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Prof. Dr. Rainer Sinn
Florian Bräunlich
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Fakultät für Mathematik und Informatik
Mathematisches Institut

ALGEBRAIC GEOMETRY

Tutorial – Tensor Product
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We saw that for localization, the canonical homomorphism $\tau\colon A \longrightarrow S^{-1}A$ helped us induce an $S^{-1}A$-module structure on the A-module $S^{-1}M$. This can be generalized using *tensor products*.

Definition. Let A be a ring and let M and N be A-modules. A *tensor product* of M and N is an A-module $M \otimes N$ together with a symmetric (A-)bilinear map $\otimes\colon M \times N \longrightarrow M \otimes N$, $(m, n) \mapsto m \otimes n$ satisfying the following universal property: *For every bilinear map $M \otimes N \longrightarrow P$ of A-modules, there exists a unique linear map $M \otimes N \longrightarrow P$ such that the following diagram commutes:*

$$\begin{array}{ccc} M \times N & \longrightarrow & P. \\ {\scriptstyle\otimes}\downarrow & \nearrow{\scriptstyle\exists!} & \\ M \otimes N & & \end{array}$$

It is easy to see that the tensor product is unique, if it exists, as well as that taking the tensor product is symmetric and associative up to an isomorphism.

T2.1 (a) Prove that a tensor product exists.

Hint. Consider the free module $F = A^{(M \times N)}$ with basis $M \times N$. Let K be the submodule of F generated by elements of the form

$$(a_1 m_1 + a_2 m_2, n) - a_1(m_1, n) - a_2(m_2, n),$$
$$(m, a_1 n_1 + a_2 n_2) - a_1(m, n_1) - a_2(m, n_2),$$

with $a_1, a_2 \in A$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$. Now take $M \otimes N := F/K$ and show the rest.

(b) Show that $M \otimes N$ is generated by the elements $m \otimes n$, $m \in M$, $n \in N$.

(c) Show that for A-modules M, N and P there is a canonical isomorphism of A-modules

$$\begin{aligned} \operatorname{Bil}_A(M, N; P) &\longrightarrow \operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)) \\ \mu &\longmapsto (m \mapsto \mu(m, -)) \\ ((m, n) \mapsto \varphi(m)(n)) &\longleftarrow\!\shortmid \varphi \end{aligned}$$

(d) Use (d) to show the Hom-$\otimes$-*adjunction:* There is a natural isomorphism

$$\operatorname{Hom}_A(M \otimes N, P) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)).$$

(e) Show that the tensor product commutes with arbitrary direct sums (using universal properties).

T2.2 In this exercise we want to make sense of the comment regarding localization and tensor products.

Let $\varphi\colon A \longrightarrow B$ a ring homomorphism.

Any B-module N becomes an A-module by defining

$$a \cdot n := \varphi(a)n.$$

In this way, any B-linear map $N' \longrightarrow N$ is also A-linear. *If you know categories, this means that* φ *defines a functor* $\mathsf{Mod}(B) \longrightarrow \mathsf{Mod}(A)$. This is called *restriction of scalars along* $\varphi\colon A \longrightarrow B$.

Conversely, if M is an A-module, then $B \otimes_A M$ is a B-module via

$$b \cdot (c \otimes m) = (bc) \otimes m.$$

Any A-module homomorphism $\psi\colon M \longrightarrow M'$ yields a homomorphism of B-modules $\mathrm{id}_B \otimes \psi\colon B \otimes_A M \longrightarrow B \otimes_A M'$. This yields a functor $\mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(B)$. This functor is called *extension of scalars along* $\varphi\colon A \longrightarrow B$.

(a) Show that localization is an example of extension of scalars by considering the "obvious" bilinear map $S^{-1}A \times M \longrightarrow S^{-1}M$ yielding a map

$$f\colon S^{-1}A \otimes_A M \longrightarrow S^{-1}M.$$

Show that f is an isomorphism.

(b) Prove the *restiction-extension-adjunction*: for an A-module M and a B-module N there is a natural isomorphism

$$\begin{aligned} \mathrm{Hom}_B(B \otimes M, N) &\cong \mathrm{Hom}_A(M, N). \\ \alpha &\mapsto (m \mapsto \alpha(1 \otimes m)) \\ (b \otimes m \mapsto b\beta(m)) &\leftarrow\!\shortmid \beta \end{aligned}$$

(c) Use (b) to show that extension of scalars preserves tensor products: Let M and N be A-modules. Then, there is a canonical isomorphism of B-modules

$$(B \otimes_A M) \otimes_B (B \otimes_A N) \cong B \otimes_A (M \otimes_A N).$$