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ALGEBRAIC GEOMETRY

Tutorial – Exact Sequences
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In this tutorial we conclude our excursion into commutative algebra by exploring *exactness* of the functors $\mathrm{Hom}_A(M, -)$, $S^{-1}(-)$ and $- \otimes N$. For this, we will briefly repeat the definition of an *exact sequence* of modules (or rings, (abelian) groups,...) and discuss the action of our functors on them.
If this sheet looks particularly long, do not be scared. In particular, it is enough to do the non-starred exercises.

Definition. Let A be a ring. A *sequence* of A-modules is a diagram

$$\dots \xrightarrow{\varphi_{i-1}} M_{i-1} \xrightarrow{\varphi_i} M_i \xrightarrow{\varphi_{i+1}} M_{i+1} \xrightarrow{\varphi_{i+2}} \dots$$

of A-modules M_i and A-linear maps $\varphi_i\colon M_{i-1} \to M_i$. A sequence (M_i, φ_i) is called *exact at* M_i, if $\mathrm{im}(\varphi_{i-1}) = \ker(\varphi_i)$. The sequence is *exact*, if it is exact at M_i for all i.
A *short exact sequence* is an exact sequence of the form

$$0 \to M' \to M \to M'' \to 0.$$

A functor $F\colon \mathsf{Mod}(A) \to \mathsf{Mod}(B)$ is called *left-exact*, if for all short exact sequences $0 \to M' \to M \to M'' \to 0$, the sequence

$$0 \to F(M') \to F(M) \to F(M'')$$

is exact, and *right-exact*, if

$$F(M') \to F(M) \to F(M'') \to 0$$

is exact for all short exact sequences. F is *exact*, if it is left and right exact. Equivalently, F is exact if for all exact sequences $M' \to M \to M''$ the sequence $F(M') \to F(M) \to F(M'')$ is exact.
If G is a contravariant functor, then G is *left-exact*, if

$$G(M'') \to G(M) \to G(M') \to 0$$

is exact for all short exact sequences, and *right-exact*, if

$$0 \to G(M'') \to G(M) \to G(M')$$

is exact.

T3.1 Let $\varphi\colon M \to M'$ be an A-linear map.

(a) Show that φ is injective, if and only if $0 \to M \to M'$ is exact.

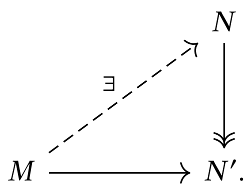
(b) Show that φ is surjective, if and only if $M \longrightarrow M' \longrightarrow 0$ is exact.

Let M be an A-module and consider the functor $\mathrm{Hom}_A(M, -)\colon \mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(A)$.

T3.2 (a) Show that $\mathrm{Hom}_A(M, -)$ is left-exact.

(b) Give an example of a ring A, an A-module M and a short exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ of A-modules, such that corresponding Hom-sequence is not exact.

(c)* Show that $\mathrm{Hom}_A(M, -)$ is exact, if and only if M satisfies the following property: *For every surjective homomorphism $N \twoheadrightarrow N'$ and every homomorphism $M \longrightarrow N'$ there exists a homomorphism $M \longrightarrow N$ making the following diagram commute*:



Such modules are called *projective*.

(d)* What are the analogous statements for $\mathrm{Hom}_A(-, M)$? Prove them.

Now, let N be an A-module and consider $- \otimes_A N$.

T3.3 (a) Show that $- \otimes_A N$ is right-exact.
Hint: First, show that the sequence $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of A-modules is exact, if and only if the sequence $0 \longrightarrow \mathrm{Hom}_A(M'', P) \longrightarrow \mathrm{Hom}(M, P) \longrightarrow \mathrm{Hom}_A(M', P)$ is exact for all A-modules P.

(b) Find an example that shows that $- \otimes_A N$ need not be exact.

An A-module N is called *flat*, if the functor $- \otimes_A N$ is (left-)exact.

(c) Show that finite-dimensional vector spaces are flat k-modules, where k is a field.

Finally, lets deal with localization. Let $S \subset A$ be a multiplicative subset.

T3.4 (a) Show that localization preserves tensor products, that is for two A-modules M and N, we have

$$S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N.$$

(b) Show that $S^{-1}(-)$ is exact.

We already saw that localization preserves direct sums. Let us collect some more similar properties: Let M be an A-module

(c) Let $(N_i)_{i \in I}$ be a family of submodules $N_i \subset M$. Then $S^{-1}(\sum N_i) \cong \sum S^{-1}N_i$.

(d) Let N and P be submodules of M. Then $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.

(e) Canonically, $S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$.