nag_elliptic_integral_rf (s21bbc)

1. Purpose

nag_elliptic_integral_rf (s21bbc) returns a value of the symmetrised elliptic integral of the first kind.

2. Specification

```
#include <nag.h>
#include <nags.h>

double nag_elliptic_integral_rf(double x, double y, double z, NagError *fail)
```

3. Description

This function calculates an approximation to the integral

$$R_F(x,y,z) = \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

where x, y, $z \geq 0$ and at most one is zero.

The basic algorithm, which is due to Carlson (1978) and Carlson (1988), is to reduce the arguments recursively towards their mean by the rule: $x_0 = \min(x,y,z), z_0 = \max(x,y,z)$, $y_0 =$ remaining third intermediate value argument. (This ordering, which is possible because of the symmetry of the function, is done for technical reasons related to the avoidance of overflow and underflow.)

$$\begin{array}{rcl}
\mu_n & = & (x_n + y_n + z_n)/3 \\
X_n & = & 1 - x_n/\mu_n \\
Y_n & = & 1 - y_n/\mu_n \\
Z_n & = & 1 - z_n/\mu_n \\
\lambda_n & = & \sqrt{x_n y_n} + \sqrt{y_n z_n} + \sqrt{z_n x_n} \\
x_{n+1} & = & (x_n + \lambda_n)/4 \\
y_{n+1} & = & (y_n + \lambda_n)/4 \\
z_{n+1} & = & (z_n + \lambda_n)/4
\end{array}$$

$\epsilon_n = \max(|X_n|,|Y_n|,|Z_n|)$ and the function may be approximated adequately by a 5th-order power series:

$$R_F(x,y,z) = \frac{1}{\sqrt{\mu_n}}\left(1 - \frac{E_2}{10} + \frac{E_2^2}{24} - \frac{3E_2E_3}{44} + \frac{E_3}{14}\right)$$

where $E_2 = X_nY_n + Y_nZ_n + Z_nX_n, E_3 = X_nY_nZ_n$.

The truncation error involved in using this approximation is bounded by $\epsilon_n^6/4(1-\epsilon_n)$ and the recursive process is stopped when this truncation error is negligible compared with the ***machine precision***.

Within the domain of definition, the function value is itself representable for all representable values of its arguments. However, for values of the arguments near the extremes the above algorithm must be modified so as to avoid causing underflows or overflows in intermediate steps. In extreme regions arguments are pre-scaled away from the extremes and compensating scaling of the result is done before returning to the calling program.

4. Parameters

x
y
z

Input: the arguments x, y and z of the function.
Constraint: **x**, **y**, **z** ≥ 0.0 and only one of **x**, **y** and **z** may be zero.

fail

The NAG error parameter, see the Essential Introduction to the NAG C Library.

5. Error Indications and Warnings

NE_REAL_ARG_LT

On entry, **x** must not be less than 0.0: **x** = ⟨*value*⟩.
On entry, **y** must not be less than 0.0: **y** = ⟨*value*⟩.
On entry, **z** must not be less than 0.0: **z** = ⟨*value*⟩.
The function is undefined.

NE_REAL_ARG_EQ

On entry, ⟨*parameters*⟩ must not be equal to 0.0: ⟨*parameters*⟩ = ⟨*value*⟩.
On entry, two or more of **x**, **y** and **z** are zero; the function is undefined and the function returns zero.

6. Further Comments

If two arguments are equal, the function reduces to the elementary integral R_C, computed by nag_elliptic_integral_rc (s21bac).

Symmetrised elliptic integrals returned by functions nag_elliptic_integral_rf, nag_elliptic_integral_rc (s21bac), nag_elliptic_integral_rd (s21bcc) and nag_elliptic_integral_rj (s21bdc) can be related to the more traditional canonical forms (see Abramowitz and Stegun (1968)), as described in the Chapter Introduction.

6.1. Accuracy

In principle the function is capable of producing full ***machine precision***. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the ***machine precision***.

6.2. References

Abramowitz M and Stegun I A (1968) *Handbook of Mathematical Functions* Dover Publications, New York ch 17.

Carlson B C (1978) *Computing Elliptic Integrals by Duplication* Department of Physics, Iowa State University (Preprint).

Carlson B C (1988) A Table of Elliptic Integrals of the Third Kind *Math. Comp.* **51** 267–280.

7. See Also

nag_elliptic_integral_rc (s21bac)
nag_elliptic_integral_rd (s21bcc)
nag_elliptic_integral_rj (s21bdc)

8. Example

This example program simply generates a small set of non-extreme arguments which are used with the function to produce the table of low accuracy results.

8.1. Program Text

```
/* nag_elliptic_integral_rf(s21bbc) Example Program
 *
 * Copyright 1990 Numerical Algorithms Group.
 *
 * Mark 2 revised, 1992.
 *
 * Mark 3 revised, 1994.
 */

#include <nag.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nags.h>

main()
{
  double rf, x, y, z;
  Integer ix;

  Vprintf("s21bbc Example Program Results\n");
  Vprintf("     x      y      z       s21bbc\n");
  for (ix=1; ix<=3; ix++)
    {
      x = ix*0.5;
      y = (ix+1)*0.5;
      z = (ix+2)*0.5;
      rf = s21bbc(x, y, z, NAGERR_DEFAULT);
      Vprintf("%7.2f%7.2f%7.2f%12.4f\n", x, y, z, rf);
    }
  exit(EXIT_SUCCESS);
}
```

8.2. Program Data

None.

8.3. Program Results

```
s21bbc Example Program Results
     x      y      z       s21bbc
   0.50   1.00   1.50      1.0281
   1.00   1.50   2.00      0.8260
   1.50   2.00   2.50      0.7116
```