Fourier Series on the n-dimensional Torus

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Abstract

In this paper we give a short overview of Fourier series in the context of n-dimensional tori, thus including the classical case of a function defined on $[0, 2\pi)$. We will focus on convergence and divergence issues.

1 Orthonormal Bases

Definition 1 *If H is a Hilbert space then we will call a subset $S \subset H$ such that*
1.) $\forall x, y \in S, x \neq y : (x, y) = 0$ *(orthogonality)*
2.) $\forall x \in S : \|x\|_H = 1$ *(normed vectors)*
an orthonormal system *in H. We will call an orthonormal system* complete *(or* orthonormal basis*) if $H = \overline{\text{span}}(S)$.*

It is important to note that an orthonormal basis is not a vector space basis (unless H is finite-dimensional). However, of course every orthonormal basis can be enlarged to become a vector space basis.

There are other possible definitions of the completeness of an orthonormal system. Our choice is based on the wish to keep proofs as short as possible.

Theorem 2 *Every separable Hilbert space admits a countable, complete orthonormal system.*

Theorem 3 (Generalized Pythagoras) *Let $x, y \in H$ be orthogonal vectors, that is $(x, y) = 0$. Then*

$$\|x\|_H^2 + \|y\|_H^2 = \|x + y\|_H^2 \tag{1}$$

This follows from a simple computation with the scalar product.

Theorem 4 (Bessel Inequality) *If H is a Hilbert space, $S = \{e_1, e_2, ...\}$ a countable orthonormal system, then for all $x \in H$ we have*

$$\sum_{k=1}^{\infty} |(x, e_k)|^2 \leq \|x\|_H^2 \ (< \infty) \tag{2}$$

This is an equality exactly if S is complete.

Proof. Choose an arbitrary $N \geq 1$. Setting

$$x_N = x - \sum_{k=1}^{N} (x, e_k) e_k \tag{3}$$

we easily see that x_N is orthogonal to e_j for $j = 1, 2, \ldots, N$ since

$$\begin{aligned} (x_N, e_j) &= \left(x - \sum_{k=1}^{N} (x, e_k) e_k, e_j \right) = (x, e_j) - \sum_{k=1}^{N} (x, e_k)(e_k, e_j) \quad (4) \\ &= (x, e_j) - (x, e_j) = 0 \quad (5) \end{aligned}$$

Hence, the Pythagoras equality gives us

$$\begin{aligned} \|x\|_H^2 &= \|x_N\|_H^2 + \left\| \sum_{k=1}^{N} (x, e_k) e_k \right\|_H^2 \quad (6) \\ &= \|x_N\|_H^2 + \underbrace{\sum_{k=1}^{N} |(x, e_k)|^2}_{\text{since } \|e_k\|=1} \quad (7) \\ &\geq \sum_{k=1}^{N} |(x, e_k)|^2 \quad (8) \end{aligned}$$

However, our choice of a specific N was arbitrary. This implies the desired inequality. We postpone the proof of the case of equality until the next theorem. ■

Now we come to the main theorem about orthonormal systems in Hilbert spaces.

Theorem 5 *Let H be a Hilbert space and $S = \{e_1, e_2, \ldots\} \subset H$ an orthonormal system. Then the following statements are equivalent:*
1.) $H = \overline{\text{span}}(S)$
2.) $\forall x \in H : x = \sum_{e \in S} (x, e) e$
3.) $\forall x, y \in H : (x, y) = \sum_{e \in S} (x, e)(e, y)$
4.) $\forall x \in H : \|x\|_H^2 = \sum_{e \in S} |(x, e)|^2$ *(Parseval formula)*

The second statement clarifies why a complete orthonormal system is called an orthonormal *basis*.

Proof. (1.) $\Rightarrow$ (2.) Choose some fixed $x \in H$. Since we assume $\text{span}(S)$ to be dense in H, we know that there are sequences $(\beta_{k,n})$ such that

$$x_n := \sum_{k=1}^{C(n)} \beta_{k,n} \cdot e_k \quad \text{with} \quad x_n \to x \tag{9}$$

Computing the scalar product of x_n and some $e_j \in S$, we obtain

$$(x_n, e_j) = \sum_{k=1}^{C(n)} \beta_{k,n} \cdot (e_k, e_j) = \beta_{j,n} \tag{10}$$

Using this formula, the Bessel inequality (cf. Thm. 4) now guarantees that

$$\begin{aligned}
\left\| \sum_{k=1}^{C(n)} (x, e_k) e_k - x_n \right\|_H^2 &= \left\| \sum_{k=1}^{C(n)} (x, e_k) e_k - \sum_{k=1}^{C(n)} (x_n, e_k) e_k \right\|_H^2 && (11) \\
&= \left\| \sum_{k=1}^{C(n)} (x - x_n, e_k) e_k \right\|_H^2 \\
&= \sum_{k=1}^{C(n)} |(x - x_n, e_k)|^2 \\
&\leq \|x - x_n\|_H^2 \text{ (via Bessel)} && (12)
\end{aligned}$$

Since we know that $x_n \to x$, this gives us the desired result (via triangle inequality)

$$\lim_{n \to \infty} \sum_{k=1}^{n} (x, e_k) e_k = x \tag{13}$$

(2.) $\Rightarrow$ (3.) A simple computation now reveals that for all $x, y \in H$

$$\begin{aligned}
(x, y) &= \left(\sum_{k_1} (x, e_{k_1})\, e_{k_1}, \sum_{k_2} (y, e_{k_2})\, e_{k_2} \right) && (14) \\
&= \sum_{k_1} (x, e_{k_1}) \left(e_{k_1}, \sum_{k_2} (y, e_{k_2})\, e_{k_2} \right) \\
&= \sum_{k_1} \sum_{k_2} (x, e_{k_1})\, \overline{(y, e_{k_2})}\, (e_{k_1}, e_{k_2}) \\
&= \sum_{k} (x, e_k)\, (e_k, y) && (15)
\end{aligned}$$

(3.) $\Rightarrow$ (4.) Simply set $x = y$.
(4.) $\Rightarrow$ (1.) (we will not need this implication) Assume $\overline{\text{span}}(S) \neq H$. Then there must be a $h \in H$ with $h \notin \overline{\text{span}}(S)$. We define

$$h' := h - \sum_{k} (h, e_k)\, e_k. \tag{16}$$

We know $h' \neq 0$ since $h \notin \overline{\text{span}}(S)$ and it is clear that h' is orthogonal to all (e_k). Now the Parseval identity gives us

$$\|h'\|_H^2 = \sum_k |(h', e_k)|^2 = 0 \tag{17}$$

However, we know that $h' \neq 0$. This contradiction implies the desired result. ■

Theorem 6 (Fischer-Riesz) *For every separable, complex Hilbert space H of infinite dimension and every countable orthonormal basis $S = \{e_1, e_2, \ldots\} \subset H$ there is an isometric isomorphism*

$$\begin{aligned} \mathcal{F} : H &\rightarrow l^2(S, \mathbb{C}) \\ x &\mapsto ((x, e))_{e \in S} \end{aligned} \tag{18}$$

Proof. For every $x \in H$ we define

$$\mathcal{F}(x) := ((x, e))_{e \in S} \tag{19}$$

The Bessel Inequality (cf. Thm. 4) guarantees that $\mathcal{F}(x) \in l^2(S, \mathbb{C})$. It is clear that $\mathcal{F}$ is linear and Parseval's formula (cf. Thm. 5) assures us that $\mathcal{F}$ is isometric (and continuous). It remains to show that $\mathcal{F}$ is bijective: Take an arbitrary $(y_e)_{e \in S} \in l^2(S, \mathbb{C})$ and define

$$x := \sum_{e \in S} y_e e \tag{20}$$

This series converges as $(y_e)_{e \in S} \in l^2(S, \mathbb{C})$ and clearly $\mathcal{F}(x) = (y_e)_{e \in S}$. This completes the proof (continuity of the inverse map is clear by isometry). ■

2 Convergence in $L^2(\mathbb{T}^n, \mathbb{C})$

The Fourier series of a given complex-valued function $f \in L^2(\mathbb{T}^n, \mathbb{C})$ is actually defined to be the orthonormal basis representation of f with respect to a special orthonormal basis, which we are going to define now.

Theorem 7 (Fourier Orthonormal System) [1] *The complex-valued functions $e_k : \mathbb{T}^n \rightarrow \mathbb{C}$ with $k \in \mathbb{Z}^n$, given by*

$$e_k(x) := (2\pi)^{-\frac{n}{2}} e^{ikx} = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} e^{ik_j x_j} \tag{21}$$

[1] We will read $\mathbb{T}^n$ as the space $(\mathbb{R}/2\pi\mathbb{Z})^n$, where we will identify equivalence classes with their canonical representatives in $[0, 2\pi)^n$. This supplies us with a coordinate system on $\mathbb{T}^n$.
We will interpret $|k|$ with $k \in \mathbb{Z}^n$ as $|k|_\infty$ (maximum norm). For multi-indices $\alpha \in \mathbb{N}^n$ we will read $|\alpha|$ as $|\alpha|_1$.

constitute an orthonormal basis of $L^2(\mathbb{T}^n, \mathbb{C})$, which we will call the Fourier basis.

Remark 8 *From a historical perspective the Fourier basis was originally defined to be a system of sine and cosine functions. However, thanks to Euler's formula we can express such functions as sums of exponential functions, which turns out to give a much more comfortable theory as it simplifies formulas significantly.*

Proof. *(Step 1)* First we will show that the functions $\{e_k\}$ are an orthonormal system in $L^2(\mathbb{T}^n, \mathbb{C})$. To do this, let $k, l \in \mathbb{Z}^n$ be arbitrary. We compute straightforwardly

$$\begin{aligned} (e_k, e_l)_{L^2} &= (2\pi)^{-n} \int\limits_{\mathbb{T}^n} \prod_{j=1}^n e^{ik_j x_j} \cdot \prod_{j=1}^n \overline{e^{il_j x_j}} dx \\ &= (2\pi)^{-n} \int\limits_{\mathbb{T}^n} \prod_{j=1}^n e^{i(k_j - l_j)x_j} dx, \end{aligned}$$

where we have exploitet the relation $\overline{e^{ix}} = e^{-ix}$. Now we apply Fubini's Theorem and get

$$\begin{aligned} &= (2\pi)^{-n} \prod_{j=1}^n \int\limits_0^{2\pi} e^{i(k_j - l_j)x_j} dx_j \\ &= (2\pi)^{-n} \prod_{j=1}^n \begin{cases} 2\pi & \text{for} \quad k_j = l_j \\ 0 & \text{for} \quad k_j \neq l_j \end{cases} \\ &= \begin{cases} 1 & \text{for} \quad k = l \\ 0 & \text{for} \quad k \neq l \end{cases} \end{aligned}$$

Hence, $\{e_k\}$ is an orthonormal system of $L^2(\mathbb{T}^n, \mathbb{C})$. It remains to show that this orthonormal system is also complete.
(Step 2) The completeness can be proven by the denseness of $\text{span}_{\mathbb{C}}(e_k)$ in $C(\mathbb{T}^n, \mathbb{C})$ (which is again dense in $L^2(\mathbb{T}^n, \mathbb{C})$), which is a consequence of the Stone-Weierstraß theorem[2]. We will not go into any details, since this will also follow from later results. ■

Now we are in the position to define Fourier series.

Definition 9 (Fourier Series in the L^2-sense) *Suppose $f \in L^2(\mathbb{T}^n, \mathbb{C})$. For every $k \in \mathbb{Z}^n$ we define the k-th* Fourier coefficient *of f to be the complex number*

$$\hat{f}_k := (2\pi)^{-n} \int\limits_{\mathbb{T}^n} f(x) e^{-ikx} dx \tag{22}$$

[2] which requires the functions to be a unital $\mathbb{C}$-algebra, being point-separating and having complex conjugates.

(which equals $(2\pi)^{-\frac{n}{2}}(f, e_k)$*) and we will call*

$$\sum_{k\in\mathbb{Z}^n} \hat{f}_k \cdot e^{ikx} \tag{23}$$

the corresponding Fourier series*. The Fourier series of* f *converges to* f *w.r.t. the norm of* $L^2(\mathbb{T}^n, \mathbb{C})$ *(which follows from Thm. 5). Additionally we have the (classical)* Parseval formula*, which states*

$$\int_{\mathbb{T}^n} |f(x)|^2 \, dx = (2\pi)^n \sum_{k\in\mathbb{Z}^n} \left|\hat{f}_k\right|^2 \tag{24}$$

Remark 10 *For notational convenience it is usual to scale the coefficients obtained by the Fischer-Riesz isomorphism with a factor of* $(2\pi)^{-\frac{n}{2}}$*, hereby avoiding the factor to appear in the Fourier series. We follow this convention (as Roe does).*

Remark 11 *The Fischer-Riesz isomorphism provides the mathematical background for the interesting insight that it makes no difference whether we approximate a given function* f *by least-squares-method for its values or for its frequency distribution.*

2.0.1 Pointwise Convergence

It is important to note that convergence in the L^2-norm does NOT imply pointwise convergence almost everywhere. So what about pointwise convergence in L^2 (or more generally L^p)? We have the following results:

Theorem 12 (Kolmogorov, 1926) [3] *In* $\mathcal{L}^1(\mathbb{T}^1)$ *there exists a function with a Fourier series that diverges almost everywhere.*

After this result Lusin conjectured that the theorem would carry over to $\mathcal{L}^2$. This became known as LUSIN CONJECTURE, which was generally expected to be true (for example by Zygmund).

Theorem 13 (Carleson, 1966) [4] *In* $\mathcal{L}^2(\mathbb{T}^1)$ *the Fourier series of every function converges pointwise almost everywhere.*

This result was a big surprise; Carleson actually found his proof as a result of an unsuccessful attemp to disprove it. Richard Hunt generalized the result.

[3] A. N. Kolmogorov "Une série de Fourier-Lebesgue divergente presque partout", C. R. Acad. Sci. Paris 183 (1926), 2 pages

[4] L. Carleson "On convergence and growth of partial sums of Fourier series", Acta Math. 116, (1966), 22 pages

Theorem 14 (Carleson-Hunt, 1967) [5] *In $\mathcal{L}^p(\mathbb{T}^1)$ with $p > 1$ the Fourier series of every function converges pointwise almost everywhere.*

A generalization of these results to the higher-dimensional case seems to be difficult and is still subject of research. If partial sums are taken over a "growing" polygon of fixed shape (in $\mathbb{Z}^2$), then the result carries over to the 2-dimensional case. However, C. Fefferman has given a polygon of varying shape (a rectangle indeed), for which the result becomes wrong.[6]

There is an "8-page-proof" of the Carleson theorem[7], however the step from what the authors actually prove to the Carleson result is not really immediate.

3 Convergence in $C^m(\mathbb{T}^n, \mathbb{C})$ $(m \geq 1)$

Now we will consider the situation to reconstruct a function f in $C^m(\mathbb{T}^n, \mathbb{C})$ from given Fourier coefficients. Note that the norm of $C^m(\mathbb{T}^n, \mathbb{C})$ is very different from the norm of $L^2(\mathbb{T}^n, \mathbb{C})$, so it is no wonder that convergence problems arise. For our first observation, assume that $\{g_k\}_{k \in \mathbb{Z}^n}$ are arbitrary complex numbers. We will try to interpret them as Fourier coefficients of a (hopefully existing) unknown function on $\mathbb{T}^n$. The p-th partial sum of the Fourier series is

$$f_p(x) = \sum_{\max(|k_j|) \leq p} g_k \cdot e^{ikx} \tag{25}$$

Observe that the α-th partial derivative of f_p consequently satisfies

$$D^\alpha f_p(x) = \sum_{\max(|k_j|) \leq p} g_k \cdot (ik)^\alpha \cdot e^{ikx} \tag{26}$$

By the Fischer-Riesz isomorphism (Thm. 6) we know that all "coefficients" $g \in l^2(\mathbb{Z}^n, \mathbb{C})$ have a corresponding function $f \in L^2(\mathbb{Z}^n, \mathbb{C})$, which has exactly these Fourier coefficients. We will start with the question, how far this reconstruction of a function f from given "coefficients" works in $C^m(\mathbb{T}^n, \mathbb{C})$.

Theorem 15 (Reconstruction) *Suppose $\{g_k\}_{k \in \mathbb{Z}^n}$ are arbitrary complex numbers. If there is a natural number $m \geq 0$, such that for all $|\alpha| \leq m$ the inequality*

$$\sum_{k \in \mathbb{Z}^n} |k^\alpha| \cdot |g_k| < \infty \tag{27}$$

holds, then there is a complex-valued function $f \in C^m(\mathbb{T}^n, \mathbb{C})$ which has Fourier coefficients $\hat{f}_k = g_k$.

[5] R. Hunt "On the convergence of Fourier series", Southern Illinois Univ. Press, Carbondale, 1968, 20 pages

[6] I quote these two results from Ash, J.M. "Review of S. Krantz 'A Panorama of Harmonic Analysis" (2000), who himself quotes from Krantz.

[7] M. Lacey, C.M. Thiele "Convergence of Fourier series", preprint, 8 pages (available online)

Proof. The proof of this theorem is rather straightforward, although it is an existence proof [8]. Surely (27) is a Cauchy sequence. Hence, for every $\varepsilon > 0$ there is a $N(\varepsilon)$, such that for all $q > p \geq N(\varepsilon)$ the inequality

$$\sum_{p<\max(|k_j|)\leq q} |k^\alpha| \cdot |g_k| < \varepsilon \tag{28}$$

holds. Now we use this inequality to give an estimate for the Fourier series with coefficients g_k. The triangle inequality guarantees that

$$\left|\sum_{p<\max(|k_j|)\leq q} (ik)^\alpha \cdot g_k \cdot e^{ikx}\right| \leq \sum_{p<\max(|k_j|)\leq q} \left|(ik)^\alpha \cdot g_k \cdot e^{ikx}\right| \tag{29}$$

$$= \sum_{p<\max(|k_j|)\leq q} |k^\alpha| \cdot |g_k| < \varepsilon. \tag{30}$$

Now a direct computation of the norm in C^m gives us

$$\|f_p - f_q\|_{C^m} = \sum_{|\alpha|\leq m} \|D^\alpha(f_p - f_q)\|_{C^0} \tag{31}$$

$$= \sum_{|\alpha|\leq m} \sup_{x\in\mathbb{T}^n} \left|\sum_{p<\max(|k_j|)\leq q} (ik)^\alpha \cdot g_k \cdot e^{ikx}\right| \tag{32}$$

$$< (m+1)^n \cdot \varepsilon. \tag{33}$$

Hence, f_p is a Cauchy sequence and by completeness of $C^m(\mathbb{T}^n, \mathbb{C})$ we obtain the existence of a function f. It is clear that f has the Fourier coefficients $\{g_k\}$ (compute them!). ■

We originally intended to show the convergence of the Fourier series of a function $f \in C^m(\mathbb{T}^n, \mathbb{C})$. We will do this by showing that the Fourier coefficients of such a function satisfy the conditions of the preceding theorem, which then implies the desired result. Quite surprisingly $f \in C^m(\mathbb{T}^n, \mathbb{C})$ does not turn out to suffice for convergence of the Fourier series in $C^m(\mathbb{T}^n, \mathbb{C})$. We will need to require stronger regularity conditions on f in the form of higher differentiability.[9]

Theorem 16 (Convergence for C^m) *Let $m \geq 0$ be some natural number. Suppose $f \in C^{m+\lceil \frac{n+1}{2} \rceil}(\mathbb{T}^n, \mathbb{C})$. Then*

$$f(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}_k \cdot e^{ikx} \tag{34}$$

w.r.t. the norm of $C^m(\mathbb{T}^n, \mathbb{C})$.

[8] This is because we delegate the "hard work" to show existence to the completeness of C^m.

[9] In the one-dimensional case there are several criteria that suffice for convergence in C^0 which are weaker that differentiability; for example it is enough to require f to be of bounded variation (= the difference of two monotonously increasing functions).

Proof. For the Fourier coefficients of f we have

$$\hat{f}_k = (2\pi)^{-n} \int_{\mathbb{T}^n} f(x) \cdot e^{-ikx} dx \tag{35}$$

Clearly the partial derivatives of f (as far as they exist) can also be expressed as Fourier series. Assume $|\alpha| \leq m + \lceil \frac{n+1}{2} \rceil$. We will denote the Fourier coefficients of $D^\alpha f$ with $\hat{f}_k^\alpha$. For these coefficients we compute

$$\begin{aligned}
\hat{f}_k^\alpha &= (2\pi)^{-n} \int_{\mathbb{T}^n} D^\alpha f(x) \cdot e^{-ikx} dx && (36)\\
&= (2\pi)^{-n} \underbrace{\int_{\mathbb{T}^n} f(x) \cdot (-ik)^\alpha \cdot e^{-ikx} dx}_{\text{partial integration}} && (37)\\
&= (-ik)^\alpha \cdot \left[(2\pi)^{-n} \int_{\mathbb{T}^n} f(x) \cdot e^{-ikx} dx \right] && (38)\\
&= (-ik)^\alpha \cdot \hat{f}_k && (39)
\end{aligned}$$

From this, we conclude the following inequality, which will be slightly more convenient for our purposes.

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} |k|^{2\left(m+\lceil \frac{n+1}{2} \rceil\right)} \cdot \left|\hat{f}_k\right|^2 &= \sum_{k\in\mathbb{Z}^n} \max(|k_1|, \ldots, |k_n|)^{2\left(m+\lceil \frac{n+1}{2} \rceil\right)} \cdot \left|\hat{f}_k\right|^2 && (40)\\
&\leq \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}^n} |k_j|^{2\left(m+\lceil \frac{n+1}{2} \rceil\right)} \cdot \left|\hat{f}_k\right|^2 && (41)\\
&\leq \underset{\text{(with } \alpha=(0,\ldots,m+\lceil \frac{n+1}{2} \rceil,\ldots,0) \text{ at } j\text{-th place)}}{\sum_{j=1}^{n} \sum_{k\in\mathbb{Z}^n} |k^\alpha|^2 \cdot \left|\hat{f}_k\right|^2} && (42)\\
&= \sum_{j=1}^{n} \underbrace{\sum_{k\in\mathbb{Z}^n} \left|(-ik)^\alpha \cdot \hat{f}_k\right|^2}_{=\|\hat{f}^\alpha\|_{l^2}^2} && (43)\\
&= \sum_{j=1}^{n} \|D^\alpha f\|_{L^2}^2 < \infty \text{ (by Fischer-Riesz)} && (44)
\end{aligned}$$

It is time to work on the boundedness conditions that we will need to invoke Thm. 15. Assume $|\beta| \leq m$. We get

$$\sum_{\substack{k\in\mathbb{Z}^n\\k\neq 0}} \left|k^\beta\right| \cdot \left|\hat{f}_k\right| = \sum_{\substack{k\in\mathbb{Z}^n\\k\neq 0}} \max(\left|k_1^{\beta_1}\right|, \ldots, \left|k_n^{\beta_n}\right|) \cdot \left|\hat{f}_k\right|$$

$$\leq \sum_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} |k|^m \cdot \left|\hat{f}_k\right| \tag{45}$$

$$= \sum_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} \underbrace{\frac{1}{|k|^{\lceil \frac{n+1}{2} \rceil}} \cdot |k|^{m+\lceil \frac{n+1}{2} \rceil}}_{=|k|^m} \cdot \left|\hat{f}_k\right| \tag{46}$$

$$= \left\| \left(\left(\frac{1}{|k|^{\lceil \frac{n+1}{2} \rceil}} \right) \cdot \left(|k|^{m+\lceil \frac{n+1}{2} \rceil} \cdot \left|\hat{f}_k\right| \right) \right) \right\|_{l^1} \tag{47}$$

Then the Cauchy-Schwarz inequality supplies us with the estimate

$$\leq \sqrt[2]{\sum_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} \frac{1}{|k|^{2\lceil \frac{n+1}{2} \rceil}}} \cdot \sqrt[2]{\sum_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} |k|^{2\left(m+\lceil \frac{n+1}{2} \rceil\right)} \cdot \left|\hat{f}_k\right|^2} \tag{48}$$

One sees that the first sum is constant over "boxes" of same extent. Hence, for some constant C_n (only depending on dimension) we get

$$\leq \underbrace{\sqrt[2]{C_n \cdot \sum_{\substack{\hat{k}\in\mathbb{Z} \\ \hat{k}>0}} \frac{1}{\hat{k}^{2\lceil \frac{n+1}{2} \rceil}} \cdot \hat{k}^{n-1}}}_{\text{summing over "balls" of constant value}} \cdot \sqrt[2]{\sum_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} |k|^{2\left(m+\lceil \frac{n+1}{2} \rceil\right)} \cdot \left|\hat{f}_k\right|^2} \tag{49}$$

$$\leq \sqrt[2]{C_n \cdot \sum_{\substack{\hat{k}\in\mathbb{Z} \\ \hat{k}>0}} \cdot \hat{k}^{-2}} \cdot \sqrt[2]{\sum_{\substack{k\in\mathbb{Z}^n \\ k\neq 0}} |k|^{2\left(m+\lceil \frac{n+1}{2} \rceil\right)} \cdot \left|\hat{f}_k\right|^2} \tag{50}$$

Now the first sum converges (to $\frac{C_n \pi^2}{6}$) and the second one as well by (40-44). This is the desired boundedness and Thm. 15 finishes the proof. ■

As a by-product we see

Corollary 17 *Suppose $f \in C^\infty(\mathbb{T}^n, \mathbb{C})$. Then the Fourier series of f converges to f w.r.t. the norm of C^m and we have*

$$\sum_{k\in\mathbb{Z}^n} |k|^m \cdot \left|\hat{f}_k\right|^2 < \infty \tag{51}$$

for any m of our choice. Hence the Fourier series of smooth functions are also very well-behaved.

4 Convergence in $C^m(\mathbb{T}^n, \mathbb{C})$ $(m = 0)$

Why do we need f to be of higher differentiability order than m for an m-times continuously differentiable Fourier series? We will investigate the case for

$m = 0$. So we focus on the more special question: How can the Fourier series of a continuous function fail to converge? Let us take a closer look at how the inversion of a Fourier series works. We can actually interpret it as a composition of two maps; the first mapping the function f to its Fourier coefficients $\hat{f}$, the second mapping these coefficients to a function f_{inv} (which we hope to coincide with f).

$$f \overset{\int_{\mathbb{T}^n} \ldots du}{\longrightarrow} \hat{f} \overset{\sum_{\mathbb{Z}^n} \ldots}{\longrightarrow} f_{inv} \tag{52}$$

In order to see why convergence may not work, looking at the composition of these two maps reveals to be a good approach. We know that this gives the identity map in $L^2(\mathbb{T}^n, \mathbb{C})$, however, is this also the case in $C(\mathbb{T}^n, \mathbb{C})$? Suppose we are given a function $f \in C(\mathbb{T}^n, \mathbb{C})$. Then the partial sums of the Fourier series have the form

$$\begin{aligned}
s_g(x) &= \sum_{\max(|k_j|) \leq g} \hat{f}_k \cdot e^{ikx} && (53)\\
&= \sum_{\max(|k_j|) \leq g} (2\pi)^{-n} \int_{\mathbb{T}^n} f(u) e^{-iku} du \cdot e^{ikx} && (54)\\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} \sum_{\max(|k_j|) \leq g} f(u) e^{-iku} \cdot e^{ikx} du && (55)\\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} \sum_{\max(|k_j|) \leq g} f(x-u) e^{iku} du && (56)\\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} f(x-u) \cdot \underbrace{\sum_{\max(|k_j|) \leq g} e^{iku}}_{=: D_g \text{ (Dirichlet Kernel)}} du && (57)\\
&= (2\pi)^{-n} (D_g * f). && (58)
\end{aligned}$$

Definition 18 (Dirichlet Kernel) *For every $g \geq 0$ we define the* Dirichlet kernel $D_g : \mathbb{R}^n \to \mathbb{C}$ *to be*

$$D_g(u) := \sum_{-g \leq k_j \leq g} e^{iku} = (2g+1)^c \cdot \prod_{\substack{j=1 \\ u_j \neq 0}}^{n} \frac{\sin\left(\left(g + \frac{1}{2}\right) \cdot u_j\right)}{\sin\left(\frac{1}{2} \cdot u_j\right)} \tag{59}$$

where c is the number of components of $u \in \mathbb{R}^n$ with $u_j = 0$.

Proof. Here we need to prove the equality of the two given formulas. This is a rather straightforward computation based on the summation of geometric series. We obtain

$$\sum_{-g\le k_j\le g} e^{iku} = \prod_{j=1}^{n}\sum_{k_j=-g}^{g} e^{ik_ju_j} \tag{60}$$

$$= \prod_{j=1}^{n} e^{-igu_j}\sum_{k_j=0}^{2g} e^{ik_ju_j} \tag{61}$$

$$= (2g+1)^c\cdot\prod_{\substack{j=1\\ u_j\neq 0}}^{n} e^{-igu_j}\underbrace{\frac{e^{i(2g+1)u_j}-1}{e^{iu_j}-1}}_{\text{geometric series}} \tag{62}$$

$$= (2g+1)^c\cdot\prod_{\substack{j=1\\ u_j\neq 0}}^{n}\frac{e^{i(2g+1-g)u_j}-1\cdot e^{-igu_j}}{e^{iu_j}-1}\cdot\underbrace{\frac{e^{-i\frac{1}{2}u_j}}{e^{-i\frac{1}{2}u_j}}}_{=1} \tag{63}$$

$$= (2g+1)^c\cdot\prod_{\substack{j=1\\ u_j\neq 0}}^{n}\frac{e^{i(g+\frac{1}{2})u_j}-e^{-i(g+\frac{1}{2})u_j}}{e^{i\frac{1}{2}u_j}-e^{-i\frac{1}{2}u_j}} \tag{64}$$

$$= (2g+1)^c\cdot\prod_{\substack{j=1\\ u_j\neq 0}}^{n}\frac{\sin\left(\left(g+\frac{1}{2}\right)\cdot u_j\right)}{\sin\left(\frac{1}{2}\cdot u_j\right)}, \tag{65}$$

which is exactly what we have claimed to be true. ■

As we would like the Fourier series of every function $f\in C(\mathbb{T}^n,\mathbb{C})$ to converge to f, we hope that $(D_g * f)$ converges to f w.r.t. the norm of $C(\mathbb{T}^n,\mathbb{C})$. Thus $\{D_g\}_{g\to\infty}$ needs to be an approximate identity. The sequence D_g induces a sequence of operators $(D_g*\cdot): C(\mathbb{T}^n,\mathbb{C})\to C(\mathbb{T}^n,\mathbb{C})$. We will define pointwise evaluation operators $F_g: C(\mathbb{T}^n,\mathbb{C})\to\mathbb{C}$ by applying $(D_g*\cdot)$ and then evaluating (D_g*f) at some fixed point of $\mathbb{T}^n$ (the choice of the point is arbitrary, of course, we will take $(0,0,\ldots,0)\in\mathbb{T}^n$). Hence,$F_g(f) := \int_{\mathbb{T}^n} D_g(-u)\cdot f(u)du$

Lemma 19 *The norm of these operators turns out to be*

$$\int_{\mathbb{T}^n} |D_g(u)|\, du \tag{66}$$

Proof. *(Step 1)* It is rather easy to find an upper bound for the operator norm.

A direct computation leads to

$$
\begin{aligned}
|F_g(f)|_{\mathbb{C}} := & \left|\int\limits_{\mathbb{T}^n} D_g(-u)\cdot f(u)du\right| && (67)\\
\leq & \int\limits_{\mathbb{T}^n} |D_g(u)|\cdot|f(u)|\,du && (68)\\
\leq & \int\limits_{\mathbb{T}^n} |D_g(u)|\,du \cdot \sup_{x\in\mathbb{T}^n}|f(x)| && (69)\\
= & \int\limits_{\mathbb{T}^n} |D_g(u)|\,du \cdot \|f\|_{C(\mathbb{T}^n,\mathbb{C})} && (70)\\
\Rightarrow & \quad \|F_g\|_{L(C(\mathbb{T}^n,\mathbb{C}),\mathbb{C})} \leq \int\limits_{\mathbb{T}^n} |D_g(u)|\,du && (71)
\end{aligned}
$$

(Step 2) Obtaining a lower bound reveals to be a more complicated issue. We need to construct a sequence of functions that exhausts our desired bound, showing that no lower number than our claim can be the operator norm. Let $\varepsilon > 0$ be given. We set

$$
f_\varepsilon(u) := \frac{\overline{D_g(u)}}{|D_g(u)|+\varepsilon} \tag{72}
$$

($\frac{\overline{D_g(u)}}{|D_g(u)|}$ would be perfect, but unfortunately it fails to be continuous as the Dirichlet kernel has zeros) and it is easy to see that $f_\varepsilon \in C(\mathbb{T}^n,\mathbb{C})$ and $\|f_\varepsilon\|_{C(\mathbb{T}^n,\mathbb{C})} \leq 1$. Next, we compute

$$
\begin{aligned}
F_g(f_\varepsilon) &= \int\limits_{\mathbb{T}^n} D_g(-u)\cdot\frac{\overline{D_g(u)}}{|D_g(u)|+\varepsilon}du && (73)\\
&= \int\limits_{\mathbb{T}^n} D_g(u)\cdot\frac{\overline{D_g(u)}}{|D_g(u)|+\varepsilon}du && (74)\\
&= \int\limits_{\mathbb{T}^n} \frac{\overline{D_g(u)}\cdot D_g(u)}{|D_g(u)|+\varepsilon}du && (75)\\
&= \int\limits_{\mathbb{T}^n} \frac{|D_g(u)|^2}{|D_g(u)|+\varepsilon}du && (76)
\end{aligned}
$$

Here we integrate a positive function, thus

$$
\geq \int\limits_{\mathbb{T}^n} \frac{|D_g(u)|^2-\varepsilon^2}{|D_g(u)|+\varepsilon}du
$$

The third binomial formula leads to

$$= \int_{\mathbb{T}^n} \frac{(|D_g(u)| - \varepsilon)(|D_g(u)| + \varepsilon)}{|D_g(u)| + \varepsilon} du \tag{77}$$

$$= \int_{\mathbb{T}^n} |D_g(u)| - \varepsilon du \tag{78}$$

This completes our proof if we let $0 < \varepsilon \to 0$. ■

We will need a little more machinery. Without proof we state the following theorem:

Theorem 20 (Banach-Steinhaus) [10] *Let X be a Banach space, Y a normed space, $\mathcal{I}$ some index set and $F_i \in L(X,Y)$ (with $i \in \mathcal{I}$). If we have*

$$\sup_{i \in \mathcal{I}} \|F_i(x)\|_Y < \infty \text{ (for all } x) \tag{79}$$

then this implies (even)

$$\sup_{i \in \mathcal{I}} \|F_i\|_{L(X,Y)} < \infty \tag{80}$$

for the operator norms.

Now let us assume that all Fourier series of continuous functions converge uniformly (this is what we are looking to find a contradiction for). Hence they also converge pointwise[11]. Consequently, the (ordinary complex) norm of the sequence $F_g(f)$ must be bounded for every function f (by definition and assumption $F_g(f)$ converges to $f(0)$ as $g \to \infty$). The Banach-Steinhaus theorem assures us that even

$$\sup_g \|F_g\| < \infty \tag{81}$$

holds. However, we will now explicitly compute a lower bound for $\|F_g\|$, which will tend to infinity as $g \to \infty$. This is the desired contradiction.

[10] also known as 'uniform boundedness principle'. Proofs can be found in any book about functional analysis.

[11] The whole argument actually also works for showing the stronger result that not even pointwise convergence is sure

Via our formula (66) for the operator norm of F_g, we compute

$$\begin{aligned}
\|F_g\| &= \int_{\mathbb{T}^n} |D_g(u)|\,du && (82)\\
&= \int_{\mathbb{T}^n} (2g+1)^c \cdot \prod_{\substack{j=1\\u_j\neq 0}}^{n} \left| \frac{\sin\left(\left(g+\frac{1}{2}\right)\cdot u_j\right)}{\sin\left(\frac{1}{2}\cdot u_j\right)} \right| du && (83)\\
&= \int_{\mathbb{T}^n} \prod_{j=1}^{n} \left| \frac{\sin\left(\left(g+\frac{1}{2}\right)\cdot u_j\right)}{\sin\left(\frac{1}{2}\cdot u_j\right)} \right| du && (84)\\
&= \prod_{j=1}^{n} \int_0^{2\pi} \left| \frac{\sin\left(\left(g+\frac{1}{2}\right)\cdot u_j\right)}{\sin\left(\frac{1}{2}\cdot u_j\right)} \right| du_j && (85)
\end{aligned}$$

The simple estimate $\sin(x) \leq x$ (for $x \geq 0$) admits the following lower bound

$$\begin{aligned}
&\geq \prod_{j=1}^{n} \int_0^{2\pi} \frac{\left|\sin\left(\left(g+\frac{1}{2}\right)\cdot u_j\right)\right|}{\frac{1}{2}\cdot |u_j|} du_j && (86)\\
&= \prod_{j=1}^{n} \int_0^{2\pi\cdot\left(g+\frac{1}{2}\right)} \frac{|\sin(u_j)|}{\frac{1}{2}\cdot|u_j|} du_j \text{ (substitution)} && (87)\\
&= 2^n \prod_{j=1}^{n} \int_0^{2\pi\cdot\left(g+\frac{1}{2}\right)} \frac{|\sin(u_j)|}{|u_j|} du_j && (88)\\
&\geq 2^n \prod_{j=1}^{n} \sum_{k=1}^{g} \int_{2\pi(k-1)}^{2\pi k} \frac{|\sin(u_j)|}{|u_j|} du_j && (89)
\end{aligned}$$

We have introduced the sum splitting of the integrals in order to estimate the single summands individually.

$$\begin{aligned}
&\geq 2^n \prod_{j=1}^{n} \sum_{k=1}^{g} \int_{2\pi(k-1)}^{2\pi k} \frac{|\sin(u_j)|}{2\pi k} du_j && (90)\\
&= \pi^{-n} \prod_{j=1}^{n} \sum_{k=1}^{g} \frac{1}{k} \int_{2\pi(k-1)}^{2\pi k} |\sin(u_j)|\,du_j && (91)\\
&= \pi^{-n} \left(\sum_{k=1}^{g} \frac{4}{k} \right)^n && (92)
\end{aligned}$$

We see that this tends to infinity even for $n = 1$. There also exist constructive counter-examples. The first one was found by DuBois-Reymond in 1876.

Remark 21 *There are several ways to "repair" this defect of Fourier series. The first solution which became widely known was due to* L. Fejer *(around 1900). He sums the Fourier series by means of* Cesaro *summation (use the limit of the averages of the partial sums of the Fourier series instead of the partial sums themselves). This leads to the so-called Fejér-Kernel, which has significantly nicer properties than the Dirichlet-Kernel (they are positive kernels and have totally bounded operator norm. The positivity of the kernels makes them into a* Dirac family*.* S. Lang *gave a proof in his recent lecture that such families approximate a unit element in the convolution algebra. This argument can be used to show convergence for the Fejér kernel (Lang actually had it on the board, however, he did not mention the name Fejér)). Using Fejér's method, every continuous function can be reconstructed from its (Cesàro-) Fourier series.*

˜ The End ˜