nag_zeros_complex_poly (c02afc)

1. Purpose

nag_zeros_complex_poly (c02afc) finds all the roots of a complex polynomial equation, using a variant of Laguerre's Method.

2. Specification

```
#include <nag.h>
#include <nagc02.h>

void nag_zeros_complex_poly(Integer n, Complex a[], Boolean scale, Complex z[],
     NagError *fail)
```

3. Description

The function attempts to find all the roots of the nth degree complex polynomial equation

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n = 0.$$

The roots are located using a modified form of Laguerre's Method, originally proposed by Smith (1967).

The method of Laguerre (see Wilkinson (1965)) can be described by the iterative scheme

$$L(z_k) = z_{k+1} - z_k = \frac{-nP(z_k)}{P'(z_k) \pm \sqrt{H(z_k)}},$$

where $H(z_k) = (n-1)[(n-1)(P'(z_k))^2 - nP(z_k)P''(z_k)]$, and z_0 is specified.

The sign in the denominator is chosen so that the modulus of the Laguerre step at z_k, viz. $|L(z_k)|$, is as small as possible. The method can be shown to be cubically convergent for isolated roots (real or complex) and linearly convergent for multiple roots.

The function generates a sequence of iterates $z_1, z_2, z_3, \ldots$, such that $|P(z_{k+1})| < |P(z_k)|$ and ensures that $z_{k+1} + L(z_{k+1})$ 'roughly' lies inside a circular region of radius $|F|$ about z_k known to contain a zero of $P(z)$; that is, $|L(z_{k+1})| \leq |F|$, where F denotes the Féjer bound (see Marden (1966)) at the point z_k. Following Smith (1967), F is taken to be $\min(B, 1.445nR)$, where B is an upper bound for the magnitude of the smallest zero given by

$$B = 1.0001 \times \min(\sqrt{n}L(z_k), |r_1|, |a_n/a_0|^{1/n}),$$

r_1 is the zero X of smaller magnitude of the quadratic equation

$$(P''(z_k)/(2n(n-1)))X^2 + (P'(z_k)/n)X + \frac{1}{2}P(z_k) = 0$$

and the Cauchy lower bound R for the smallest zero is computed (using Newton's Method) as the positive root of the polynomial equation

$$|a_0|z^n + |a_1|z^{n-1} + |a_2|z^{n-2} + \ldots + |a_{n-1}|z - |a_n| = 0.$$

Starting from the origin, successive iterates are generated according to the rule $z_{k+1} = z_k + L(z_k)$ for $k = 1, 2, 3, \ldots$ and $L(z_k)$ is 'adjusted' so that $|P(z_{k+1})| < |P(z_k)|$ and $|L(z_{k+1})| \leq |F|$. The iterative procedure terminates if $P(z_{k+1})$ is smaller in absolute value than the bound on the rounding error in $P(z_{k+1})$ and the current iterate $z_p = z_{k+1}$ is taken to be a zero of $P(z)$. The deflated polynomial $\tilde{P}(z) = P(z)/(z - z_p)$ of degree $n-1$ is then formed, and the above procedure is repeated on the deflated polynomial until $n < 3$, whereupon the remaining roots are obtained via the 'standard' closed formulae for a linear ($n = 1$) or quadratic ($n = 2$) equation.

4. Parameters

n

Input: the degree of the polynomial, n.
Constraint: $\mathbf{n} \geq \mathbf{1}$.

a[n+1]

Input: **a**[i]**.re** and **a**[i]**.im** must contain the real and imaginary parts of a_i (i.e., the coefficient of z^{n-i}), for $i = 0, 1, \ldots, n$.
Constraint: **a[0].re** $\neq$ **0.0** or **a[0].im** $\neq$ **0.0**.

scale

Input: indicates whether or not the polynomial is to be scaled. The recommended value is **TRUE**. See Section 6 for advice on when it may be preferable to set **scale** = **FALSE** and for a description of the scaling strategy.

z[n]

Output: the real and imaginary parts of the roots are stored in **z**[i]**.re** and **z**[i]**.im** respectively, for $i = 0, 1, \ldots, n-1$.

fail

The NAG error parameter, see the Essential Introduction to the NAG C Library.

5. Error Indications and Warnings

NE_INT_ARG_LT

On entry, **n** must not be less than 1: **n** = ⟨*value*⟩.

NE_COMPLEX_ZERO

On entry, the complex variable **a[0]** has zero real and imaginary parts.

NE_POLY_NOT_CONV

The iterative procedure has failed to converge. This error is very unlikely to occur. If it does, please contact NAG immediately, as some basic assumption for the arithmetic has been violated.

NE_POLY_UNFLOW

The function cannot evaluate $P(z)$ near some of its zeros without underflow. Please contact NAG immediately.

NE_POLY_OVFLOW

The function cannot evaluate $P(z)$ near some of its zeros without overflow. Please contact NAG immediately.

NE_ALLOC_FAIL

Memory allocation failed.

6. Further Comments

If **scale** = **TRUE**, then a scaling factor for the coefficients is chosen as a power of the base b of the machine so that the largest coefficient in magnitude approaches $thresh = b^{e_{\max}-p}$. Users should note that no scaling is performed if the largest coefficient in magnitude exceeds $thresh$, even if **scale** = **TRUE**. (For definition of b, $e_{\max}$ and p see Chapter Introduction x02.)

However, with **scale** = **TRUE**, overflow may be encountered when the input coefficients $a_0, a_1, a_2, \ldots, a_n$ vary widely in magnitude, particularly on those machines for which b^{4p} overflows. In such cases, **scale** should be set to **FALSE** and the coefficients scaled so that the largest coefficient in magnitude does not exceed $b^{e_{\max}-2p}$.

Even so, the scaling strategy used in nag_zeros_complex_poly is sometimes insufficient to avoid overflow and/or underflow conditions. In such cases, the user is recommended to scale the independent variable (z) so that the disparity between the largest and smallest coefficient in magnitude is reduced. That is, use the function to locate the zeros of the polynomial $d \times P(cz)$ for some suitable values of c and d. For example, if the original polynomial was $P(z) = 2^{-100}i + 2^{100}z^{20}$,

then choosing $c = 2^{-10}$ and $d = 2^{100}$, for instance, would yield the scaled polynomial $i + z^{20}$, which is well-behaved relative to overflow and underflow and has zeros which are 2^{10} times those of $P(z)$.

If the function fails with **NE_POLY_NOT_CONV**, **NE_POLY_UNFLOW** or **NE_POLY_OVFLOW**, then the real and imaginary parts of any roots obtained before the failure occurred are stored in **z** in the reverse order in which they were found. More precisely, **z[n−1].re** and **z[n−1].im** contain the real and imaginary parts of the 1st root found, **z[n−2].re** and **z[n−2].im** contain the real and imaginary parts of the 2nd root found, and so on. The real and imaginary parts of any roots not found will be set to a large negative number, specifically $-1.0/(\sqrt{2.0}*\text{X02AMC})$.

6.1. Accuracy

All roots are evaluated as accurately as possible, but because of the inherent nature of the problem complete accuracy cannot be guaranteed.

6.2. References

Marden M (1966) *Geometry of Polynomials. Mathematical Surveys* Am. Math. Soc., Providence, Rhode Island, USA.

Smith B T (1967) *ZERPOL: A Zero Finding Algorithm for Polynomials Using Laguerre's Method* Technical Report, Department of Computer Science, University of Toronto, Canada.

Wilkinson J H (1965) *The Algebraic Eigenvalue Problem* Clarendon Press, Oxford.

7. See Also

nag_zeros_real_poly (c02agc)

8. Example

To find the roots of the polynomial $a_0z^5+a_1z^4+a_2z^3+a_3z^2+a_4z+a_5 = 0$, where $a_0 = (5.0+6.0i)$, $a_1 = (30.0 + 20.0i)$, $a_2 = -(0.2 + 6.0i)$, $a_3 = (50.0 + 100000.0i)$, $a_4 = -(2.0 - 40.0i)$ and $a_5 = (10.0 + 1.0i)$.

8.1. Program Text

```
/* nag_zeros_complex_poly(c02afc) Example Program
 *
 * Copyright 1991 Numerical Algorithms Group.
 *
 * Mark 2, 1991.
 */

#include <nag.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nagc02.h>

#define MAXDEG 100

main()
{

  Complex  a[MAXDEG+1], z[MAXDEG];
  Integer  i, n;
  Boolean  scale;

  Vprintf("c02afc Example Program Results\n");
  /*  Skip heading in data file */
  Vscanf("%*[^\n]");
  Vscanf("%ld", &n);
  scale = TRUE;
  if (n>0 && n<=MAXDEG)
    {
      for (i=0; i<=n; i++)
        Vscanf("%lf%lf", &a[i].re, &a[i].im);

      c02afc(n, a, scale, z, NAGERR_DEFAULT);
```

```
      Vprintf("\nDegree of polynomial = %4ld\n\n", n);
      Vprintf("Roots of polynomial\n\n");
      for (i=0; i<n; ++i)
        Vprintf("z = %12.4e   %+14.4e\n", z[i].re, z[i].im);
    }
  else
    {
      Vfprintf(stderr, "Error: n is out of range: n = %3ld\n", n);
      exit(EXIT_FAILURE);
    }
  exit(EXIT_SUCCESS);
}
```

8.2. Program Data

```
c02afc Example Program Data
 5
    5.0       6.0
   30.0      20.0
   -0.2      -6.0
   50.0  100000.0
   -2.0      40.0
   10.0       1.0
```

8.3. Program Results

```
c02afc Example Program Results

Degree of polynomial =    5

Roots of polynomial

z =  -2.4328e+01      -4.8555e+00
z =   5.2487e+00      +2.2736e+01
z =   1.4653e+01      -1.6569e+01
z =  -6.9264e-03      -7.4434e-03
z =   6.5264e-03      +7.4232e-03
```