{fppf}}(V_{\overline{\mathbb{F}}_q}, \mu_{l^e})$.

ii) For l a prime number and M an abelian group, the notation $M_{l\text{-pow}}$ shall be used for the l-power torsion subgroup of M. Similarly, we will write $M_{l\text{-div}} \subseteq M_{l\text{-pow}}$ for the subgroup of infinitely l-divisible elements.

iii) We will denote by M^{Frob} and M_{Frob} the invariants, respectively coinvariants, under the operation of Frob on the abelian group M. The coinvariants may have torsion even when M is torsion-free. Write M'_{Frob} for the torsion-free quotient.

Proposition 6. *Let V be a $K3$ surface over a finite field $\mathbb{F}_q$ and l be any prime. Write Φ for the characteristic polynomial of* Frob *on the étale cohomology of $V_{\overline{\mathbb{F}}_q}$ and ρ for the multiplicity of q as a zero of Φ.*

i) *Then, the Brauer group* $\mathrm{Br}(V)$ *is a torsion group. The quotient*

$$\mathrm{Br}_0(V, l) := \mathrm{Br}(V)_{l\text{-pow}} / \mathrm{Br}(V)_{l\text{-div}}$$

is a finite group of square order.

ii) *Further, $H^2_{\text{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu)^{\text{Frob}}$ is a free $\mathbb{Z}_l$-module of rank ρ.*

iii) *Denote by Δ_l the discriminant of the bilinear form*

$$H^2_{\text{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu)^{\text{Frob}} \times H^2_{\text{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu)^{\text{Frob}} \longrightarrow \mathbb{Z}_l$$

defined by Poincaré duality. Then,

$$\nu_l(\Delta_l) = \nu_l\left(\frac{\lim\limits_{T\to q}\frac{\Phi(T)}{(T-q)^\rho}}{q^{21-\rho}\#\mathrm{Br}_0(V,l)}\right).$$

Proof. i) Finiteness of $\mathrm{Br}_0(V,l)$ follows immediately from [8, (8.9)]. Further, there is a non-degenerate alternating pairing $\mathrm{Br}_0(V,l)\times\mathrm{Br}_0(V,l)\to\mathbb{Q}_l/\mathbb{Z}_l$ constructed in [19, Lemma 3.4.1]. This ensures that the group order is a perfect square.

ii) and iii) We denote the zeroes of Φ by $r_1,\dots,r_{22}$.

First case. $l\neq p$. Here, $H:=H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q},T_l\mu)=H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q},\mathbb{Z}_l(1))$ is the same as l-adic étale cohomology. It is a free $\mathbb{Z}_l$-module of rank 22. In the present case, the operation of Frob on H is known to be semi-simple [4, Corollary 1.10]. The eigenvalues are $r_1/q,\dots,r_{22}/q$. Assertion ii) follows immediately from this.

Further, we have $\nu_l(\Delta_l)=\nu_l(\#\mathrm{coker}(H^{\mathrm{Frob}}\to\mathrm{Hom}(H^{\mathrm{Frob}},\mathbb{Z}_l)))$, the map being induced by Poincaré duality. Identifying $\mathrm{Hom}(H,\mathbb{Z}_l)$ with H, the module $\mathrm{Hom}(H^{\mathrm{Frob}},\mathbb{Z}_l)$ goes over into H'_{Frob}. Here, as shown in [19, Proposition 1.4.2], $(H_{\mathrm{Frob}})_{\mathrm{tors}}\cong\mathrm{Br}_0(V,l)$. Further, the order of the cokernel of the canonical homomorphism $H^{\mathrm{Frob}}\to H_{\mathrm{Frob}}$ is equal to the l-primary part of $\prod\limits_{r_j\neq q}(1-r_j/q)$. Altogether, this implies the claim.

Second case. $l=p$. Here, some modifications are necessary which are described in [11]. More concretely, the short exact sequence

$$0\to\mathrm{Pic}(V_{\overline{\mathbb{F}}_q})\otimes_{\mathbb{Z}}\mathbb{Z}_p\to H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q},T_p\mu)\to\varprojlim\mathrm{Br}(V_{\overline{\mathbb{F}}_q})_{p^n}\to 0$$

immediately shows that $H:=H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q},T_p\mu)$ is a torsion-free $\mathbb{Z}_p$-module. Otherwise, its structure is rather different from the previous case. The rank of H is, in general, less than 22. Eigenvalues of Frob are only those r_j/q which are units in $\overline{\mathbb{Q}}_p$ [11, 1.4]. But this is enough to show ii).

Generally, there are unipotent connected quasi-algebraic groups U^d and étale group schemes D^d_n for $d=2,3$ and $n\gg 0$ which provide short exact sequences $0\to U^d(\overline{\mathbb{F}}_q)\to H^d_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q},\mu_{p^n})\to D^d_n(\overline{\mathbb{F}}_q)\to 0$. For varying n, the vector groups $U^3(\overline{\mathbb{F}}_q)$ are connected by identities. Further, $D^3_n=0$. Hence, if $\dim U^3=s$ then $\#H^3_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q},T_p\mu)^{\mathrm{Frob}}=q^s$ the operation of Frob being semi-simple. Actually, one has $s=0$ except when V is supersingular.

Poincaré duality is available [12, Theorem 5.2 and Corollary 2.7.c)] only at the level of torsion coefficients. Thereby, $U^2(\overline{\mathbb{F}}_q)$ and $U^3(\overline{\mathbb{F}}_q)$ are dual to each other. One has $\varprojlim U^2(\overline{\mathbb{F}}_q)=0$ and $R^1\varprojlim U^2(\overline{\mathbb{F}}_q)=0$ as the connecting homomorphisms are zero. Hence, $H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q},T_p\mu)\cong\varprojlim D^2_n(\overline{\mathbb{F}}_q)$. Further, it turns out that the homomorphism $H_{\mathrm{Frob}}\to\mathrm{Hom}(H^{\mathrm{Frob}},\mathbb{Z}_p)$ does not need to be bijective. It has a cokernel exactly of order q^s (cf. [11, Lemma 5.2]).

Summarizing, we find that Δ_p has the same p-adic valuation as

$$q^s\cdot\prod_{\substack{\nu_p(r_j/q)=0\\ r_j\neq q}}(1-r_j/q).$$

For iii), it remains to show the following. Up to p-adic units, the product of the remaining factors, i.e. $\prod\limits_{\nu_p(r_j/q)\neq 0}(1-r_j/q)$, equals q^{s-1}. This is worked out in [11, sec. 7]. $\square$

Remark 7. The Tate conjecture implies $H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu)^{\mathrm{Frob}} \cong \mathrm{Pic}(V_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Further, it is equivalent to $\mathrm{Br}(V)_{l\text{-div}} = 0$. Thus, Proposition 6 goes over into the Artin-Tate formula in its usual form. However, the Tate conjecture is unknown in general, even for $K3$ surfaces. For this reason, we prefer to apply the version of the Artin-Tate formula which holds unconditionally.

4 The rank-1 condition

Let V be a $K3$ surface of degree d over a finite field $\mathbb{F}_q$. Assume that q is a simple zero of the characteristic polynomial of Frob. Then, the Tate conjecture is true for V and the arithmetic Picard rank is equal to 1. The discriminant of $\mathrm{Pic}(V)$ is equal to d. A comparison with the analytic discriminant computed via the Artin-Tate formula leads to a non-trivial condition for hypothetical Weil polynomials.

Remarks 8. a) This is a condition for rank-1 surfaces of a given degree d. It is not a condition for $K3$ surfaces, in general.

b) The degree of a $K3$ surface may be any even integer greater than zero. On the other hand, when the arithmetic Picard rank is 1, the number $(-q)$ is necessarily among the Frobenius eigenvalues. Hence, the Artin-Tate formula can generate only even numbers.

c) The Artin-Tate conjecture implies the inequality $\#\mathrm{Br}(V)|\varDelta| \leq 2^{22-\rho}q$. Thus, the left hand side is $\mathrm{O}(q)$. Observe the following striking consequence. Over the field $\mathbb{F}_q$, there is no $K3$ surface of a square-free degree $d > 2^{21}q$ and arithmetic Picard rank 1.

Remark 9. The rank-1 condition may be extended to other situations where a subgroup of the Picard group is known. For this, one has to compare the predicted ranks and discriminants with the known ones.

5 The field extension condition

Notation 10. For q a positive integer, let $\varPhi$ be a q^2-Weil polynomial. Then, we will write

$$E^{(c)}_{\varPhi} := \prod_{r_j \neq q} \frac{q^c - r_j^c}{q - r_j} \bigg/ q^{(c-1)(21-\rho)} \, .$$

Here, r_j runs over all the zeroes of $\varPhi$. Further, ρ is the multiplicity of the zero q.

Observation 11 (Field extension for the characteristic polynomial). Let V be any smooth, projective variety over $\mathbb{F}_q$ and $\prod_j(T - r_j)$ the characteristic polynomial of Frob on $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. Then, the corresponding polynomial for $V_{\mathbb{F}_{q^d}}$ is $\prod_j(T - r_j^d)$.

Theorem 12. *Let V be a $K3$ surface over $\mathbb{F}_q$. Further, let c be a positive integer. Then, for $\varPhi$ the characteristic polynomial of* Frob*, the expression $E^{(c)}_{\varPhi}$ is a perfect square in $\mathbb{Q}$.*

Proof. If there is an $r_j \neq q$ such that $r_j^c = q^c$ then $E^{(c)}_{\varPhi} = 0$. Otherwise, for every prime l, $H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu)^{\mathrm{Frob}_{q^c}}$ is a sublattice of finite index in $H^2_{\mathrm{fppf}}(V_{\overline{\mathbb{F}}_q}, T_l\mu)^{\mathrm{Frob}_q}$. In particular, the discriminants differ by a factor being a perfect square. Dividing the Artin-Tate formulas for $V_{\mathbb{F}_{q^c}}$ and $V_{\mathbb{F}_q}$ through each other yields that $\nu_l(E^{(c)}_{\varPhi})$ is even for every l. Finally, it is easy to see that $E^{(c)}_{\varPhi} > 0$. $\square$

Remark 13. Assume the Tate conjecture. Then, $E^{(c)}_{\varPhi}$ is non-zero if and only if $\operatorname{rk}\operatorname{Pic}(V_{\mathbb{F}_q}) = \operatorname{rk}\operatorname{Pic}(V_{\mathbb{F}_{q^c}})$.

Definition 14. We will call the condition on $E^{(c)}_{\varPhi}$ to be a perfect square, the *field extension condition* for the field extension $\mathbb{F}_{q^c}/\mathbb{F}_q$.

Explicit computation of the expression $\boldsymbol{E^{(c)}_{\varPhi}}$. Our goal is now to describe the square class of $E^{(c)}_{\varPhi}$ more explicitly. It will turn out that, for an arbitrary Weil polynomial, $E^{(c)}_{\varPhi}$ may be a non-square. In other words, Theorem 12 provides a non-trivial condition.

Remark 15. A priori, there are infinitely many conditions, one for each value of c. The main result of this section is that there is in fact only one condition. Further, this condition may be checked easily.

Lemma 16. *Let $f \in \mathbb{Q}[T]$ be a q^2-Weil polynomial. Suppose $f(q) \neq 0$ and $f(-q) \neq 0$. Then, for $r_1, \ldots, r_{2l}$ the zeroes of f,*

$$\prod_{j=1}^{2l} \frac{q^c - r_j^c}{q - r_j} \in \begin{cases} (\mathbb{Q}^*)^2 \cup \{0\} & \text{for } c \text{ odd,} \\ f(-q)(\mathbb{Q}^*)^2 \cup \{0\} & \text{for } c \text{ even.} \end{cases}$$

Further, the left hand side is actually in $f(-q)(\mathbb{Q}^*)^2$ for $c = 2$.

Proof. First observe that, for $c = 2$, the numerators $q^2 - r_j^2$ are all non-zero according to the assumption. Hence, the additional assertion is clear once we showed the main one.

For that, let us start with the contribution of one pair of complex conjugate roots. Put $r_j = q(u + iv)$. Then, the corresponding factor is

$$\begin{aligned} \frac{(q^c - r_j^c)(q^c - \overline{r}_j^c)}{(q - r_j)(q - \overline{r}_j)} &= \frac{(q^c - q^c(u+iv)^c)(q^c - q^c(u-iv)^c)}{(q - q(u+iv))(q - q(u-iv)} \\ &= q^{2(c-1)} \prod_{k=1}^{c-1} (1 - \zeta_c^k (u+iv))(1 - \zeta_c^k (u-iv)) \,. \end{aligned}$$

Using $(u + iv)(u - iv) = 1$, we get

$$q^{2(c-1)} \prod_{k=1}^{c-1} (1 - 2\zeta_c^k u + \zeta_c^{2k}) \,.$$

Next, for $k \neq c/2$, let us multiply the factors for k and $c-k$. This yields

$$(1-2\zeta_c^k u+\zeta_c^{2k})(1-2\zeta_c^{c-k}u+\zeta_c^{2c-2k}) = 2+4u^2-8u\operatorname{Re}(\zeta_c^k)+2\operatorname{Re}(\zeta_c^{2k})\,.$$

As $\operatorname{Re}(\zeta_c^{2k}) = 2\operatorname{Re}(\zeta_c^k)^2-1$, the latter term is the same as

$$4u^2-8u\operatorname{Re}(\zeta_c^k)+4\operatorname{Re}(\zeta_c^k)^2 = (2u-2\operatorname{Re}(\zeta_c^k))^2\,.$$

Multiplying over all k such that $1 \leq k < c/2$, we find a square in $\mathbb{Q}(u)$. Consequently, up to the factor for $k = c/2$, if present, the contribution of the pair $\{r_j, \overline{r}_j\}$ is a square in the resolvent algebra A of f.

Multiplying over all l pairs means to form a norm for the extension $A/\mathbb{Q}$. As the norm of a square is a square, the result is a perfect square in $\mathbb{Q}$. For c odd, this completes the argument.

For c even, the factors for $k = c/2$ are still missing. These are the ones for $\zeta_c^k = -1$. We find the product

$$\prod_{j=1}^{l}(1+r_j/q)(1+\overline{r}_j/q) = q^{-2l}f(-q)\,.$$

The assertion follows. □

Proposition 17. *Let Φ be a q^2-Weil polynomial of even degree. Then,*

$$E_\Phi^{(c)} \in \begin{cases} (\mathbb{Q}^*)^2 \cup \{0\} & \text{for } c \text{ odd,} \\ q\Phi(-q)(\mathbb{Q}^*)^2 \cup \{0\} & \text{for } c \text{ even.} \end{cases}$$

For $c = 2$, we actually have $E_\Phi^{(c)} \in q\Phi(-q)(\mathbb{Q}^*)^2$.

Proof. *First case:* c is odd.

Then, the denominator $q^{(c-1)(21-\rho)}$ is a perfect square. The zeroes $(-q)$ contribute factors q^{c-1} which are squares, too. Finally, the contribution to $E_\Phi^{(c)}$ of the zeroes not being real is a perfect square according to Lemma 16.

Second case: c is even.

If $(-q)$ is a zero of Φ then $E_\Phi^{(c)} = 0$. This coincides with the claim as $\Phi(-q) = 0$. Otherwise, write $\Phi(T) = (T-q)^\rho f(T)$ where $f(q) \neq 0$ and $f(-q) \neq 0$. By assumption, ρ is even. Hence, $q^{(c-1)(21-\rho)}$ is in the square class of q. Further, the zeroes of Φ differing from q are exactly the zeroes of f. Their contribution is in $f(-q)(\mathbb{Q}^*)^2$ for $c = 2$ and in $f(-q)(\mathbb{Q}^*)^2 \cup \{0\}$, in general. As ρ is even, $f(-q)(\mathbb{Q}^*)^2$ is the same class as $\Phi(-q)(\mathbb{Q}^*)^2$. The assertion follows. □

Corollary 18. *Let $f \in \mathbb{Z}[T]$ be a q^2-Weil polynomial.*

i) *Then, all field extension conditions for $\mathbb{F}_{q^c}/\mathbb{F}_q$ are satisfied if only if the condition for the quadratic extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ does hold.*

ii) *For extensions of odd degree, the field extension condition is always satisfied.*

iii) *If $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ lead to different Picard ranks then all the field extension conditions are satisfied.*

Remark 19. One might want to study the field extension conditions for $\mathbb{F}_{q^{ac}}/\mathbb{F}_{q^a}$, i.e., for an extended ground field. Our calculations show that this does not lead to new conditions.

Simplification of the field extension test. Denote by ϕ_n the n-th cyclotomic polynomial. Correspondingly, there is the monic polynomial ψ_n given by $\psi_n(T) := q^{\varphi(n)}\phi_n(T/q)$. This is a q^2-Weil polynomial.

Lemma 20. *Let $n > 1$ be an integer. Then,*

$$\psi_n(-q) \in \begin{cases} (\mathbb{Q}^*)^2 & \text{if } n \text{ is not a power of } 2\,, \\ 2(\mathbb{Q}^*)^2 & \text{for } n = 2^m, m \geq 2\,, \\ \{0\} & \text{for } n = 2\,. \end{cases}$$

Proof. It is well known (see, e.g., [14, sec. 3]) that $\phi_n(-1) = 1$ unless n is a power of 2. Further, the formula $\phi_{2^e}(t) = t^{2^{e-1}} + 1$ shows $\phi_2(-1) = 0$ and $\phi_{2^e}(-1) = 2$ for $e > 1$. Observe, finally, that $\varphi(n)$ is always even for $n > 2$. $\square$

Remark 21. The result used here is a very special case of the value of a cyclotomic polynomial at a root of unity.

Theorem 22. *Let $\Phi \in \mathbb{Z}[T]$ be a q^2-Weil polynomial of even degree. Factorize Φ as*

$$\Phi(T) = (T-q)^r(T+q)^s\psi_{n_1}(T)\cdot\ldots\cdot\psi_{n_k}(T)\Phi_1(T)$$

such that Φ_1 has no root being a root of unity multiplied by q. Denote by M the number of the powers of 2 among the $n_1, \ldots, n_k$. Then,

i) *if c is odd then $E_\Phi^{(c)} \in (\mathbb{Q}^*)^2 \cup \{0\}$.*

ii) *If c is even and $s > 0$ then $E_\Phi^{(c)} = 0$ for every c.*

iii) *Finally, if c is even and $s = 0$ then $E_\Phi^{(c)} \in 2^M q\Phi_1(-q)(\mathbb{Q}^*)^2 \cup \{0\}$. Furthermore, for $c = 2$, one actually has*

$$E_\Phi^{(2)} \in 2^M q\Phi_1(-q)(\mathbb{Q}^*)^2\,.$$

Proof. i) and ii) are immediate consequences from Proposition 17. For iii), observe the assumption implies that r is even. In particular, $(-2q)^r$ is a perfect square. The assertion now follows from Proposition 17 together with Corollary 20. $\square$

Remark 23. Suppose $\Phi \in \mathbb{Z}[T]$ is a q^2-Weil polynomial of degree 22. In order to show that Φ may not be the characteristic polynomial of the Frobenius for a $K3$ surface over $\mathbb{F}_q$, it suffices to verify that $s = 0$ and $2^M q\Phi_1(-q)$ is a non-square.

Example 24. As an example, we look at $K3$ surfaces of Picard rank 18 such that the Picard group is defined over an extension of odd degree. Then, $(-q)$ is not an eigenvalue of the Frobenius. The transcendental part of the characteristic polynomial is given by $(T^4 + aT^3 + bT^2 + aq^2T + q^4)$. Hence, the field extension condition usually requires that $(2q^2 - 2aq + b)q$ is a perfect square. If, however, the cyclotomic factors contain an odd number of type ψ_{2^n} then $2(2q^2 - 2aq + b)q$ is required to be a square.

6 The special case of a degree-2 surface – Twisting

When a $K3$ surface has a non-trivial automorphism, one can hope to get more conditions by inspecting the corresponding twist. This is the case for degree-2 surfaces.

The Twist. Let the $K3$ surface V be given by the equation

$$w^2 = f_6(x, y, z) \,.$$

Then, for n a non-square in $\mathbb{F}_q$, consider the twist $\widetilde{V}$ of V given by

$$nw^2 = f_6(x, y, z) \,.$$

Fact 25. *Assume that $q, r_2, \ldots, r_{22}$ are the eigenvalues of* Frob *for V. Then, the eigenvalues for $\widetilde{V}$ are $q, -r_2, \ldots, -r_{22}$.*

Proof. For e even, $V_{\mathbb{F}_{q^e}}$ and $\widetilde{V}_{\mathbb{F}_{q^e}}$ are isomorphic. When e is odd, we have

$$\#V(\mathbb{F}_{q^e}) + \#\widetilde{V}(\mathbb{F}_{q^e}) = 2 \cdot \#\mathbf{P}^2(\mathbb{F}_{q^e}) = 2q^{2e} + 2q^e + 2 \,.$$

It is easy to check that the Lefschetz trace formula, applied to the eigenvalues $q, -r_2, \ldots, -r_{22}$, implies exactly this relation. $\square$

Proposition 26. *Let V be a $K3$ surface of degree 2 over $\mathbb{F}_q$. Denote by Φ the characteristic polynomial of* Frob *for V and by $\widetilde{\Phi}$ the corresponding polynomial for the twist $\widetilde{V}$.*

i) *Then, Φ has a simple zero at q if and only if $\widetilde{\Phi}$ does not have a zero at $(-q)$. I.e., the rank-1 condition can be applied to the one precisely when the field extension condition is non-empty for the other one.*

ii) *The two conditions are equivalent to each other.*

Proof. i) immediately follows from Fact 25.

ii) By assumption, we can write $\Phi(T) = (T - q)(T + q)^{2n-1} f(T)$. Here both, $f(q)$ and $f(-q)$ are non-zero. Fact 25 shows, the corresponding polynomial for the twist is $\widetilde{\Phi}(T) = (T - q)^{2n} f(-T)$. Using these two formulas, one can make the conditions explicit. The rank-1 condition for Φ simply means $(2q)^{2n-1} f(q) = 2$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ which is equivalent to saying that $qf(q)$ is a perfect square. This is precisely the field extension condition for $\widetilde{\Phi}$. $\square$

7 Examples

Let us show in detail the data for a few examples. Our goal is to illustrate how the Artin-Tate conditions work in practice.

Example 27 (A $K3$ surface of degree 2 over $\mathbb{F}_7$). Consider the surface V over $\mathbb{F}_7$, given by

$$w^2 = y^6 + 3z^6 + 5xz^5 + 5x^2y^4 + x^2z^4 + 3x^3y^3 + x^3z^3 + 5x^4y^2 + x^4z^2 + 5x^5y + 2x^6 \,.$$

Over $\mathbb{F}_7, \ldots, \mathbb{F}_{7^9}$, there are exactly 66, 2 378, 118 113, 5 768 710, 282 535 041, 13 841 275 877, 678 223 852 225, 33 232 944 372 654, and 1 628 413 551 007 224 points. We claim that $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_7}) = 2$.

Assuming the characteristic polynomial of the Frobenius has more than two zeroes of the form 7 times a root of unity, [6, Algorithm 22] leaves us with three candidates Φ_1, Φ_2, Φ_3.

$$\begin{aligned}\Phi_i(t) \,=\, & t^{22} - 16\,t^{21} + 140\,t^{20} - 1\,029\,t^{19} + 5\,831\,t^{18} - 36\,015\,t^{17} + 268\,912\,t^{16} \\ & - 1\,882\,384\,t^{15} + 11\,529\,602\,t^{14} - 46\,118\,408\,t^{13} + a_i t^{12} + b_i t^{11} + c_i t^{10} \\ & + (-1)^{j_i}[-110\,730\,297\,608\,t^9 + 1\,356\,446\,145\,698\,t^8 - 10\,851\,569\,165\,584\,t^7 \\ & + 75\,960\,984\,159\,088\,t^6 - 498\,493\,958\,544\,015\,t^5 + 3\,954\,718\,737\,782\,519\,t^4 \\ & - 34\,196\,685\,556\,119\,429\,t^3 + 227\,977\,903\,707\,462\,860\,t^2 \\ & - 1\,276\,676\,260\,761\,792\,016\,t + 3\,909\,821\,048\,582\,988\,049]\end{aligned}$$

for

$$\begin{aligned} j_1 &= 0, & (a_1, b_1, c_1) &= (161\,414\,428, -1\,129\,900\,996, \quad 7\,909\,306\,972)\,, \\ j_2 &= 1, & (a_2, b_2, c_2) &= (\ 80\,707\,214, \qquad 0, -3\,954\,653\,486)\,, \\ j_3 &= 1, & (a_3, b_3, c_3) &= (121\,060\,821, \qquad 0, -5\,931\,980\,229)\,. \end{aligned}$$

Each of the three polynomials leads to an upper bound of 4 for the rank of the geometric Picard group. All three have roots of absolute value 7, only. Applying the Artin-Tate formula, we find the following.

Table 1. Hypothetical ranks and discriminants

polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
Φ_1	$\mathbb{F}_7$	2	58
	$\mathbb{F}_{49}$	2	4524
Φ_2	$\mathbb{F}_7$	1	4
	$\mathbb{F}_{49}$	2	1996
Φ_3	$\mathbb{F}_7$	1	6
	$\mathbb{F}_{49}$	2	2997

The polynomial Φ_1 is excluded by the field extension condition as the two values in the rightmost column define different square classes. On the other hand, the rank-1 condition excludes Φ_2 and Φ_3 since we have a degree-2 example. Thus, relative to the Tate conjecture, geometric Picard rank 2 is proven.

Example 28 (continuation). On the same surface, point counting over $\mathbb{F}_{7^{10}}$ leads to a number of 79 792 267 067 823 523. For the characteristic polynomial of the Frobenius, we find the two candidates Φ_4, Φ_5,

$$\begin{aligned}\Phi_i(t) \,=\, & t^{22} - 16\,t^{21} + 140\,t^{20} - 1\,029\,t^{19} + 5\,831\,t^{18} - 36\,015\,t^{17} + 268\,912\,t^{16} \\ & - 1\,882\,384\,t^{15} + 11\,529\,602\,t^{14} - 46\,118\,408\,t^{13} + 40\,353\,607\,t^{12} + a_i t^{11} \\ & + (-1)^{j_i}[\,-1\,977\,326\,743\,t^{10} + 110\,730\,297\,608\,t^9 - 1\,356\,446\,145\,698\,t^8 \\ & + 10\,851\,569\,165\,584\,t^7 - 75\,960\,984\,159\,088\,t^6 + 498\,493\,958\,544\,015\,t^5 \\ & - 3\,954\,718\,737\,782\,51\,9t^4 + 34\,196\,685\,556\,119\,429\,t^3 \\ & - 227\,977\,903\,707\,462\,860\,t^2 + 1\,276\,676\,260\,761\,792\,016\,t \\ & - 3\,909\,821\,048\,582\,988\,049]\end{aligned}$$

for $j_4 = 0$, $a_4 = 0$, $j_5 = 1$, and $a_5 = 564\,950\,498$. Φ_4 corresponds to the minus sign in the functional equation, Φ_5 to the case of the plus sign. Both candidates, according to the Tate conjecture, imply geometric Picard rank 2.

To decide which sign is the right one, one would first check the absolute values of the roots. Unfortunately, both polynomials only have roots of absolute value 7. The Artin-Tate formula provides the picture given in the table below.

Table 2. Hypothetical ranks and discriminants

polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
Φ_4	$\mathbb{F}_7$	1	2
	$\mathbb{F}_{49}$	2	997
Φ_5	$\mathbb{F}_7$	2	55
	$\mathbb{F}_{49}$	2	4125

Thus, Φ_5 is excluded by the field extension condition. The minus sign in the functional equation is correct.

Example 29 (A $K3$ *surface of degree* 8 *over* $\mathbb{F}_3$*).* Consider the complete intersection V of the three quadrics in $\mathbf{P}^5_{\mathbb{F}_3}$, given by q_1, q_2, and q_3,

$$\begin{aligned} q_1 &:= -xy + xz + xu + xv + xw - y^2 - yz - yv + yw \\ &\qquad\qquad + z^2 + zu + zw - u^2 - uw + v^2 + w^2 \,, \\ q_2 &:= -x^2 + xy + xz - xv + xw - y^2 + yz - yu - yv \\ &\qquad\qquad + yw - zu - zw + uw - v^2 + vw \,, \\ q_3 &:= xu - yz \,. \end{aligned}$$

V is smooth and, therefore, a $K3$ surface. As q_3 is of rank 4, V carries an elliptic fibration. There are precisely 14, 98, 794, 6 710, 59 129, 532 460, 4 784 990, 43 049 510, and 387 374 024 points over $\mathbb{F}_3, \ldots, \mathbb{F}_{3^9}$. From these data, let us check whether one can prove $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_3}) = 2$.

Assume that the characteristic polynomial of the Frobenius has more than two zeroes of the form 3 times a root of unity. Then, [6, Algorithm 22] leaves us with five polynomials $\Psi_1, \ldots, \Psi_5$,

$$\begin{aligned} \Psi_i(t) = {} & t^{22} - 4\,t^{21} + 27\,t^{18} + 81\,t^{17} - 243\,t^{16} + 6\,561\,t^{13} + a_i t^{12} + b_i t^{11} + c_i t^{10} \\ & + (-1)^{j_i}[531\,441\,t^9 - 14\,348\,907\,t^6 + 43\,046\,721\,t^5 + 129\,140\,163\,t^4 \\ & \qquad - 13\,947\,137\,604\,t + 31\,381\,059\,609] \end{aligned}$$

for

$$\begin{aligned} j_1 &= 0, & (a_1, b_1, c_1) &= (-59\,049, \ 236\,196, -531\,441)\,, \\ j_2 &= 0, & (a_2, b_2, c_2) &= (\quad 0, -118\,098, \quad 0)\,, \\ j_3 &= 0, & (a_3, b_3, c_3) &= (\ 19\,683, -236\,196, \ 177\,147)\,, \\ j_4 &= 1, & (a_4, b_4, c_4) &= (-59\,049, \quad 0, \ 531\,441)\,, \\ j_5 &= 1, & (a_5, b_5, c_5) &= (-39\,366, \quad 0, \ 354\,294)\,. \end{aligned}$$

Applying the Artin-Tate formula to these polynomials, we obtain the following data.

Table 3. Hypothetical ranks and discriminants

polynomial	field	arithmetic Picard rank	$\#\mathrm{Br}(V)\lvert\Delta\rvert$
Ψ_1	$\mathbb{F}_3$	2	24
	$\mathbb{F}_9$	4	1116
Ψ_2	$\mathbb{F}_3$	2	27
	$\mathbb{F}_9$	2	81
Ψ_3	$\mathbb{F}_3$	2	28
	$\mathbb{F}_9$	2	112
Ψ_4	$\mathbb{F}_3$	3	144
	$\mathbb{F}_9$	4	1152
Ψ_5	$\mathbb{F}_3$	1	2
	$\mathbb{F}_9$	2	65

Observe that an elliptic surface of Picard rank 2 automatically has a discriminant of the form $(-n^2)$ for n an integer. We may therefore exclude everything except for Ψ_4. Note that Ψ_2 is, in addition, incompatible with the field extension condition.

Thus, using the numbers of points over the fields up to $\mathbb{F}_{3^9}$, we only obtain that, either the geometric Picard rank is equal to 2, or Ψ_4 is the characteristic polynomial of the Frobenius in which case it is 4.

Example 30 (continuation). The number of points over $\mathbb{F}_{3^{10}}$ is 34 871 648 631. This additional information reproduces Ψ_1 and Ψ_4 as possible characteristic polynomials of Frob. Consequently, the minus sign holds in the functional equation and the geometric Picard rank of V is equal to 4.

8 Statistics

We tested the Artin-Tate conditions on samples of $K3$ surfaces of degrees 2, 4, 6, and 8. The possibilities of computing are limited by the fact that point counting over large finite fields is slow. In degree 2, decoupling [6, Algorithm 17] (see also [5]) leads to a substantial speed-up. In higher degrees, one may focus on elliptic $K3$ surfaces and exploit the fact that point counting on the elliptic fibers is fast. The numbers and particularities of the examples treated are listed in Table 4.

Table 4. Numbers of examples computed

	$p = 2$	$p = 3$	$p = 5$	$p = 7$
$d = 2$	1000 rand	1000 rand	1000 dec	1000 dec
$d = 4$	1000 rand	1000 ell		
$d = 6$	1000 rand	1000 ell		
$d = 8$	1000 rand	1000 ell		

dec = decoupled, ell = elliptic, rand = random

The remaining parameters of the surfaces were chosen by a random number generator. We stored the equations and the numbers of points over $\mathbb{F}_p, \ldots, \mathbb{F}_{p^{10}}$ in a file.

Results I. Point counting until $\mathbb{F}_{p^9}$. First, we tried to show that the geometric Picard-rank was equal to 2 only using the numbers of rational points over $\mathbb{F}_p, \ldots, \mathbb{F}_{p^9}$. I.e., we applied [6, Algorithm 22]. This algorithm produces a list of hypothetical Weil polynomials for each surface. If one is able to exclude all of them then, relative to the Tate conjecture, rank 2 is proven. To exclude a particular polynomial, we first checked whether the roots are of absolute value p. When a surface was known to be elliptic over $\mathbb{F}_p$, we checked in addition that the predicted Picard rank over $\mathbb{F}_p$ was at least equal to 2.

Then, we applied the Artin-Tate conditions to the polynomials. We checked the field extension condition and the rank-1 condition. For surfaces known to be elliptic over $\mathbb{F}_p$, we observed the fact that arithmetic Picard rank 2 forces the discriminant to be minus a perfect square. The results are summarized in Table 5.

Table 5. Distribution of the remaining hypothetical characteristic polynomials

	Number of polynomials	0	1	2	3	4	5	6
$d = 2,\ p = 2$	without	84	479	312	89	21	12	3
	with A-T conditions	149	598	218	28	7	0	0
$d = 2,\ p = 3$	without	116	480	285	88	24	4	3
	with A-T conditions	214	573	193	20	0	0	0
$d = 2,\ p = 5$	without	85	581	209	96	25	4	0
	with A-T conditions	158	651	169	20	2	0	0
$d = 2,\ p = 7$	without	92	534	232	98	37	7	0
	with A-T conditions	214	611	154	21	0	0	0
$d = 4,\ p = 2$	without	40	532	303	87	29	8	1
	with A-T conditions	81	638	249	27	5	0	0
$d = 4,\ p = 3$	without	22	669	242	57	9	1	0
	with A-T conditions	53	785	161	1	0	0	0
$d = 6,\ p = 2$	without	39	549	312	70	22	6	2
	with A-T conditions	83	645	257	14	1	0	0
$d = 6,\ p = 3$	without	16	713	217	47	7	0	0
	with A-T conditions	50	797	148	5	0	0	0
$d = 8,\ p = 2$	without	25	657	268	38	8	4	0
	with A-T conditions	29	723	239	5	4	0	0
$d = 8,\ p = 3$	without	12	720	236	27	4	1	0
	with A-T conditions	20	803	175	2	0	0	0

Results II. Point counting until $\mathbb{F}_{p^{10}}$. Using data up to $\mathbb{F}_{p^{10}}$, one obtains two hypothetical Weil polynomials for each of the surfaces. The two polynomials correspond to the possible signs in the functional equation (1). One has to exclude one of them. For this, we first checked the absolute values of the roots. For surfaces known to be elliptic over $\mathbb{F}_p$, we then tested whether the predicted arithmetic Picard rank is at least 2. Then, we applied the Artin-Tate conditions. We checked the field extensions and the rank-1 condition. For elliptic surfaces, supposed to be of arithmetic Picard rank 2, we tested, in addition, whether the predicted discriminant was minus a square.

Table 6 shows the number of surfaces with known signs. In the case that the sign is not known, we computed the numbers of points predicted over further extensions of $\mathbb{F}_p$. Comparing these numbers for both hypothetical polynomials indicates whether further point counting would lead to a decision of the sign. We count how often which fields had to be considered in order to decide the sign.

Table 6. Sign decision in the functional equation

p	2	3	5	7	2	3	2	3	2	3
d	2	2	2	2	4	4	6	6	8	8
Known signs without A-T	768	843	864	869	761	876	790	888	822	897
Known signs using A-T	863	940	940	961	863	943	868	933	867	944
Remaining unknown signs	137	60	60	39	137	57	132	67	133	56
Data up to $\mathbb{F}_{p^{11}}$ insufficient	84	23	15	12	69	19	77	25	72	21
Data up to $\mathbb{F}_{p^{12}}$ insufficient	41	11	2	1	39	3	42	11	47	7
Data up to $\mathbb{F}_{p^{13}}$ insufficient	22	5	1	0	24	2	20	2	24	2
Data up to $\mathbb{F}_{p^{14}}$ insufficient	13	2	0	0	12	0	13	1	8	0
Data up to $\mathbb{F}_{p^{15}}$ insufficient	7	0	0	0	8	0	7	0	5	0
Data up to $\mathbb{F}_{p^{16}}$ insufficient	4	0	0	0	3	0	2	0	4	0
Data up to $\mathbb{F}_{p^{17}}$ insufficient	4	0	0	0	2	0	2	0	0	0
Data up to $\mathbb{F}_{p^{18}}$ insufficient	4	0	0	0	0	0	1	0	0	0
Data up to $\mathbb{F}_{p^{19}}$ insufficient	2	0	0	0	0	0	1	0	0	0
Data up to $\mathbb{F}_{p^{20}}$ insufficient	0	0	0	0	0	0	0	0	0	0

Using these data, we repeated our attempt to prove that the geometric Picard rank is equal to 2. More precisely, we checked whether only two roots of the characteristic polynomial are of the form p times a root of unity. The numbers of surfaces for which we succeeded are listed in Table 7.

Table 7. Numbers of rank-2 cases using $\mathbb{F}_{p^{10}}$-data

		rank 2 proven	rank 2 possible
$p=2,\ d=2$	without	271	330
	with A-T conditions	278	301
$p=3,\ d=2$	without	397	460
	with A-T conditions	409	428
$p=5,\ d=2$	without	353	425
	with A-T conditions	360	382
$p=7,\ d=2$	without	460	511
	with A-T conditions	464	476
$p=2,\ d=4$	without	132	197
	with A-T conditions	138	163
$p=3,\ d=4$	without	79	114
	with A-T conditions	79	81
$p=2,\ d=6$	without	145	183
	with A-T conditions	152	163
$p=3,\ d=6$	without	74	101
	with A-T conditions	74	81
$p=2,\ d=8$	without	65	93
	with A-T conditions	65	74
$p=3,\ d=8$	without	23	47
	with A-T conditions	23	25

Conclusion. The Artin-Tate conditions usually halve the number of cases with unknown signs. Furthermore, they double the number of cases where geometric Picard rank 2 may be proven only using data up to $\mathbb{F}_{p^9}$. Comparing Table 5 with Table 7, we see, however, that still only about one half of the cases with Picard rank 2 may be detected when counting until $\mathbb{F}_{p^9}$.

Remark 31. Let us finally mention that the Artin-Tate conditions came to us as a big surprise. It is astonishing that the Artin-Tate formula may be incompatible with itself under field extensions. Thus, it seems not entirely unlikely that there are even more constraints and one can still do better.

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