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ALGEBRAIC GEOMETRY

Tutorial – Localization
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The goal of these exercises is to revise the basics of localization. They are optional and should *not* be handed in. We begin with the necessary definitions:

Definition. Let A be a (commutative) ring. A subset $S \subset A$ is called *multiplicative*, if

(i) $1 \in S$ and

(ii) S is closed under multiplication, i.e. $fg \in S$ for all $f, g \in S$.

Now, let $S \subset A$ be multiplicative. On $A \times S$, consider the equivalence relation (!) given by

$$(*) \qquad (a, s) \sim (b, t) :\Leftrightarrow \text{ex. } u \in S \text{ with } u(at - bs) = 0.$$

The *localization* $S^{-1}A$ of A at S is now defined as the quotient

$$S^{-1}A := (A \times S)/\sim .$$

We suggestively write $\frac{a}{s}$ for the equivalence class of (a, s). Addition and multiplication are then defined as in $\mathbb{Q}$: Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$, then

$$\frac{a}{s} + \frac{b}{t} := \frac{at + bs}{st},$$
$$\frac{a}{s} \cdot \frac{b}{s} := \frac{ab}{st}.$$

(You might want to check that this is well-defined.)
We then obtain the canonical homomorphism $\tau\colon A \longrightarrow S^{-1}A, a \mapsto \frac{a}{1}$.

Example. Two important examples are the following:

- Let $f \in A$ be any element. Then $S = \{1, f, f^2, \dots\}$ is a multiplicative set and we write

$$A_f := S^{-1}A = \left\{ \frac{a}{f^i} \mid a \in A, i \in \mathbb{N}_0 \right\}.$$

- Let $\mathfrak{p} \subset A$ be a prime ideal. Then $A \setminus \mathfrak{p}$ is multiplicative (!) and we write

$$A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1}A \overset{!}{=} \left\{ \frac{a}{s} \mid a \in A, s \notin \mathfrak{p} \right\}.$$

T.1 (a) Show that the relation $(*)$ in the definition is indeed an equivalence relation.
(b) Why can the element $u \in S$ in the definition of the equivalence relation not be omitted?

(c) Show that $\tau(S) \subset (S^{-1}A)^*$.

(d) When is τ injective?

(e) Suppose A is a domain and write $\mathrm{Quot}(A) := S^{-1}A$ for the localization of A at $S = A \setminus \{0\}$. Show that $\mathrm{Quot}(A)$ is a field.

T.2 Let A be a ring and $S \subset A$ be a multiplicative subset. Consider the localization $S^{-1}A$ and the canonical map $\tau\colon A \to S^{-1}A, a \mapsto \frac{a}{1}$. We already know that $\tau(S) \subset (S^{-1}A)^*$.

(a) Show that $S^{-1}A$ together with τ is universal with respect to the following property: *For any ring B and any ring homomorphism $\varphi\colon A \to B$ with $\varphi(S) \subset B^*$ there is a unique homomorphism $\overline{\varphi}\colon S^{-1}A \to B$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & B. \\ {\scriptstyle\tau}\big\downarrow & \nearrow_{\exists!\overline{\varphi}} & \\ S^{-1}A & & \end{array}$$

In particular, show that up to a unique isomorphism, the pair $(S^{-1}A, \tau)$ is uniquely determined by this property.

(b) Let $f \in A$ be an element. Show that there is a canonical isomorphism

$$A[t]/(1 - ft) \cong A_f.$$

(c) Let M be an A-module. We define the A-module morphism $\varphi\colon M \to S^{-1}M$ as being initial among the A-module morphisms $M \to N$ such that $s \times \cdot\colon N \to N$ is an isomorphism. (What is the correct category in which φ is an initial object?)

That is, every such map $\alpha\colon M \to N$ factors uniquely through φ:

$$\begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & N. \\ {\scriptstyle\varphi}\big\downarrow & \nearrow_{\exists!} & \\ S^{-1}M & & \end{array}$$

This extends the A-module structure on $S^{-1}M$ to an $S^{-1}A$-module structure.

Show that such a map $\varphi\colon M \to S^{-1}M$ exists by explicitly constructing it.

(d) Show that localization commutes with finite products and arbitrary direct sums.

(e) The same cannot be said about *arbitrary products*:

Show that the obvious (!) map $S^{-1}(\prod_i M_i) \to \prod_i S^{-1}M_i$ need not be an isomorphism by considering $\prod_{i\in\mathbb{Z}} \mathbb{Q}$.