On the distribution of small points on abelian and toric varieties

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ABSTRACT

We give a unified proof for a theorem of S. Zhang and a theorem of Yu. Bilu. First, if A is a polarized abelian variety over $\overline{\mathbb{Q}}$ and $X \subset A$ a closed subvariety containing a generic sequence of points with Néron-Tate height tending to zero then X is torsion. Secondly, if $X \subset \mathbf{P}^n_{\overline{\mathbb{Q}}}$ contains a generic sequence of points whose naive heights tend to zero then X is preperiodic under coordinate-wise formation of d-th powers for every $d \geqslant 2$.

1. Introduction

Classical results. In 1972, J.-P. Serre proved the following remarkable result.

THEOREM 1.1 (Serre). *Let K be an algebraic number field and E be an elliptic curve over K without complex multiplication. Then, for almost every prime number l, the Galois group* $\mathrm{Gal}(\overline{K}/K)$ *acts transitively on the l-torsion points of E.*

Notation 1.2. For $x \in E(\overline{K})$, we denote by δ_x the Dirac measure associated to its Galois orbit. I.e., if $x \in E(F)$ for some number field $F \supseteq K$ then $\delta_x(\varphi) := \frac{1}{\sharp\,\mathrm{Gal}(F/K)} \sum\limits_{\sigma\colon F \hookrightarrow \mathbb{C}} \varphi(\sigma(x))$ for each $\varphi \in C^0(E(\mathbb{C}))$.

1.3. In this language, Serre's theorem immediately implies the following.

COROLLARY. *Let K be an algebraic number field and E be an elliptic curve over K without complex multiplication. Further, let $(x_i)_{i\in\mathbb{N}}$ be a sequence of torsion points in E not containing any constant subsequence.*

Then, the associated sequence $(\delta_{x_i})_{i\in\mathbb{N}}$ of measures on $E(\mathbb{C})$ converges in the weak sense to the Haar measure normalized to volume one.

1.4. In 1997, L. Szpiro, E. Ullmo and S. Zhang [SUZ] showed a generalization from torsion points (which are of Néron-Tate height zero) to points of a small positive height. Their result is as follows.

THEOREM 1.5 (Szpiro, Ullmo, Zhang). *Let A be an abelian variety over a number field K and $(x_i)_{i\in\mathbb{N}}$ be a sequence of closed points in A. Assume that $(x_i)_{i\in\mathbb{N}}$ converges to the generic point of A in the sense of the Zariski topology. Suppose further that* $\lim\limits_{i\to\infty} \mathrm{h}_{NT}(x_i) = 0$.

Then, the associated sequence $(\delta_{x_i})_{i\in\mathbb{N}}$ of measures on $A(\mathbb{C})$ converges in the weak sense to the Haar measure of volume one.

1.6. Somewhat surprisingly, there is a completely parallel theorem for the naive height on projective space due to Yu. Bilu [Bi97].

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Theorem 1.7 (Bilu). *Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of closed points in $\mathbf{P}^n_{\mathbb{Q}}$. Assume that $(x_i)_{i\in\mathbb{N}}$ converges to the generic point in the sense of the Zariski topology. Assume further that $\lim\limits_{i\to\infty} \mathrm{h}_{\mathrm{naive}}(x_i) = 0$.*

Then, the associated sequence $(\delta_{x_i})_{i\in\mathbb{N}}$ of measures on $\mathbf{P}^n(\mathbb{C})$ converges weakly to the Haar measure of volume one on $(S^1)^n \subset \mathbb{C}^n \subset \mathbf{P}^n(\mathbb{C})$.

Remark 1.8. For $n = 1$, Yu. Bilu proved this result already in 1988 [Bi88].

Canonical heights.

Definition 1.9. Let P be a projective variety over $\mathbb{Q}$ equipped with an ample invertible sheaf $\mathscr{L} \in \mathrm{Pic}(P)$. Further, we assume there is given a self-map $f\colon P \to P$ such that there is an isomorphism $\Phi\colon \mathscr{L}^{\otimes d} \stackrel{\cong}{\longrightarrow} f^*\mathscr{L}$.

Then, the *canonical* height function $\mathrm{h}_{f,\mathscr{L}}$ on P is given by

$$\mathrm{h}_{f,\mathscr{L}}(x) := \lim_{n\to\infty} \frac{1}{d^n} \mathrm{h}_{\mathscr{L}}\big(f^{(n)}(x)\big)$$

for $x \in P$ a closed point. Here, $f^{(n)}$ means the n-fold iteration of f. By $\mathrm{h}_{\mathscr{L}}$, we denote the height defined by the ample invertible sheaf $\mathscr{L}$.

Remark 1.10. The limit process defining $\mathrm{h}_{f,\mathscr{L}}$ is a generalization of the classical one for the Néron-Tate height [CS, Chapter VI, §4]. We will show in Example 1.15.ii) that the naive height is a canonical height, too.

Remark 1.11. The height $\mathrm{h}_{\mathscr{L}}$ is defined only up to a bounded summand. Nevertheless, the canonical height function $\mathrm{h}_{f,\mathscr{L}}$ is independent of the choice of a particular representative. Indeed, if $|\mathrm{h}^{(1)}_{\mathscr{L}}(x) - \mathrm{h}^{(2)}_{\mathscr{L}}(x)| \leqslant C$ for every $x \in P$ then

$$\left| \frac{1}{d^n}\,\mathrm{h}^{(1)}_{\mathscr{L}}\big(f^{(n)}(x)\big) - \frac{1}{d^n}\,\mathrm{h}^{(2)}_{\mathscr{L}}\big(f^{(n)}(x)\big) \right| \leqslant \frac{C}{d^n}\,.$$

An equidistribution result. The goal of the present article is to prove a common generalization of Theorems 1.5 and 1.7. We will give an Arakelovian approach for a situation covering the case of the Néron-Tate height as well as that of the naive height.

Situation 1.12. Let P be a regular projective variety over $\mathbb{Q}$ containing a group scheme $G \subseteq P$ as an open dense subset. Further, let a morphism $f\colon P \to P$ be given and assume that

i) the unit e is a repelling fixed point. I.e., all eigenvalues of the tangent map $T_ef\colon T_eG \to T_eG$ are of absolute value strictly bigger than 1.

ii) $f|_G$ is a group homomorphism $f|_G\colon G \to G$.

iii) There is some compact, Zariski dense subgroup $K \subseteq G(\mathbb{C})$ which is both backward and forward invariant under $f_{\mathbb{C}}$.

Finally, let $\mathscr{L} \in \mathrm{Pic}(P)$ be ample and $\Phi\colon \mathscr{L}^{\otimes d} \stackrel{\cong}{\longrightarrow} f^*\mathscr{L}$ be an isomorphism. Then, there is the canonical height $\mathrm{h}_{f,\mathscr{L}}$ on P.

Theorem 1.13 (Equidistribution on $P(\mathbb{C})$). *Let P, f, $\mathscr{L}$, and Φ be as in 1.12. Further, let $(x_i)_{i\in\mathbb{N}}$ be a sequence of closed points in P which is convergent to the generic point of P in the sense of the Zariski topology. Finally, suppose that $\mathrm{h}_{f,\mathscr{L}}(x_i) \to 0$.*

Then, the associated sequence $(\delta_{x_i})_{i\in\mathbb{N}}$ of measures on $P(\mathbb{C})$ converges in the weak sense to the measure τ on $P(\mathbb{C})$ being the zero measure on $P(\mathbb{C})\setminus K$ and the Haar measure of volume one on K.

Remark 1.14. For a general self-map f, iterating will lead to a fractal. Unfortunately, not much is known about the corresponding fractal heights.

Examples 1.15. The condition that e is a repelling fixed point is fulfilled, in particular, when G is commutative and $f\colon g\mapsto g^l$ is the homomorphism raising to the l-th power for $l\geqslant 2$. This includes the two standard cases. Namely,

i) let $P=G=A$ be an abelian variety and $\mathscr{L}\in\operatorname{Pic}(A)$ a symmetric, ample invertible sheaf. One has

$$f=[l]\colon A\to A$$

and puts $K=A(\mathbb{C})$. Then, independently of the choice of l, $\mathrm{h}_{f,\mathscr{L}}$ is the Néron-Tate height corresponding to $\mathscr{L}$.

ii) Let $P=\mathbf{P}^n, G=\mathbb{G}_m^{n+1}/\mathbb{G}_m\cong\mathbb{G}_m^n$ and $\mathscr{L}=\mathscr{O}(1)$. We have

$$f=[l]\colon(x_0,\dots,x_n)\mapsto(x_0^l,\dots,x_n^l)$$

and put $K:=U(1)^n$. Here, $\mathrm{h}_{f,\mathscr{L}}$ is the naive height.

iii) One may combine abelian varieties and projective spaces and consider a *split semi-abelian variety* $A\times\mathbf{P}^n$. Here, $G=A\times\mathbb{G}_m^n$, $K=A\times U(1)$, and $\mathscr{L}=\mathscr{S}\boxtimes\mathscr{O}(p)$ for $\mathscr{S}$ a symmetric, ample invertible sheaf on A. One may put $f:=[l]\times[l^2]$.

A. Chambert-Loir's equidistribution result [CL] for quasi-split semi-abelian varieties is easily deduced from Theorem 1.13.

iv) Consider a free abelian group N of finite rank and a finite rational polyhedral decomposition $\{\sigma_\alpha\}_\alpha$ of $N\otimes_{\mathbb{Z}}\mathbb{R}$. These data define a proper toric variety X.

The multiplication map $N\to N$, $n\mapsto ln$ induces a self-morphism $f=[l]\colon X\to X$. Further, consider a function $g\colon N\otimes_{\mathbb{Z}}\mathbb{R}\to\mathbb{R}$ which is strictly convex in the sense of [KKMS, Chapter I, Theorem 13]. Then, $\mathscr{L}:=\mathscr{F}_g$ is a T-invariant ample invertible sheaf on X. One has $[l]^*\mathscr{F}_g\cong\mathscr{F}_{lg}=\mathscr{F}_g^{\otimes l}$. In this situation, $P=X$, $G=T\cong\mathbb{G}_m^n$ is the Zariski dense torus in X, and $K\subset T$ is its maximal compact subgroup.

We have a canonical height on the toric variety X such that equidistribution is fulfilled for sequences which are generic and small.

Remark 1.16. The latter example clearly contains the case of the naive height on projective space. Note that it also contains the blow-up of $\mathbf{P}^2$ in one point and the canonical height defined by the ample divisor $nH-mE$ for $n,m\in\mathbb{N},n>m$.

Remark 1.17. For abelian varieties, our proof will coincide with that of Szpiro, Ullmo, and Zhang. In the case of the projective space, it seems to us that our proof is different from Bilu's.

2. The dynamics of f

Arakelovian interpretation of the canonical height. Let P, f, $\mathscr{L}$, and Φ be as in Definition 1.9. In order to understand the canonical height better, we have to use the language of Arakelov geometry [GS90].

First, recall that in Definition 1.9 we may replace $\mathrm{h}_{\mathscr{L}}$ by any other height function which differs from $\mathrm{h}_{\mathscr{L}}$ only by a bounded summand. For example, we may work with $\mathrm{h}'_{\mathscr{L}}(x) := \frac{1}{m}\,\mathrm{h}_{l^2}\big(i_{\mathscr{L}^{\otimes m}}(x)\big)$. Further, there is a projective model $\mathscr{P}$ of P over $\operatorname{Spec}\mathbb{Z}$ to which $\mathscr{L}^{\otimes m}$ extends. Indeed, $\mathscr{L}^{\otimes m}$ defines a closed embedding $i_{\mathscr{L}^{\otimes m}}\colon P \to \mathbf{P}^N$ and one may put $\mathscr{P} := \overline{i_{\mathscr{L}^{\otimes m}}(P)}$.

Then, $\underline{\mathscr{L}} := \mathscr{O}(1)|_{\mathscr{P}}$ is an extension of $\mathscr{L}^{\otimes m}$ to $\mathscr{P}$. We equip $\underline{\mathscr{L}}$ with the restriction of the Fubini-Study metric. This guarantees that, for every closed point $x \in P$,

$$\mathrm{h}_{\mathscr{L},0}(x) := \mathrm{h}_{\mathscr{L}}(x) = \frac{1}{m}\,\mathrm{h}_{(\underline{\mathscr{L}},\|\,.\|)}(x)\,.$$

Here, $\mathrm{h}_{(\underline{\mathscr{L}},\|\,.\|)}$ is the absolute height defined [BGS] by the smooth Hermitian line bundle $(\underline{\mathscr{L}},\|\,.\|)$.

The recursion is given by $\mathrm{h}_{\mathscr{L},i+1}(x) := \frac{1}{d}\,\mathrm{h}_{\mathscr{L},i}\big(f(x)\big)$. Unfortunately, the isomorphism $\Phi\colon \mathscr{L}^{\otimes d} \xrightarrow{\cong} f^*\mathscr{L}$ does not extend to $\mathscr{P}$. If it would then we could put $\|\,.\|_0 := \|\,.\|$ and, recursively,

$$\|\,.\|_i := \Phi^{-1} f^* \|\,.\|_{i-1}^{1/d}\,.$$

This would then lead to $\mathrm{h}_{\mathscr{L},i}(x) = \frac{1}{m}\,\mathrm{h}_{(\underline{\mathscr{L}},\|\,.\|_i)}(x)$.

2.1. To make the idea above precise, one has to work in the more flexible context of adelic Picard groups. This is a theory due to S. Zhang [Zh95a]. In an appendix, we will recall it shortly and fix notation.

There is the adelic metric $\|\,.\|_\sim^{(0)}$ on $\mathscr{L}^{\otimes m}$ induced by $\underline{\mathscr{L}}$,

$$(\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(0)}) := i_{\mathscr{P}}(\underline{\mathscr{L}}, \|\,.\|_0)\,.$$

Since $\underline{\mathscr{L}}$ is very ample and $\|\,.\|_0$ is a restriction of the Fubini-Study metric, we see that $(\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(0)}) \in C_P^{\geqslant 0}$. Further, we define, recursively,

$$\|\,.\|_\sim^{(i)} := (\Phi^{\otimes m})^{-1} f^* \big[\|\,.\|_\sim^{(i-1)}\big]^{\frac{1}{d}}\,.$$

We have $(\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(i)}) \in C_P^{\geqslant 0}$ for every $i \in \mathbb{N}$. We also observe that the isomorphism Φ does extend to an open neighbourhood of the generic fiber of $\mathscr{P}$. Therefore, the sequence $(\|\,.\|_\sim^{(i)})_{i\in\mathbb{N}}$ is actually constant at all but finitely many valuations. Furthermore,

$$\delta\big((\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(i)}), (\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(i+1)})\big) \leqslant \frac{1}{d^i}\cdot\delta\big((\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(0)}), (\mathscr{L}^{\otimes m}, \|\,.\|_\sim^{(1)})\big)\,.$$

This means, for every $\nu \in \mathrm{Val}(\mathbb{Q})$, the sequence $(\|\,.\|_\nu^{(i)})_{i\in\mathbb{N}}$ of metrics is uniformly convergent. The limit $\|\,.\|_\sim$ clearly fulfills all the requirements of Definition A.4. I.e., $\|\,.\|_\sim$ is an adelic metric on $\mathscr{L}^{\otimes m}$. Even more,

$$\overline{\mathscr{L}} := (\mathscr{L}^{\otimes m}, \|\,.\|_\sim) \in \overline{C_P^{\geqslant 0}}\,.$$

Finally, we have $\mathrm{h}_{f,\mathscr{L}} = \frac{1}{m}\,\mathrm{h}_{\overline{\mathscr{L}}} = \frac{1}{m}\,\mathrm{h}_{(\mathscr{L}^{\otimes m},\|\,.\|_\sim)}$.

Lemma 2.2. *f is automatically finite of degree $d^{\dim P}$.*

Proof. P is a projective variety over $\mathbb{Q}$ and $f\colon P \to P$ is a morphism such that $\mathscr{L}^{\otimes d} \cong f^*\mathscr{L}$ for some ample $\mathscr{L} \in \operatorname{Pic}(P)$. In particular, $f^*\mathscr{L}$ is ample which implies f is quasi-finite. As f is a projective morphism, this is sufficient for finiteness. Further, we have $\deg_{\mathscr{L}} P \neq 0$ and $\deg_{f^*\mathscr{L}} P = \deg_{\mathscr{L}^{\otimes d}} P = d^{\dim P} \deg_{\mathscr{L}} P$. $\square$

Lemma 2.3. i) $p_{\mathbb{Z}}^P\big((\mathscr{L}^{\otimes m}, \|\,.\|_\sim), \ldots, (\mathscr{L}^{\otimes m}, \|\,.\|_\sim)\big) = 0$,

ii) $e_1(h_{\overline{\mathscr{L}}}) = \ldots = e_{\dim P+1}(h_{\overline{\mathscr{L}}}) = 0$.

Proof. i) For every $i \in \mathbb{N}$, write

$$P_i := p^P_{\mathrm{Z}}\big((\mathscr{L}^{\otimes m}, \|\,.\|^{(i)}_\sim), \ldots, (\mathscr{L}^{\otimes m}, \|\,.\|^{(i)}_\sim)\big)\,.$$

Then, since f is of degree $d^{\dim P}$,

$$d^{\dim P} P_{i-1} := p^P_{\mathrm{Z}}\big((f^*\mathscr{L}^{\otimes m}, f^*\|\,.\|^{(i-1)}_\sim), \ldots, (f^*\mathscr{L}^{\otimes m}, f^*\|\,.\|^{(i-1)}_\sim)\big) = 0\,.$$

Applying the isomorphism $\Phi^{\otimes m}$ and dividing each of the $(\dim P + 1)$ operators by d yields

$$\frac{P_{i-1}}{d} = p^P_{\mathrm{Z}}\big((\mathscr{L}^{\otimes m}, \|\,.\|^{(i)}_\sim), \ldots, (\mathscr{L}^{\otimes m}, \|\,.\|^{(i)}_\sim)\big) = P_i\,.$$

$(P_i)_{i\in\mathbb{N}}$ is, therefore, a zero sequence.

ii) There is a constant C such that $\mathrm{h}_{\mathscr{L},0}(x) > C$ for every $x \in P$. Consequently, $\mathrm{h}_{\mathscr{L},i}(x) > C/d^i$ and $\mathrm{h}_{\overline{\mathscr{L}}}(x) \geqslant 0$ for every $x \in P$. This means $e_{\dim P+1}(h_{\overline{\mathscr{L}}}) \geqslant 0$. Theorem A.14 implies the claim. □

Remark 2.4. As the construction of $\|\,.\|_\sim$ depends on the iterated pull-backs under f, one has to understand the dynamics of the system $(f^i)_{i\geqslant 1}$. For our purposes, it will be sufficient to pay attention to the infinite place. I.e., to the dynamics of that system on $P(\mathbb{C})$.

Analyzing the dynamics of f. Let us analyze the special situation described in 1.12.

i) $G \subseteq P$ is an open dense subset being a group scheme. The self-map f has a restriction $m := f|_G\colon G \to G$ which is a homomorphism of group schemes. Therefore, $\ker m$ is a group scheme of finite order $d^{\dim P}$.

As f is surjective, $\operatorname{im} m$ is a priori Zariski dense. Being a homomorphism, m is surjective, too.

There is a compact subgroup $K \subseteq G(\mathbb{C})$ which is both forward and backward invariant under $m_{\mathbb{C}}$. In particular, all the finite groups $\ker(m_{\mathbb{C}} \circ \ldots \circ m_{\mathbb{C}})$ are contained in K.

ii) All eigenvalues of the tangent map $T_e m$ have absolute value strictly bigger than 1. Therefore, on $G(\mathbb{C})$, there is a left invariant Riemannian metric μ such that $q := \|(T_e m_{\mathbb{C}})^{-1}\|_{\max} < 1$.

Indeed, based on the Jordan normal form, one easily finds a Hermitian scalar product on $T_e G(\mathbb{C})$ satisfying the analogous inequality. We take its real part. By transport of structure, we find a left invariant Riemannian metric μ on $G(\mathbb{C})$.

We will denote by δ the metric on $G(\mathbb{C})$ given by the lengths of the μ-shortest paths.

Convention 2.5. In order to simplify notation, we will usually write f and m instead of $f_{\mathbb{C}}$ and $m_{\mathbb{C}}$ when there is no danger of confusion.

Lemma 2.6. a) *m induces a diffeomorphism $\overline{m}\colon G(\mathbb{C})/K \to G(\mathbb{C})/K$.*

b) *$\overline{m}$ is expanding for the Riemannian metric $\overline{\mu}$ on $G(\mathbb{C})/K$ induced by μ. More precisely,*

$$\overline{\delta}\big(\overline{m}_{\mathbb{C}}(z_1), \overline{m}_{\mathbb{C}}(z_2)\big) \geqslant \frac{1}{q}\,\overline{\delta}(z_1, z_2)$$

for any $z_1, z_2 \subseteq G(\mathbb{C})/K$.

Proof. a) $\overline{m}$ is well-defined since K is forward invariant under m. Backward invariance of K implies that $\overline{m}$ is even an injection. Further, as $m\colon G(\mathbb{C}) \to G(\mathbb{C})$ is a surjection, the same is then true for $\overline{m}$. Finally, the tangent map $T_e\overline{m}\colon T_e G(\mathbb{C})/K \to T_e G(\mathbb{C})/K$ is expanding, too. In particular, $T_e\overline{m}$ is invertible. This yields that $\overline{m}$ is a local diffeomorphism.

b) The preimage $\overline{m}^{-1}(w)$ of a path w in $G(\mathbb{C})/K$ is a path, too. All tangent maps of m are expanding by a factor $\geqslant \frac{1}{q}$. Therefore, $\ell\big(\overline{m}^{-1}(w)\big) \leqslant q \cdot \ell(w)$. This implies the claim. □

Corollary 2.7. *Let $\overline{G(\mathbb{C})} = G(\mathbb{C}) \cup \{\infty\}$ be the one-point compactification of $G(\mathbb{C})$. Then, for each $x \in G(\mathbb{C})\backslash K$, the sequence $(x_i)_{i\in\mathbb{N}}$ given by $x_0 = x$ and $x_{i+1} = m(x_i)$ converges to ∞. In other words, K is the Julia set of the dynamical system $(m^i)_{i\geqslant 1}$ on $G(\mathbb{C})$.*

Proof. We have to show that, for each compact set $A \subseteq G(\mathbb{C})$, there are only finitely many $i \in \mathbb{N}$ such that $x_i \in A$. Replacing A by KA, if necessary, we may assume that A is left K-invariant. This means, we are given a compact set $\overline{A} \subseteq G(\mathbb{C})/K$ and have to verify that $\overline{x}_i \in \overline{A}$ for only finitely many values of i. As $\overline{m}\colon G(\mathbb{C})/K \to G(\mathbb{C})/K$ is expanding and $\overline{x}_0 \neq [K]$, we have $\overline{\delta}(\overline{x}_i, [K]) \to \infty$. However, $\overline{A}$ is a compact set. In particular, $\overline{A}$ is bounded. $\square$

Corollary 2.8. *The union over all the groups $\ker(m \circ \ldots \circ m)$ is dense in K.*

Proof. Otherwise, the topological closure of $K' := \bigcup_{i\in\mathbb{N}} \ker(\overbrace{m \circ \ldots \circ m}^{i \text{ times}})$ would be a compact Lie group properly contained in K. It is obvious that K' is forward and backward invariant under $f_{\mathbb{C}}$. Therefore, the map induced by f on the compact homogeneous space K/K' would be injective and expanding. $\square$

Notation 2.9. For $N \in \mathbb{R}$, we will write $U_N(K) := \{x \in G(\mathbb{C}) \,|\, \delta(x, K) \leqslant N\}$.

Proposition 2.10. *Let μ_i be the measure induced by the smooth differential form*

$$c_1(\mathscr{L}, \|\,.\,\|_\infty^{(i)}) \wedge \ldots \wedge c_1(\mathscr{L}, \|\,.\,\|_\infty^{(i)})$$

of type $(\dim P, \dim P)$ on $P(\mathbb{C})$. Then, the sequence $(\mu_i)_{i\in\mathbb{N}}$ converges weakly to the measure μ which is the zero measure on $P(\mathbb{C})\backslash K$ and the Haar measure of volume $\deg_{\mathscr{L}} P$ on K.

Proof. First step. Generalities.

Each μ_i is of volume $\deg_{\mathscr{L}} P$. Further, the sequence $(\mu_i)_{i\in\mathbb{N}}$ obeys the recursive low $\mu_{i+1} = \frac{1}{d^{\dim P}}\, m^*\mu_i$.

Second step. $\mu_i|_{P(\mathbb{C})\backslash K} \to 0$.

Let $g \in C_0(P(\mathbb{C}) \setminus K)$ be a continuous function with compact support. We consider the sequence $(g_i)_{i\in\mathbb{N}}$ given by $g_0 := g$ and $g_{i+1} := \frac{1}{d^{\dim P}}\, f_* g_i$. As usual, the push-forward of a function is given by summation over the fibers of the morphism f. By definition, $\mu_i(g) = \mu_0(g_i)$. Thus, let us show $\mu_0(g_i) \to 0$ for $i \to \infty$.

For this, we note that $|g|$ is bounded by some constant C. This implies $|g_i| \leqslant C$ for every i. Further, there is some $\varepsilon > 0$ such that $\operatorname{supp}(g) \subseteq P(\mathbb{C})\backslash U_\varepsilon(K)$. Consequently,

$$|\mu_i(g)| = |\mu_0(g_i)| \leqslant C \int\limits_{P(\mathbb{C})} \chi_{(P(\mathbb{C})\backslash U_{\varepsilon/q^i})}\, d\mu_0\,.$$

Here, χ_A denotes the characteristic function of a measurable set A. The integrand converges monotonically to $\chi_{P(\mathbb{C})\backslash G(\mathbb{C})}$ which is of integral zero. The theorem of Beppo Levi yields $\mu_i(g) \to 0$.

Third step. The assertion in general.

Let $g \in C_0(G(\mathbb{C}))$ be a continuous function with compact support. As above, we put $g_0 := g$ and $g_{i+1} := \frac{1}{d^{\dim P}}\, m_* g_i$. Thus, we have $\mu_i(g) = \mu_0(g_i)$. The sequence $(g_i)_{i\in\mathbb{N}}$ is uniformly bounded. Hence, it suffices to show that it converges pointwise to the constant

$$I := \frac{1}{\operatorname{vol} K} \int_K g\, d\rho\,.$$

We claim that the convergence is actually uniform in every compact set $A \subseteq G(\mathbb{C})$.

For this, we may assume without loss of generality that $A = U_N(K)$ for some $N \in \mathbb{R}$. Further, note that

$$g_i(x) = \frac{1}{\sharp \ker m^i} \sum_{k \in \ker m^i} g(ky)$$

for any y such that $f(y) = x$. Here, we clearly have $(m^i)^{-1}(A) \subseteq U_{Nq^i}(K)$ and $Nq^i \to 0$ for $i \to \infty$.

In addition, g is uniformly continuous on A and m is expanding. Thus, for each $\varepsilon > 0$, there is some $i \in \mathbb{N}$ such that, for two arbitrary points $x, x' \in A$, one can find y, y' such that $f(y) = x$, $f(y') = x'$, and $\delta(y, y') < \varepsilon$. Consequently, $(\max g_i|_A - \min g_i|_A)$ tends to zero for $i \to \infty$. Finally, for every $i \in \mathbb{N}$, I is the mean value of g_i on K. □

Some further observations. It is not true in general that the curvature current, given by $c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_\infty) := \frac{1}{2\pi i}\partial\overline{\partial} \log \|s\|^2$ for a non-zero rational section of $\mathscr{L}_\mathbb{C}$, vanishes on $P(\mathbb{C}) \setminus K$. Nevertheless, one has at least the following.

Lemma 2.11. *The curvature current $c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_\infty)$ has the properties below.*

a) *The restriction $c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_\infty)|_{G(\mathbb{C})}$ is left and right K-invariant.*

b) *It satisfies the equation $f^* c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_\infty) = d \cdot c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_\infty)$.*

Proof. a) By construction, $c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_{\infty,i})$ is invariant under $\ker(m^i_\mathbb{C})$. Thus, $c_1(\mathscr{L}_\mathbb{C}, \|\,.\|_\infty)$, being the weak limit of that sequence of currents, is invariant under $\bigcup_{i \in \mathbb{N}} \ker(m^i_\mathbb{C})$. Invariance under any $k \in K$ follows as $k_i \to k$ implies that the test forms $k_i\omega$ converge to $k\omega$ in the Schwartz space.

b) is clear. □

3. Perturbing almost semiample metrics

3.1. One of the most natural ways to produce a new adelic metric from an old one is to replace $\|\,.\|_\infty$ by $\|\,.\|_\infty \cdot \exp(-g)$ for some continuous function $g \in C(P(\mathbb{C}))$. For the new metric, we want to use the theorem of successive minima A.14 as a fundamental tool. It is, therefore, necessary to understand for which g this metric is *almost semiample*, i.e., the limit of a uniformly convergent sequence of metrics with positive curvature.

Notation 3.2. In this section, we will continue to use the notation of 1.12. Further, we will denote by S the set of all continuous functions $g \in P(\mathbb{C})$ such that $\|\,.\|_\infty \cdot \exp(-g)$ is almost semiample.

Lemma 3.3. a) *One has $C \in S$ for every constant C.*

b) *Let $g_1, g_2 \in S$ and $0 < a < 1$. Then, $ag_1 + (1-a)g_2 \in S$.*

c) *If $g \in S$ then $\frac{1}{d} f^* g \in S$.*

d) *Let $g \in S$ be a function such that $\operatorname{supp} g \subseteq G(\mathbb{C})$. Then, for each $k \in K$, one has $g \cdot k \in S$ for*

$$(g \cdot k)(x) := \begin{cases} g(kx) & \text{if } x \in G(\mathbb{C})\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

e) *S is closed under uniform convergence.*

Proof. e) is trivial.

a) $\|\,.\,\|_\infty$ is almost semiample. Thus, there is a sequence $(\|\,.\,\|^i_\infty)_{i\in\mathbb{N}}$ of smooth Hermitian metrics with strictly positive curvature on $\mathscr{L}_\mathbb{C}$ which is uniformly convergent to $\|\,.\,\|_\infty$. The metrics $\|\,.\,\|_{\infty,i}\cdot\exp(-C)$ have the same curvatures and converge uniformly to $\|\,.\,\|_\infty\cdot\exp(-C)$.

b) Let $(\|\,.\,\|'_i)_{i\in\mathbb{N}}$ and $(\|\,.\,\|''_i)_{i\in\mathbb{N}}$ be sequences uniformly convergent versus $\|\,.\,\|_\infty\cdot\exp(-g_1)$ and $\|\,.\,\|_\infty\cdot\exp(-g_2)$, respectively. Put $\|\,.\,\|_i := \|\,.\,\|'^{\,a}_i\cdot\|\,.\,\|''^{\,(1-a)}_i$. These metrics have positive curvature and, uniformly,

$$\|\,.\,\|_i \to \|\,.\,\|_\infty\cdot\exp\big(-ag_1-(1-a)g_2\big)\,.$$

c) Put $\|\,.\,\|'_\infty := \|\,.\,\|_\infty\cdot\exp(-g)$. Then,

$$\Phi^{-1}f^*\|\,.\,\|'^{\frac{1}{d}}_\infty = \Phi^{-1}f^*\|\,.\,\|^{\frac{1}{d}}_\infty\cdot\exp\Big(-\frac{1}{d}f^*g\Big) = \|\,.\,\|_\infty\cdot\exp\Big(-\frac{1}{d}f^*g\Big)\,.$$

Therefore, if $(\|\,.\,\|'_{\infty,i})_{i\in\mathbb{N}}$ is a sequence of metrics with positive curvature uniformly convergent to $\|\,.\,\|'_\infty$ then $(\Phi^{-1}f^*\|\,.\,\|'^{\frac{1}{d}}_{\infty,i})_{i\in\mathbb{N}}$ converges uniformly to $\|\,.\,\|_\infty\cdot\exp(-\frac{1}{d}f^*g)$.

d) We denote by $e_k\colon G(\mathbb{C})\to G(\mathbb{C})$ the multiplication by k from the left. By e), we may assume that $k\in\ker(m^j)$ for some $j\in\mathbb{N}$. To show $\|\,.\,\|_\infty\cdot\exp(-g\cdot k)$ is almost semiample, it suffices to consider the Hermitian metric $\|\,.\,\|^{d^j}_\infty\cdot\exp(-d^jg\cdot k)$ on $\mathscr{L}^{\otimes d^j}_\mathbb{C}$. Let us consider its restriction to $G(\mathbb{C})$, first. Iterated application of Φ induces an isomorphism

$$e_k^*\mathscr{L}^{\otimes d^j}_\mathbb{C}|_{G(\mathbb{C})} \cong e_k^*(f^j)^*\mathscr{L}_\mathbb{C}|_{G(\mathbb{C})} \cong (f^j)^*\mathscr{L}_\mathbb{C}|_{G(\mathbb{C})} \cong \mathscr{L}^{\otimes d^j}_\mathbb{C}|_{G(\mathbb{C})}\,. \tag{1}$$

Here, the isomorphism in the middle is canonical as $f^j\circ e_k = f^j$.

By construction 2.1, $\|\,.\,\|_\infty$ is the uniform limit of a sequence $(\|\,.\,\|^{(i)}_\infty)_{i\in\mathbb{N}}$ such that, for every $i\geqslant j$,

$$e_k^*\big[\|\,.\,\|^{(i)}_\infty\big]^{d^j} = \big[\|\,.\,\|^{(i)}_\infty\big]^{d^j}$$

under the identification (1). By consequence, $e_k^*\|\,.\,\|^{d^j}_\infty = \|\,.\,\|^{d^j}_\infty$, too, which shows

$$e_k^*\big[\|\,.\,\|^{d^j}_\infty\cdot\exp(-d^jg)\big] = \|\,.\,\|^{d^j}_\infty\cdot\exp(-d^jg\cdot k)\,.$$

Clearly, $\|\,.\,\|^{d^j}_\infty\cdot\exp(-d^jg\cdot k)$ is almost semiample on $G(\mathbb{C})$. In fact, almost semiampleness is preserved under holomorphic pull-back.

On the other hand, there is some neighbourhood U of $P(\mathbb{C})\setminus G(\mathbb{C})$ such that $g|_U = 0$. This implies $-d^jg\cdot k = 0$ and, therefore, $\|\,.\,\|^{d^j}_\infty\cdot\exp(-d^jg\cdot k) = \|\,.\,\|^{d^j}_\infty$ on U. In particular, $\|\,.\,\|^{d^j}_\infty\cdot\exp(-d^jg\cdot k)$ is almost semiample on U. The assertion follows from Sublemma 3.4, below. $\square$

Sublemma 3.4. *Let M be a projective complex manifold, $\mathscr{G}\in\operatorname{Pic}(M)$ be ample, and $\{U_1,\dots,U_n\}$ an open covering. Further, let $\|\,.\,\|_1,\dots,\|\,.\,\|_n$ be continuous Hermitian metrics on $\mathscr{G}|_{U_1},\dots,\mathscr{G}|_{U_n}$ such that, for some $\delta>0$, the sets $D_{1,\delta},\dots,D_{n,\delta}$, given by*

$$D_{i,\delta} := \{x\in U_i \mid \|\,.\,\|_{i,(x)}\leqslant(1+\delta)\cdot\min(\|\,.\,\|_{1,(x)},\dots,\|\,.\,\|_{n,(x)})\}\subseteq U_i$$

are compact. Assume, each $\|\,.\,\|_i$ is almost semiample. Then, the continuous Hermitian metric $\|\,.\,\| := \min_i\|\,.\,\|_i$ is almost semiample.

Proof. We may assume that $\mathscr{G}$ is very ample. For every i, let $(\|\,.\,\|_{ij})_{j\in\mathbb{N}}$ be a sequence of smooth and positively curved metrics on $\mathscr{G}|_{U_i}$ which converges uniformly to $\|\,.\,\|_i$. Put

$$\|\,.\,\|^-_j := \min(\|\,.\,\|_{1j},\dots,\|\,.\,\|_{nj})\,.$$

For $j\gg 0$, $\|\,.\,\|^-_j$ is a continuous Hermitian metric on the whole of $\mathscr{G}$. The sequence $(\|\,.\,\|^-_j)_j$ converges uniformly to $\|\,.\,\|$.

We choose an embedding $i\colon M \hookrightarrow \mathbf{P}^N$ such that $i^*\mathscr{O}(1) \cong \mathscr{G}$. Further, we extend the smooth metrics $\|\,.\|_{ij}$ to smooth metrics $\|\,.\|'_{ij}$ on $\mathscr{O}(1)$ which are defined on open subsets $W_i \supseteq U_i$ of $\mathbf{P}^N$.

Under the tautological action $\mathrm{PGL}_n(\mathbb{C}) \times \mathbf{P}^N \to \mathbf{P}^N$, each $\gamma \in \mathrm{PGL}_n(\mathbb{C})$ defines an automorphism $e_\gamma\colon \mathbf{P}^N \to \mathbf{P}^N$ such that there is a natural identification $e_\gamma^*\mathscr{O}(1) \cong \mathscr{O}(1)$. We find an open neighbourhood O of $e \in \mathrm{PGL}_n(\mathbb{C})$ such that, for every $\gamma \in O$ and every $i \in \{1, \ldots, n\}$, the pull-back $e_\gamma^*\,\|\,.\|'_{ij}$ is well-defined on $D_{i,\delta}$ and $e_\gamma^*\|\,.\|'_{ij}\,|_{U_i \cap e_\gamma^{-1}(W_i)}$ has strictly positive curvature in every point of $D_{i,\delta}$.

We claim that, for each $\gamma \in O$, the perturbation

$$\|\,.\|_j^{-,\gamma} := e_\gamma^*\big(\min(\|\,.\|'_{1j}, \ldots, \|\,.\|'_{nj})\big)|_M$$

of $\|\,.\|_j^-$ has a positive curvature current on each holomorphic curve inside M.

Indeed, this is a local statement. Let $x \in M$. Fix a holomorphic section $0 \neq s \in \Gamma(U, \mathscr{G})$ defined in some neighbourhood U of x. Then,

$$\begin{aligned} c_1(\mathscr{G}, \|\,.\|_j^{-,\gamma})|_U &= -dd^c \log\big(\|s\|_j^{-,\gamma}\big)^2 \\ &= dd^c\big(\max(-\log e_\gamma^*\|s\|'^2_{1j}, \ldots, -\log e_\gamma^*\|s\|'^2_{nj})\big). \end{aligned}$$

Observe that the maximum of a finite system of plurisubharmonic functions is plurisubharmonic, again.

Now, we use Sobolev's averaging procedure. For every non-negative smooth function $\varphi \neq 0$ on $\mathrm{PGL}_n(\mathbb{C})$ such that $\operatorname{supp}\varphi \subseteq O$, we get the approximation

$$\|\,.\|_j^{-,\varphi} := \int\limits_{\mathrm{PGL}_n(\mathbb{C})} \varphi(\gamma)\cdot\|\,.\|_j^{-,\gamma}\,d\rho_\gamma \Bigg/ \int\limits_{\mathrm{PGL}_n(\mathbb{C})} \varphi(\gamma)\,d\rho_\gamma$$

of $\|\,.\|_j^-$. Here, ρ is the left Haar measure on $\mathrm{PGL}_n(\mathbb{C})$.

Finally, consider a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of functions as above which converges weakly to the delta distribution δ_e. Every $\|\,.\|_j^{-,\varphi_k}$ is smooth and positively curved. The sequence $(\|\,.\|_k^{-,\varphi_k})_{k\in\mathbb{N}}$ converges uniformly to $\|\,.\|$. ◻

Corollary 3.5. *If $g_1, g_2 \in S$ then $g := \max\{g_1, g_2\} \in S$.*

Corollary 3.6. *Let $l \in \mathbb{N}$, $0 < a < 1$, and $0 \neq s \in \Gamma(P(\mathbb{C}), \mathscr{L}_\mathbb{C}^{\otimes l})$. Then,*

$$g := \max\{a \log\|s\|^l, 0\} \in S\,.$$

Proof. Consider the function $\overline{g}$ on $U := \{x \in P(\mathbb{C}) \mid \log\|s\|(x) > -1\}$ given by

$$\overline{g}(x) := a\log\|s\|^l(x)\,.$$

We have to show that the Hermitian metric

$$\|\,.\|'_\infty := \|\,.\|_\infty^l \cdot \exp(-\overline{g})$$

on $\mathscr{L}_\mathbb{C}^{\otimes l}|_U$ is the limit of a uniformly convergent sequence of smooth metrics with positive curvature. We put $\|\,.\|'_{\infty,i} := \|\,.\|^l_{\infty,i}/\Big(\|s\|^l_{\infty,i}\Big)^a$. It is clear that, uniformly, $\|\,.\|'_{\infty,i} \to \|\,.\|'_\infty$. Further,

$$\begin{aligned} c_1(\mathscr{L}_\mathbb{C}|_U, \|\,.\|'_{\infty,i}) &= dd^c(-\log\|s\|'^2_{\infty,i}) \\ &= dd^c\Big(-\log(\|s\|^l_{\infty,i})^{2(1-a)}\Big) \\ &= (1-a)l\cdot c_1(\mathscr{L}_\mathbb{C}|_U, \|\,.\|_{\infty,i}) > 0\,. \end{aligned}$$

□

Lemma 3.7. *Let $g \in S$ such that $\operatorname{supp} g \subseteq G(\mathbb{C})$. Consider the K-invariant function $\underline{g}$ given by*

$$\underline{g}(x) := \begin{cases} \max\limits_{k \in K} g(kx) & \text{if } x \in G(\mathbb{C})\,, \\ g(x) & \text{otherwise}\,. \end{cases}$$

Then, $\underline{g} \in S$, too.

Proof. $\underline{g}$ is the uniform limit of sequence $(g_i)_{i\in\mathbb{N}}$, defined by $g_i(x) := \max_{k\in\ker(m^i)} g(kx)$. □

Proposition 3.8. *Let $\varepsilon > 0$ and $U \subseteq G(\mathbb{C})$ be an open set containing K. Then, there is some non-negative K-invariant function $0 \neq g \in \mathbb{R}_+{\cdot}S$ such that*

i) $g(e) = 1$,

ii) $\max\limits_{x\in P(\mathbb{C})} g(x) \leqslant 1 + \varepsilon$,

iii) $\operatorname{supp} g \subseteq U$.

Proof. Put $D := P \setminus G$. For some $j \gg 0$, the coherent sheaf $\mathscr{L}^{\otimes j} \otimes \mathscr{I}_D$ has a section which does not vanish in $e \in G$. I.e., there is a section $s \in \Gamma(P, \mathscr{L}^{\otimes j})$ vanishing in D but not in e. Using Corollary 3.6, we see

$$\widetilde{g}_C := \max\left\{\frac{1}{2j}\log\|Cs\|^j, 0\right\} = \max\left\{\frac{1}{2j}\log\|s\|^j + \frac{\log C}{2j}, 0\right\} \in S$$

for every $C > 0$. It is clear that $\operatorname{supp}\widetilde{g}_C \subseteq G(\mathbb{C})$ is a compact set.

We put $A := \max_{x\in K} \frac{1}{2j}\log\|s\|^j(x)$ and $B := \max_{x\in P(\mathbb{C})} \frac{1}{2j}\log\|s\|^j(x)$. Then, we choose C such that $\log C + A > 0$ and $\frac{\log C + B}{\log C + A} \leqslant 1 + \varepsilon$. Further, let g_0 be the K-invariant function associated to $\widetilde{g}_C$. I.e.,

$$g_0 := \begin{cases} \max\limits_{k\in K} \widetilde{g}_C(kx) & \text{if } x \in G(\mathbb{C})\,, \\ g(x) & \text{otherwise}\,. \end{cases}$$

By construction, $g_0(e) > 0$ and $\max_{x\in P(\mathbb{C})} g_0(x)/g_0(e) \leqslant 1 + \varepsilon$. Further, $\operatorname{supp} g_0 \subseteq K{\cdot}\operatorname{supp}\widetilde{g}_C$ is compact. Lemma 3.7 shows that $g_0 \in S$.

Finally, we define a sequence $(g_i)_{i\in\mathbb{N}}$ of functions on $P(\mathbb{C})$ by putting, recursively, $g_{i+1} := \frac{1}{d} f^* g_i$. Then, one clearly has $g_i(e) = g_0(e) > 0$ and

$$\max_{x\in P(\mathbb{C})} g_i(x)/g_i(e) = \max_{x\in P(\mathbb{C})} g_0(x)/g_0(e) \leqslant 1 + \varepsilon$$

for every $i \in \mathbb{N}$. Lemma 3.3.c) implies that $g_i \in S$. As $\operatorname{supp} g_0$ is compact, there is some $N \in \mathbb{R}$ such that $\operatorname{supp} g_0 \subseteq U_N(K)$. This yields, by Lemma 2.6.b), that $\operatorname{supp} g_i \subseteq U_{Nq^i}(K)$. Therefore, the function g, given by $g(x) := g_i(x)/g_i(e)$ for some $i \gg 0$, has all the properties desired. □

4. Equidistribution

Definition 4.1. Let X be a projective variety over $\mathbb{Q}$. Then, a sequence $(x_i)_{i\in\mathbb{N}}$ of closed points on X is called *generic* if no infinite subsequence is contained in a proper closed subvariety of X.

Remark 4.2. In other words, $(x_i)_{i\in\mathbb{N}}$ is generic if it converges to the generic point with respect to the Zariski topology.

Definition 4.3. Let X be a projective variety over $\mathbb{Q}$ and $\overline{\mathscr{L}} \in \widetilde{\operatorname{Pic}}(X)$. Suppose that $e_{\dim X+1}(\mathrm{h}_{\overline{\mathscr{L}}}) = 0$. Then, a sequence $(x_i)_{i\in\mathbb{N}}$ of closed points on X is called *small* if $\mathrm{h}_{\overline{\mathscr{L}}}(x_i) \to 0$.

Lemma 4.4. *Let X be a projective variety over $\mathbb{Q}$ and $\overline{\mathscr{L}} \in C_X^{\geqslant 0}$. Assume $e_{\dim X+1}(\overline{\mathscr{L}}) = 0$.*

Then, $p_{\mathrm{Z}}^X(\overline{\mathscr{L}}, \ldots, \overline{\mathscr{L}}) = 0$ is equivalent to the existence of a sequence of closed points on X which is generic and small.

Proof. "$\Longrightarrow$" There are only countably many closed subvarieties $X_1, X_2, \ldots \subset X$. Further, Theorem A.14 implies $e_1(\mathrm{h}_{\overline{\mathscr{L}}}) = 0$. Therefore, we may choose a sequence $(x_i)_{i\in\mathbb{N}}$ of closed points on X such that $x_i \in X\backslash X_1\backslash X_2\backslash\ldots\backslash X_i$ and $\mathrm{h}_{\overline{\mathscr{L}}}(x_i) < \frac{1}{i}$. The sequence $(x_i)_{i\in\mathbb{N}}$ is generic and small.

"$\Longleftarrow$" Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of closed points on X which is generic and small. Let $Y \subset X$ be a closed subset of codimension one. Then, only finitely many of the x_i are contained in Y. Hence, $\inf_{x\in X\backslash Y} \mathrm{h}_{\overline{\mathscr{L}}}(x) = 0$. As this is true for every Y, we see that $e_1(\mathrm{h}_{\overline{\mathscr{L}}}) = 0$. Theorem A.14 yields $p_{\mathrm{Z}}^X(\overline{\mathscr{L}}, \ldots, \overline{\mathscr{L}}) = 0$. □

Lemma 4.5. *Let P, f, $\overline{\mathscr{L}}$, and Φ be as in 1.12. Denote by S the set of all continuous functions $g \in C(P(\mathbb{C}))$ such that $\|\,.\|_\infty \cdot \exp(-g)$ is almost semiample.*

Let $\varphi \in C(P(\mathbb{C}))$. Then, for every $\varepsilon > 0$, there exist a function $\varphi_1 \in C(P(\mathbb{C}))$ supported in $P(\mathbb{C})\backslash K$ and a function $\varphi_2 \in \mathbb{R}_+ \cdot S$ such that

$$\|\varphi - \varphi_1 - \varphi_2\|_{\max} < \varepsilon\,.$$

Proof. The set $T := \{h \in C(K) \mid h = \varphi|_K \text{ for some } \varphi \in \mathbb{R}_+ S\}$ fulfills all the assumptions of Sublemma 4.6, except for closedness under uniform convergence. Indeed, Lemma 3.3.a) and b) imply that assumptions i) and ii) are satisfied. Corollary 3.6 is general enough to guarantee assumption iii). Lemma 3.3.d) yields assumption iv). Finally, Corollary 3.5 implies that assumption v) is fulfilled.

Therefore, there is some $\varphi_2 \in \mathbb{R}_+ \cdot S$ such that $|\varphi(x) - \varphi_2(x)| < \frac{\varepsilon}{2}$ for every $x \in K$. A decomposition of the unit adapted to a suitable open covering $\{P(\mathbb{C})\backslash K, U\}$ of $P(\mathbb{C})$ yields some $\varphi_1 \in C(P(\mathbb{C}))$ such that $\operatorname{supp} \varphi_1 \subseteq P(\mathbb{C})\backslash K$ and

$$\big|\big(\varphi(x) - \varphi_2(x)\big) - \varphi_1(x)\big| < \varepsilon$$

for every $x \in P(\mathbb{C})$. □

Sublemma 4.6. *Let L be a compact topological group and $T \subseteq C(L)$ a set of continuous real-valued functions fulfilling the following conditions.*

i) *T contains all the constant functions.*

ii) *T is closed under addition and multiplication by positive constants.*

iii) *For each $x \in L$ such that $x \neq e$, there is some $g \in T$ such that $g(e) > g(x)$.*

iv) *For each $x \in L$ and $g \in T$, the shift $g \cdot x$ is in T, too.*

v) *If $g \in T$ then $g_+ \in T$ for $g_+(x) := \max\{g(x), 0\}$.*

vi) *T is closed under uniform convergence.*

Then, $T = C(L)$.

Proof. We fix a Haar measure ρ on L. Then, conditions ii,) iv) and vi) together imply the following.

(†) Let $h \in T$ and g be a non-negative, measurable, and bounded function. Then, for the convolution, we have $g{*}h \in T$.

Therefore, it suffices to show that, for each open set $U \subseteq L$ containing e, there is some non-negative function $g_U \in T$ such that $g_U \neq 0$ and $\operatorname{supp} g_U \subseteq U$. For this, by i), ii), and v), we only need a function $g \in T$ taking its maximum entirely in U.

Assume, for some open U_0, there would be no such function. Then, for every $g \in T$ taking its maximum in e, there is a non-empty set

$$A_g := \{y \in L \backslash U_0 \mid g(y) = g(e)\} \subseteq L \backslash U_0$$

where the maximum is taken, too. If $\bigcap_{g \in T} A_g = \emptyset$ then, by compactness, already a finite intersection $A_{g_1} \cap \ldots \cap A_{g_n}$ would be empty. Then, $g_1 + \ldots + g_n \in S$ had its maximum in U_0, only.

Consequently, $A_{U_0} := \bigcap_{g \in T} A_g$ is a non-empty set. Put $A := \{e\} \cup \bigcup\limits_{U \ni e, U \text{ open}} A_U$. By construction, this set has the property below.

(‡) Every function $g \in T$ taking its maximum in e is necessarily constant on A. Further, $A \subseteq L$ is the largest set containing e with this property.

The property (‡) implies that A is closed. Further, for $y \in A$ and $g \in T$, the shift $g{\cdot}y^{-1}$ has its maxima in yA. Therefore, $A = yA$ for every $y \in A$. This means that $A \subseteq L$ is a subgroup. Fix a Haar measure ρ_A on A and denote by $i\colon A \to L$ the natural inclusion. Property (‡) together with i), ii) and v) implies that there is a sequence of functions of T such that the associated distributions converge weakly to $i_*\rho_A$. Finally, property (†), together with vi), guarantees that every A-left invariant continuous function is an element of T.

Now, we use condition iii). Choose some $x_0 \in A$ different from e. There exists some $g \in T$ such that $g(e) > g(x_0)$. Shifting by an element of A, if necessary, we may assume that $g(e) = \max_{x \in A} g(x)$.

Define another continuous function M on L by $M(x) := -\max_{a \in A} g(ax)$. Obviously, M is A-invariant. Thus, $M \in T$. Consequently, $\widetilde{g} := g + M \in T$, too. However, $\widetilde{g}(x) \leqslant 0$ for every $x \in L$, $\widetilde{g}(e) = 0$, and $\widetilde{g}(x_0) < 0$ together form a contradiction to property (‡). $\square$

Proposition 4.7. *Let P, f, $\overline{\mathscr{L}}$, and Φ be as in 1.12. Further, let $(x_i)_{i \in \mathbb{N}}$ be a sequence of closed points on P which is generic and small. Assume, some non-negative $\varphi \in C(P(\mathbb{C}))$ fulfills either* $\operatorname{supp} \varphi \subseteq P(\mathbb{C}) \backslash K$ *or* $\varphi \in \mathbb{R}_+{\cdot}S$. *Then,*

$$\liminf_{i \to \infty} \int\limits_{P(\mathbb{C})} \varphi \, d\delta_{x_i} \geqslant \int\limits_{P(\mathbb{C})} \varphi \, d\tau \, .$$

Here, τ is the zero measure on $P(\mathbb{C}) \setminus K$ and the Haar measure of volume one on K.

Proof. For φ such that $\operatorname{supp} \varphi \subseteq P(\mathbb{C}) \backslash K$, the right hand side is zero. Thus, in this case, the assertion is clear.

Let $\varphi \in \mathbb{R}_+{\cdot}S$. Then, for any positive $\lambda \in \mathbb{R}$, let $\| \,.\, \|^\lambda_\sim$ be the adelic metric on $\mathscr{L}^{\otimes m}$ given by $\| \,.\, \|^\lambda_p := \| \,.\, \|_p$ for the non-archimedean valuations and by $\| \,.\, \|^\lambda_\infty := \| \,.\, \|_\infty \cdot \exp(-\lambda\varphi)$ for the archimedean valuation. For $\lambda \to 0$, the adelic metric $\| \,.\, \|^\lambda$ fulfills the assumptions of Theorem A.14. Further, one clearly has

$$\mathrm{h}_{(\mathscr{L}^{\otimes m}, \| \,.\, \|^\lambda)}(x_i) = \mathrm{h}_{\overline{\mathscr{L}}}(x_i) + \lambda \int\limits_{X(\mathbb{C})} \varphi \, d\delta_{x_i} \, .$$

As $(x_i)_{i\in\mathbb{N}}$ is generic and $\mathrm{h}_{\overline{\mathscr{L}}}(x_i)\to 0$, we obtain, according to the very definition of e_1,

$$\lambda\cdot\liminf_{i\to\infty}\int\limits_{X(\mathbb{C})}\varphi\,d\delta_{x_i}\geqslant e_1(\mathrm{h}_{(\mathscr{L}^{\otimes m},\|\,.\|^{\lambda})})\,. \tag{4}$$

On the other hand,

$$\begin{aligned}&\frac{1}{(\dim X+1)\,c_1(\mathscr{L}_{\mathbb{Q}}^{\otimes m})^{\dim X}}\,p_{\mathrm{Z}}^{X}\Big((\mathscr{L}^{\otimes m},\|\,.\|_{\sim}^{\lambda}),\ldots,(\mathscr{L}^{\otimes m},\|\,.\|_{\sim}^{\lambda})\Big)\\ &=\frac{1}{(\dim X+1)\,m^{\dim X}\deg_{\mathscr{L}}X}\,p_{\mathrm{Z}}^{X}(\overline{\mathscr{L}},\ldots,\overline{\mathscr{L}})\\ &\quad+\frac{1}{m^{\dim X}\deg_{\mathscr{L}}X}\cdot\\ &\qquad\qquad p_{\mathrm{Z}}^{X}\big((\mathscr{O}_X,\|\,.\|_{\infty}\cdot\exp(-\lambda\varphi)),(\mathscr{L}^{\otimes m},\|\,.\|_{\sim}),\ldots,(\mathscr{L}^{\otimes m},\|\,.\|_{\sim})\big)\\ &\quad+O(\lambda^2)\,.\end{aligned}$$

By virtue of Lemma 4.4, we have $p_{\mathrm{Z}}^{X}(\overline{\mathscr{L}},\ldots,\overline{\mathscr{L}})=0$. Further,

$$\begin{aligned}\frac{1}{m^{\dim X}\deg_{\mathscr{L}}X}\,p_{\mathrm{Z}}^{X}\big((\mathscr{O}_X,\|\,.\|_{\infty}\cdot\exp(-\lambda\varphi)),(\mathscr{L}^{\otimes m},\|\,.\|_{\sim}),\ldots,(\mathscr{L}^{\otimes m},\|\,.\|_{\sim})\big)&\\ =\lambda\lim_{i\to\infty}I_i(\varphi)&\end{aligned}$$

for

$$I_i(\varphi):=\frac{1}{m^{\dim X}}p_{\mathrm{Z}}^{X}\big((\mathscr{O}_X,\|\,.\|_{\infty}\cdot\exp(-\varphi)),\underbrace{(\mathscr{L}^{\otimes m},\|\,.\|_{\sim}^{(i)}),\ldots,(\mathscr{L}^{\otimes m},\|\,.\|_{\sim}^{(i)})}_{\dim X\ \text{times}}\big)\,.$$

At this level, we can make the arithmetic intersection product explicit and find

$$I_i(\varphi)=\int_{P(\mathbb{C})}\varphi c_1(\mathscr{L},\|\,.\|_{\infty}^{(i)})\wedge\ldots\wedge c_1(\mathscr{L},\|\,.\|_{\infty}^{(i)})\,.$$

Theorem A.14 therefore yields in view of Proposition 2.10

$$e_1(\mathrm{h}_{(\mathscr{L}^{\otimes m},\|\,.\|^{\lambda})})\geqslant\lambda\cdot\int\limits_{P(\mathbb{C})}\varphi\,d\tau\,. \tag{5}$$

The equations (4) and (5) together imply the assertion. □

Theorem 4.8 (Equidistribution on $P(\mathbb{C})$). *Let P, f, $\overline{\mathscr{L}}$, and Φ be as in 1.12. Then, for each sequence $(x_i)_{i\in\mathbb{N}}$ of closed points on P which is generic and small, the associated sequence of measures $(\delta_{x_i})_{i\in\mathbb{N}}$ converges weakly to the measure τ which is the zero measure on $P(\mathbb{C})\backslash K$ and the Haar measure of volume one on K.*

Proof. First step. $\delta_{x_i}|_{P(\mathbb{C})\backslash K}\to 0$.

By Proposition 3.8, for every $\varepsilon_1,\varepsilon_2>0$, there is some non-negative $g_{\varepsilon_1,\varepsilon_2}\in S$ such that $\operatorname{supp}g_{\varepsilon_1,\varepsilon_2}\subseteq U_{\varepsilon_1}(K)$,

$$\max_{x\in P(\mathbb{C})}g_{\varepsilon_1,\varepsilon_2}\leqslant 1+\varepsilon_2\,,$$

and

$$\int_{P(\mathbb{C})}g_{\varepsilon_1,\varepsilon_2}\,d\tau=1\,.$$

Then, by Proposition 4.7,

$$\liminf_{i\to\infty} \int\limits_{P(\mathbb{C})} g_{\varepsilon_1,\varepsilon_2}\, d\delta_{x_i} \geqslant 1\,.$$

This yields

$$\limsup_{i\to\infty} \int\limits_{P(\mathbb{C})} (1-g_{\varepsilon_1,\varepsilon_2})\, d\delta_{x_i} \leqslant 0\,.$$

But

$$1-g_{\varepsilon_1,\varepsilon_2}(x) \geqslant \begin{cases} 1 & \text{if } x\in P(\mathbb{C})\backslash U_{\varepsilon_1}(K)\,, \\ -\varepsilon_2 & \text{if } x\in U_{\varepsilon_1}(K)\,. \end{cases}$$

Consequently,

$$\limsup_{i\to\infty} \left[\delta_{x_i}\big(P(\mathbb{C})\backslash U_{\varepsilon_1}(K)\big) - \varepsilon_2\delta_{x_i}\big(U_{\varepsilon_1}(K)\big)\right] \leqslant 0\,.$$

This shows

$$\limsup_{i\to\infty} \delta_{x_i}(P(\mathbb{C})\backslash U_{\varepsilon_1}(K)) \leqslant \varepsilon_2\,.$$

As the latter is true for every $\varepsilon_2>0$, we have

$$\lim_{i\to\infty} \delta_{x_i}(P(\mathbb{C})\backslash U_{\varepsilon_1}(K)) = 0\,.$$

Further, this formula is true for each $\varepsilon_1>0$. Hence, $(\delta_{x_i}|_{P(\mathbb{C})\backslash K})_{i\in\mathbb{N}}$ converges weakly to the zero measure.

Second step. The assertion in general.

Let $\varphi\in C(P(\mathbb{C}))$ be an arbitrary continuous function. As we can interchange the roles of φ and $(-\varphi)$, it suffices to prove

$$\liminf_{i\to\infty} \int\limits_{P(\mathbb{C})} \varphi\, d\delta_{x_i} \geqslant \int\limits_{P(\mathbb{C})} \varphi\, d\tau.$$

By Lemma 4.5, we have continuous functions φ_1 and φ_2 on $P(\mathbb{C})$ such that $\operatorname{supp}\varphi_1 \subseteq P(\mathbb{C})\backslash K$ and $\varphi_2\in\mathbb{R}_+{\cdot}S$. The result of the first step shows

$$\lim_{i\to\infty} \int\limits_{P(\mathbb{C})} \varphi_1\, d\delta_{x_i} = 0\,.$$

Further, by Proposition 4.7,

$$\liminf_{i\to\infty} \int\limits_{P(\mathbb{C})} \varphi_2\, d\delta_{x_i} \geqslant \int\limits_{P(\mathbb{C})} \varphi_2\, d\tau\,.$$

Consequently,

$$\liminf_{i\to\infty} \int\limits_{P(\mathbb{C})} \varphi\, d\delta_{x_i} \geqslant \int\limits_{P(\mathbb{C})} \varphi\, d\tau - \varepsilon\,.$$

The assertion follows since $\varepsilon>0$ is arbitrary. $\square$

Corollary 4.9. *Let $X\subset P$ be a closed subvariety and $(x_i)_{i\in\mathbb{N}}$ be a sequence of closed points on X which is generic and small.*

Then, the sequence $(\delta_{x_i}|_{X(\mathbb{C})\backslash K})_{i\in\mathbb{N}}$ converges in the weak sense to the zero measure.

Proof. As there are only finitely many i such that $x_i \notin G$, let us assume that the whole sequence $(x_i)_{i\in\mathbb{N}}$ is contained in G. Choose some K-invariant metric on $G(\mathbb{C})$ and assume that, for some $\varepsilon > 0$, we would have

$$\liminf_{i\to\infty} \delta_{x_i}\big(X(\mathbb{C})\backslash U_\varepsilon(K)\big) = \delta > 0\,.$$

We will construct another sequence $(y_i)_{i\in\mathbb{N}}$ which is generic and small on the whole of P. For this, note first that we have $\mathrm{h}_{\overline{\mathscr{L}}}(x) = \frac{1}{\deg f}\,\mathrm{h}_{\overline{\mathscr{L}}}\big(f(x)\big)$. Consequently, $\mathrm{h}_{\overline{\mathscr{L}}}$ is invariant under shift by any torsion point $t \in K_{\mathrm{tor}} = \bigcup_{j\in\mathbb{N}} \ker(m^j)$. Further, all the sequences $(t\cdot x_i)_{i\in\mathbb{N}}$ fulfill

$$\liminf_{i\to\infty} \delta_{t\cdot x_i}\big(P(\mathbb{C})\backslash U_\varepsilon(K)\big) = \delta\,.$$

Finally, $K \subseteq G(\mathbb{C})$ is Zariski dense. Therefore, for each i, the union $\bigcup\limits_{t\in K_{\mathrm{tor}}} \{t\cdot x_i\}$ is a Zariski dense subset of P.

Recall, at this point, the fact there are only countably many proper closed subvarieties $P_0, P_1, P_2, \ldots \subset P$. We may choose a sequence $(y_i)_{i\in\mathbb{N}}$ such that, for every $i \in \mathbb{N}$, one has $y_i \in \bigcup\limits_{t\in K_{\mathrm{tor}}, i\geqslant 0} \{t\cdot x_i\}$ and $y_i \in P\backslash P_0\backslash\ldots\backslash P_i$. Then,

$$\delta_{y_i}\big(P(\mathbb{C})\backslash U_\varepsilon(K)\big) > \delta/2$$

for $i \gg 0$ and $\mathrm{h}_{\overline{\mathscr{L}}}(y_i) \to 0$. This is a contradiction to Theorem 4.8. $\square$

Remark 4.10. This corollary shows, in particular, the following. If $X \cap K = \emptyset$ then

$$p^X_{\mathrm{Z}}(\overline{\mathscr{L}}, \ldots, \overline{\mathscr{L}}) > 0\,.$$

In other words, there are no small and generic sequences on X.

Appendix A. The adelic Picard group

In this appendix, we will recall S. Zhang's adelic Picard group [Zh95a] and fix notation.

The local case. Preparations. Let K be an algebraically closed valuation field. The cases we have in mind are $K = \overline{\mathbb{Q}}_p$ for a prime number p and $K = \overline{\mathbb{Q}}_\infty = \mathbb{C}$ but, of course, there are many other examples, all of which are non-archimedean.

Definition A.1. Let X be a projective scheme over K. Then, a *metric* on an invertible sheaf $\mathscr{L} \in \mathrm{Pic}(X)$ is a system of K-norms on the K-vector spaces $\mathscr{L}(x)$ for all $x \in X(K)$.

Example A.2. Let K be non-archimedean and $\mathscr{O}_K$ the ring of integers in K. Then, a *model* of X, i.e. a flat projective scheme $\pi\colon \mathscr{X} \to \mathscr{O}_K$ such that $\mathscr{X}_K \cong X$, together with an invertible sheaf $\widetilde{\mathscr{L}} \in \mathrm{Pic}(\mathscr{X})$ with $\widetilde{\mathscr{L}}|_X \cong \mathscr{L}^{\otimes n}$ for some $n > 0$ induces a metric $\|.\|$ on $\mathscr{L}$ as follows:

The point $x \in X(K)$ has a unique extension $\overline{x}\colon \operatorname{Spec}\mathscr{O}_K \to \mathscr{X}$. Then $\overline{x}^*\widetilde{\mathscr{L}}$ is a projective $\mathscr{O}_K$-module of rank 1. Each $l \in \mathscr{L}(x)$ induces $l^{\otimes n} \in \mathscr{L}^{\otimes n}(x)$ and a rational section of $\overline{x}^*\widetilde{\mathscr{L}}$. Put

$$\|l\|_x := \Big[\inf\Big\{|a| \mid a \in K,\, l \in a\cdot \overline{x}^*\widetilde{\mathscr{L}}\Big\}\Big]^{\frac{1}{n}}\,.$$

This metric is called the metric on $\mathscr{L}$ being *induced* by the *model* $(\mathscr{X}, \widetilde{\mathscr{L}})$.

Definition A.3. Let X be a smooth, projective scheme over K.

a) Assume $K = \mathbb{C}$. Then, a metric $\|.\|$ on $\mathscr{L} \in \operatorname{Pic}(X)$ is called *continuous* respectively *smooth* if, for each $x \in X(\mathbb{C})$, there exist some open neighborhood U_x and a holomorphic section $s \in \Gamma(U_x, \mathscr{L}|_{U_x})$ such that $\|s\|$ is continuous, respectively smooth.

b) Assume K is non-archimedean. Then, a metric $\|.\|$ on $\mathscr{L} \in \operatorname{Pic}(X)$ is called *continuous* if $\|.\| = f \cdot \|.\|'$ for f a continuous function on $X(K)$ and $\|.\|'$ a metric being induced by a model.

The global case. The adelic Picard group. Let now X be a projective variety over $\mathbb{Q}$ and $\mathscr{L} \in \operatorname{Pic}(X)$ be a invertible sheaf.

Definition A.4. An *adelic metric* $\|.\|$ on $\mathscr{L}$ is a system $\{\|.\|_p\}_{p \in \{2,3,5,\dots;\infty\}}$ of continuous and bounded metrics on $\mathscr{L}_{\overline{\mathbb{Q}}_p} \in \operatorname{Pic}(X_{\overline{\mathbb{Q}}_p})$ such that

i) for each $p \in \{2,3,5,\dots;\infty\}$ the metric $\|.\|_p$ is $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-invariant,

ii) for some $n \in \mathbb{N}$, there exist a projective model $\mathscr{X}$ of X over $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ and an invertible sheaf $\widetilde{\mathscr{L}} \in \operatorname{Pic}(\mathscr{X})$ with $\widetilde{\mathscr{L}}|_X \cong \mathscr{L}$, which induces $\|.\|_p$ for almost all p.

$\mathscr{L}$ being equipped with an adelic metric is called an *adelicly metrized invertible sheaf.* The group of all adelicly metrized invertible sheaves on X will be denoted by $\operatorname{Pic}^{\mathrm{ad}}(X)$.

Remark A.5. Let $\mathscr{X} \to \operatorname{Spec} \mathbb{Z}$ be a model of X. Then, there are the following two natural homomorphisms given by the induced metrics.

$$\begin{aligned} i_{\mathscr{X}}\colon\ & \widehat{\operatorname{Pic}}(\mathscr{X}) \to \operatorname{Pic}^{\mathrm{ad}}(X) \\ a_{\mathscr{X}}\colon\ & \ker(\operatorname{Pic}(\mathscr{X}) \to \operatorname{Pic}(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Pic}^{\mathrm{ad}}(X) \end{aligned}$$

Here, $\widehat{\operatorname{Pic}}(\mathscr{X})$ denotes the arithmetic Picard group of Gillet and Soulé [GS90]. Further, one has the forgetful homomorphism $v\colon \operatorname{Pic}^{\mathrm{ad}}(X) \to \operatorname{Pic}(X)$. Note, if $\zeta \in \operatorname{im} a_{\mathscr{X}}$ then there exists some $n \in \mathbb{N}$ such that $n\zeta \in \operatorname{im} i_{\mathscr{X}}$.

Definition A.6. a) The group

$$\operatorname{Pic}^{\mathrm{in}}(X) := \langle \operatorname{im} i_{\mathscr{X}}, \operatorname{im} a_{\mathscr{X}} \rangle_{\mathscr{X} \text{ model of } X} \subseteq \operatorname{Pic}^{\mathrm{ad}}(X)$$

is called the group of the invertible sheaves on X *with induced metrics.*

b) $\zeta \in \operatorname{Pic}^{\mathrm{in}}(X)$ is called *semipositive*, $\zeta \in C_X^{\geqslant 0}$, if there are a natural number n, a model $\mathscr{X}$ of X, and $\overline{\mathscr{L}} \in \widehat{\operatorname{Pic}}(\mathscr{X})$ such that $n\zeta = i_{\mathscr{X}}(\overline{\mathscr{L}})$ where

i) $\overline{\mathscr{L}}$ has a non-negative curvature form $c_1(\overline{\mathscr{L}})$,

ii) for every prime number p, the underlying bundle $\mathscr{L}$ is of non-negative degree on each curve in $\mathscr{X}_{\mathbb{F}_p}$.

Definition A.7 (Metric on $v^{-1}(\mathscr{L}) \subseteq \operatorname{Pic}^{\mathrm{ad}}(X)$).
Let $(\mathscr{L}, \|.\|)$ and $(\mathscr{L}, \|.\|')$ be two adelicly metrized invertible sheaves with the same underlying bundle. Then, the *distance* between $(\mathscr{L}, \|.\|)$ and $(\mathscr{L}, \|.\|')$ is given by

$$\delta\left((\mathscr{L}, \|.\|), (\mathscr{L}, \|.\|')\right) := \sum_{p \in \{2,3,5,\dots;\infty\}} \sup_{x \in X(\overline{\mathbb{Q}}_p)} \left|\log \frac{\|.\|'_p}{\|.\|_p}\right|.$$

Lemma A.8. *Let $\mathscr{X}$ be a flat, projective scheme over $\mathbb{Z}$ of dimension $d+1$. Denote its generic fiber by X.*

Further, let $(\mathscr{L}', \|.\|'), (\mathscr{L}'', \|.\|''), \overline{\mathscr{L}}_1, \ldots, \overline{\mathscr{L}}_d \in \widehat{\mathrm{Pic}}(\mathscr{X})$ *where* $\mathscr{L}'$ *and* $\mathscr{L}''$ *are extensions of one and the same invertible sheaf* $\mathscr{L} \in \mathrm{Pic}(X)$*. If* $i_{\mathscr{X}}(\overline{\mathscr{L}}_1), \ldots, i_{\mathscr{X}}(\overline{\mathscr{L}}_d)$ *are semipositive then*

$$\begin{aligned} &\big|\big[\widehat{c}_1(\mathscr{L}', \|.\|') - \widehat{c}_1(\mathscr{L}'', \|.\|'')\big] \cdot \widehat{c}_1(\overline{\mathscr{L}}_1) \cdot \ldots \cdot \widehat{c}_1(\overline{\mathscr{L}}_d)\big| \leqslant \\ &\qquad \leqslant \delta\big(i_{\mathscr{X}}(\mathscr{L}', \|.\|'), i_{\mathscr{X}}(\mathscr{L}'', \|.\|'')\big) \cdot c_1(\mathscr{L}_1|_X) \cdot \ldots \cdot c_1(\mathscr{L}_d|_X). \end{aligned}$$

Proof. This result is contained in [Zh95a, Theorem 1.4]. The main ingredient is the fact that on a variety over a field the intersection product of semipositive divisors is always non-negative. $\square$

The adelic Picard group.

Definition A.9. Let $\overline{C_X^{\geqslant 0}}$ be the closure of the semipositive cone in $\mathrm{Pic}^{\mathrm{ad}}(X)$.

i) Then, the subgroup $\widetilde{\mathrm{Pic}}(X) \subseteq \mathrm{Pic}^{\mathrm{ad}}(X)$ generated by $\overline{C_X^{\geqslant 0}}$ is called the *adelic Picard group* of X.

ii) If $(\mathscr{L}, \|.\|) \in \widetilde{\mathrm{Pic}}(X)$ then $(\mathscr{L}, \|.\|)$ is said to be an *integrably metrized invertible sheaf.* The metric $\|.\|$ is called an *integrable metric* on $\mathscr{L}$.

Theorem A.10 (Zhang). *Let* X *be a scheme which is smooth, projective, and equidimensional over* $\mathbb{Q}$*.*

i) *Then, there is exactly one continuous map, the* adelic intersection product

$$p_{\mathrm{Z}}^X \colon \underbrace{\widetilde{\mathrm{Pic}}(X) \times \ldots \times \widetilde{\mathrm{Pic}}(X)}_{\dim X + 1 \text{ times}} \to \mathbb{R},$$

such that, for every model $\mathscr{X}$ *of* X*, the diagram*

$$\begin{tikzcd} \widetilde{\mathrm{Pic}}(X) \times \ldots \times \widetilde{\mathrm{Pic}}(X) \arrow[r, "p_{\mathrm{Z}}^X"] & \mathbb{R} \\ \widehat{\mathrm{Pic}}(\mathscr{X}) \times \ldots \times \widehat{\mathrm{Pic}}(\mathscr{X}) \arrow[u, "i_{\mathscr{X}}^{\dim X+1}"] \arrow[ur, "p_{\mathrm{GS}}^{\mathscr{X}}"'] & \end{tikzcd}$$

commutes. Here, $p_{\mathrm{GS}}^{\mathscr{X}}$ *is the arithmetic intersection product of Gillet and Soulé.*

ii) p_{Z}^X *is multi-linear and symmetric.*

Example A.11. For L a number field, $\widetilde{\mathrm{Pic}}(\mathrm{Spec}\, L) = \widehat{\mathrm{Pic}}(\mathrm{Spec}\, \mathscr{O}_L)$. I.e., there is an exact sequence $0 \to \mathrm{Cl}_L \to \widetilde{\mathrm{Pic}}(\mathrm{Spec}\, L) \to \mathbb{R} \to 0$.

Example A.12. Let V be a semistable projective curve over $\mathbb{Q}$. Then, there is an exact sequence $0 \to \mathbb{A} \to \widetilde{\mathrm{Pic}}(V) \to \mathrm{Pic}(V) \to 0$ where

$$\mathbb{A} = \bigoplus_{\nu \in \mathrm{Val}(\mathbb{Q})} \mathrm{C}_0(V(\overline{\mathbb{Q}}_\nu)).$$

Here, $\mathrm{C}_0(V(\overline{\mathbb{Q}}_\infty)) = \mathrm{C}(V(\mathbb{C}))^{F_\infty}$ is the space of all continuous F_∞-invariant functions on $V(\mathbb{C})$.

For p a prime number, $\mathrm{C}_0(V(\overline{\mathbb{Q}}_p))$ is given as follows. We choose a minimal semistable model $\mathscr{V}$ of V over $\mathrm{Spec}\, \mathbb{Q}_p$. Let $D_1, \ldots, D_r$ be the irreducible divisors in the special fiber. Further, for each $k \in \mathbb{N}$, we define a countable disjoint covering $\{U_i^{(k)}\}_{i \in \mathbb{N}}$ of $V(\overline{\mathbb{Q}}_p)$ by the requirement that $x, y \in V(\overline{\mathbb{Q}}_p)$ belong to the same set if they coincide modulo p^k. Then, $\mathrm{C}_0(V(\overline{\mathbb{Q}}_p))$ consists of all $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-invariant functions $V(\overline{\mathbb{Q}}_p) \to \mathbb{R}$ of the type

$$\sum_{i=1}^r \alpha_i \|1\|_{(\mathscr{V}, \mathscr{O}(D_i))} + \sum_{l=1}^\infty \sum_{ik} \beta_{ikl}\, \chi_{U_i^{(k)}}.$$

Here, $\|\,.\,\|_{(\mathscr{V},\mathscr{O}(D_i))}$ denotes the metric, induced by the model $(\mathscr{V},\mathscr{O}(D_i))$, $\chi_{U_i^{(k)}}$ is the characteristic function of $U_i^{(k)}$, and α_i as well as β_{ikl} are real numbers. We require that the inner sum is finite for each $l \in \mathbb{N}$ and that the outer series is uniformly convergent.

A.13. Finally, let us recall the theorem of successive minima. We suppose that X is a scheme which is smooth, projective, and equidimensional over $\mathbb{Q}$. Further, $\overline{\mathscr{L}} \in \widetilde{\operatorname{Pic}}(X)$.

i) Then, the *height* of an L-valued point $x \in X(L)$ for L a number field is given by

$$h_{\overline{\mathscr{L}}}(x) := \frac{1}{[L:\mathbb{Q}]} \deg \overline{\mathscr{L}}|_X .$$

For $i = 1, 2, \ldots, \dim X + 1$, put

$$e_i(\overline{\mathscr{L}}) := \sup_{\operatorname{cod} Y = i} \inf_{x \in X \setminus Y} h_{\overline{\mathscr{L}}}(x) .$$

ii) The *height* of X itself with respect to $\overline{\mathscr{L}}$ is given by the formula

$$h_{\overline{\mathscr{L}}}(X) := \frac{c_1(\overline{\mathscr{L}})^{\dim X+1}}{(\dim X + 1)\, c_1(\mathscr{L}_{\mathbb{Q}})^{\dim X}} .$$

Theorem A.14 (Zhang). *Let X be a scheme which is smooth, projective, and equidimensional over $\mathbb{Q}$ and $\overline{\mathscr{L}} \in \widetilde{\operatorname{Pic}}(X)$. Assume $\overline{\mathscr{L}} \in C_X^{\geqslant 0}$. Then,*

$$e_1(\overline{\mathscr{L}}) \geqslant h_{\overline{\mathscr{L}}}(X) \geqslant \frac{e_1(\overline{\mathscr{L}}) + \ldots + e_{\dim X+1}(\overline{\mathscr{L}})}{\dim X + 1} .$$

Proof. This is [Zh95a, Theorem 1.10]. It is obtained by an easy limit argument from an analogous statement for arithmetic varieties [Zh95b, (5.2)]. Note that the arithmetic Riemann-Roch theorem [GS92] is the main ingredient here. □

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